CHAPTER 1 ACTIVITY PACKET *solutions*

Instructions. This packet is due on Quercus no later than **11:59pm on Monday, September 8th**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the examples and lecture activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

Note: Typos/mistakes found after printing are noted in red text.

Lecture Activity 1.1. Find a system of linear equations that models each of the problems below.

P1. *Blending Problems* arise in contexts where several inputs are combined to produce a final product with specific desired properties. Given the characteristics of each input, the goal is to determine how much of each input is needed to meet the target requirements.

Suppose you are planning a meal using three foods. Each unit of food contains a different combination of protein, carbohydrates, and fat, as shown below:

Food	Protein (g)	Carbs (g)	Fat (g)
Food A	2	3	1
Food B	1	1	2
Food C	4	2	3

If you want your meal to contain exactly 30 grams of protein, 25 grams of carbs, and 20 grams of fat, how many units of each food should you include in your meal?

Solution. Let x denote the quantity of Food A, y denote the quantity of Food B, and z denote the quantity of Food C in our meal. We need to solve the system

$$\begin{cases} 2x + y + 4z = 30 \\ 3x + y + 2z = 25 \\ x + 2y + 3z = 20 \end{cases}$$

Using the methods discussed later in this chapter, we find $x = 5, y = 0, z = 5$. So, we need to include have 5 units of Food A and B in our meal, and we should not use any of Food C to achieve our desired amounts of protein, carbs, and fats.

P2. Given a collection of data, it is often useful to find a mathematical model that fits the data exactly. This process, called *interpolation*, allows us to estimate missing values and understand the underlying pattern of our data.

Find a cubic function $f(x) = ax^3 + bx^2 + cx + d$ that passes through all of the following points: $(0, -2), (1, 1), (2, 6)$, and $(3, 10)$.

Solution. Plugging each of these values into the function f yields the system of linear equations

$$\begin{cases} \hfill d = -2 \\ \hfill a + b + c + d = 1 \\ \hfill 8a + 4b + 2c + d = 6 \\ 27a + 9b + 3c + d = 10 \end{cases}$$

Using the methods discussed later in this chapter, we find $a = -\frac{1}{2}, b = \frac{5}{2}, c = 1, d = -2$ which gives the cubic function

$$f(x) = -\frac{1}{2}x^3 + \frac{5}{2}x^2 + x - 2.$$

P3. *Flow conservation* problems arise in systems where something moves through a network so that at each junction, the total amount flowing in must equal the total amount flowing out.

The University of Toronto supplies water to campus buildings from the city system through a one-way distribution network. In this system, the flow into each building must equal the flow out, unless some water is used on-site. Three connected buildings are involved: ROB (Robarts Library), SS (Sidney Smith Hall), and BA (Bahen Centre). Water is delivered between the buildings as follows:

- The city sends 120 liters of water per minute to ROB,
- ROB uses 35 liters of water per minute and sends the rest to SS,
- SS uses 40 liters of water per minute and sends the rest to BA, and
- BA uses 45 liters of water per minute and sends the rest to ROB.

Describe the rate at which water needs to flow between each of the buildings in order for the system to be conserved (that is, no water is lost or stored)?

Solution. Let x denote the amount of water sent from ROB to SS, y denote the amount of water sent from SS to BA, and z denote the amount of water sent from BA to ROB. With the information given above, we have

Buildling	Water In	Water Out
ROB	$120 + z$	$35 + x$
SS	x	$40 + y$
BA	y	$45 + z$

For the flow to be conserved throughout the system, the total water in must equal the total water out at each building. Using our table above, and rearranging our equations, gives the following system

$$\begin{cases} x - z = 85 \\ x - y = 40 \\ y - z = 45 \end{cases}$$

Using methods discussed later in this chapter, we find that $x = 85 + z$ and $y = 45 + z$. Depending on the rate at which water flows from BA, we can determine the rate at which water must flow from ROB to SS and SS to BA.

Lecture Activity 1.2. Find all solutions to the the system of linear equations that have the following augmented matrices

P1. $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right)$

Solution This augmented matrix represents the system

$$\begin{cases} x + y + z = 1 \\ \quad\;\; y - z = 1 \\ \qquad\;\; 2z = 2 \end{cases}$$

From the last equation, we can solve $z = 1$. We can then plug this into the previous equation to find that

$$y - 1 = 1 \Rightarrow y = 2.$$

Finally, we can plug our values for y, z into the first equation to get

$$x + 2 + 1 = 1 \Rightarrow x = -2.$$

So, this system has solution $\boxed{\text{(x, y, z)=(-2, 2, 1)}}$.

P2. $\left(\begin{array}{ccc|c} 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$

Solution. This augmented matrix represents the system

$$\begin{cases} 2y - z = 1 \\ \qquad\; 0 = 1 \end{cases}$$

But the last equation is never true. That is, there do not exist real numbers x, y, z so that all equations in the system above hold, and so this system has $\boxed{\text{no solutions}}$.

P3. $\left(\begin{array}{ccc|c} 2 & 2 & 1 & 2 \\ 0 & 0 & 3 & 6 \end{array}\right)$

Solution. This augmented matrix represents the system

$$\begin{cases} 2x + 2y + z = 2 \\ \qquad\qquad 3z = 6 \end{cases}$$

From the last equation, we can solve $z = 2$. Plugging this into the first equation gives

$$2x + 2y + 2 = 2 \Rightarrow x + y = 0.$$

So, any triple $(x, y, 2)$ of real numbers satisfying $x + y = 0$ will yield a solution to this system. In a future section, we'll see that we can describe the set of solutions to this equation parametrically as $\boxed{\{(x, -x, 2) : x \in \mathbb{R}\}}$.

Definition 1.13. The PIVOT of a row in a matrix is ...

the leftmost nonzero entry in that row.

Definition 1.15. A matrix is in ROW ECHELON FORM if ...

1. all rows consisting only of zeros are at the bottom, and
2. the pivot of each nonzero row in the matrix is in a column to the right of the pivot of the row above it.

Definition 1.16. A matrix is in REDUCED ROW ECHELON FORM if ...

1. the matrix is in echelon form,
2. the pivot in each nonzero row is 1, and
3. each pivot is the only nonzero entry in its column.

Lecture Activity 1.3. Determine which of the following matrices are in row echelon form, reduced row echelon form, or neither.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 3 & 4 & -1/2 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; C = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; E = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; F = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution.

A is in neither row echelon or reduced row echelon form
B is in reduced row echelon form
C is in row echelon form but not reduced row echelon form
D is in row echelon form but not reduced row echelon form
E is in neither row echelon or reduced row echelon form
F is in reduced row echelon form

Lecture Activity 1.4. Give an example of matrix A that satisfies all of the following conditions.

- A is the augmented matrix for a system of 4 linear equations in 4 variables.
- A is in reduced row echelon form.
- A has exactly three pivots.
- A contains at least one entry other than 0 or 1.

Solution. One example of a matrix satisfying these conditions is given below

$$A = \left(\begin{array}{cccc|c} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Lecture Activity 1.5. Find rref(A), where

$$A = \begin{pmatrix} 1 & 1 & 1 & 7 & -1 \\ 0 & 1 & 1 & 4 & -1 \\ -1 & 2 & 2 & 5 & -2 \end{pmatrix}.$$

Solution. We have

$$\begin{aligned} A &\sim \begin{pmatrix} 1 & 1 & 1 & 7 & -1 \\ 0 & 1 & 1 & 4 & -1 \\ 0 & 3 & 3 & 12 & -3 \end{pmatrix}, \text{ via } R_3 + R_1 \\ &\sim \begin{pmatrix} 1 & 1 & 1 & 7 & -1 \\ 0 & 1 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ via } R_3 - 3R_2 \\ &\sim \begin{pmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ via } R_1 - R_2 \end{aligned}$$

,

which is in reduced row echelon form, and so $\boxed{\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}$

Definition 1.23. A system of linear equations is called CONSISTENT if it has at least one solution. If the system has no solutions, it is called INCONSISTENT.

Definition 1.24. Let C be the coefficient matrix for a consistent system of linear equations in variables $x_1, \ldots, x_n$.

1. We say that x_i is a BASIC VARIABLE for the system if ...

the ith column of $\text{rref}(C)$ has a pivot

2. We say that x_i is a FREE VARIABLE of the system if ...

the ith column of $\text{rref}(C)$ does not have a pivot.

Lecture Activity 1.6. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 7 & -1 \\ 0 & 1 & 1 & 4 & -1 \\ -1 & 2 & 2 & 5 & -2 \end{pmatrix}$$

be the matrix from Lecture Activity 1.5, and consider the system of linear equations in variables x, y, z, w which has A as its augmented matrix.

P1. Determine the free variables and basic variables of the system.

Solution. Recall from Activity 1.5 that

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have pivots in the first and second columns, and so x, y are basic while z, w are free.

P2. Give a parametric description of the solution set to this system.

Using rref(A) we see that our system is equivalent to

$$\begin{cases} x + 3w = 0 \\ y + z + 4w = -1 \end{cases}$$

Solving for our basic variables in terms of the free variables z, w gives

$$x = -3w, \text{ and } y = -1 - z - 4w.$$

Hence, the set of solutions to this system can be described parametrically as

$$\{(-3w, -1 - z - 4w, z, w) : z, w \in \mathbb{R}\}$$

Theorem 1.27. (Rouché-Capelli). Suppose that a system of linear equation has augmented matrix A and coefficient matrix C. Then,

1. The system is inconsistent if and only if ...

the last column of $\mathrm{rref}(A)$ has a pivot

2. The system has exactly one solution if and only if ...

the last column of $\mathrm{rref}(A)$ does not have a pivot, and every column of $\mathrm{rref}(C)$ has a pivot.

3. The system has infinitely many solutions if and only if ...

the last column of $\mathrm{rref}(A)$ does not have a pivot and $\mathrm{rref}(C)$ has a column without a pivot.

Lecture Activity 1.7. Suppose that we're given a set of n data points that lie exactly on the graph of some unknown cubic function. What's the minimum number of points needed to guarantee there's *exactly one* cubic function passing through them? Explain your answer.

Solution. We need at least [four points] to guarantee a unique function. As we saw in Lecture Activity 1.1, finding a cubic function $f(x) = ax^3 + bx^2 + cx + d$ passing through n points $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)$ is equivalent to solving the system of n linear equations in four variables below

$$\begin{cases} a\alpha_1^3 + b\alpha_1^2 + c\alpha_1 + d = \beta_1 \\ a\alpha_2^3 + b\alpha_2^2 + c\alpha_2 + d = \beta_2 \\ \qquad\qquad\vdots \\ a\alpha_n^3 + b\alpha_n^2 + c\alpha_n + d = \beta_n. \end{cases}$$

Let A be the augmented matrix for this system and C its coefficient matrix. By assumption, we know that the system has a solution, and so $\mathrm{rref}(A)$ does not have a pivot in the last column. By Rouché-Capelli, the system has exactly one solution if and only if $\mathrm{rref}(C)$ has pivots in all four columns. Since a matrix can have at most one pivot in each row, this means that C needs at least four rows, and hence we must have at least four points.

Remark: The problem statement from Lecture Activity 1.7 may have been phrased a bit deceptively. Note that four points are *necessary* to guarantee a unique function, but the argument above does not prove this is *sufficient* – we should be careful to note that $\text{rref}(C)$ can have a column without a pivot even when it only contains four rows. What we've shown is that *if* there is a unique cubic function passing through n points, then $n \geq 4$.

Note that sufficiency follows as an application of the Fundamental Theorem of Algebra (since cubic polynomials can have at most three roots, we can see that two cubics can intersect in at most three points in $\mathbb{R}^2$). This is how one could demonstrate the converse of what we've shown, that *if* we have at least $n \geq 4$ points that lie exactly on a cubic, then there is a unique cubic passing through these points (but this direction is beyond the scope of this course).