# CHAPTER 11 ACTIVITY PACKET (PART 2) *solutions*

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**Instructions.** This packet is due **for extra credit** on Quercus no later than **11:59pm on Monday, December 1st**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

**Definition 11.15.** An $n \times n$ matrix $A$ is called ORTHOGONALLY DIAGONALIZABLE if ...

> there exists an orthogonal matrix $Q$ and a diagonal matrix $D$ so that $Q^\top AQ = D$.

**Theorem 11.17** (The Spectral Theorem)**.** An $n \times n$ matrix $A$ is orthogonally diagonalizable if and only if ...

> it is symmetric; that is, $A = A^\top$.

**Lecture Activity 11.7.** Determine which of the following matrices are orthogonally diagonalizable. For those that are, find an orthonormal basis of eigenvectors.

P1. $A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$, given that $\chi_A(x) = (x-2)(x-7)$

*Solution.* Observe that $A = A^\top$, and so by the Spectral Theorem $A$ is orthogonally diagonalizable. We see that $A$ has eigenvalues 2 and 7 and calculate its eigenspaces to be

$$E_7 = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), E_2 = \operatorname{Span}\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right).$$

Observe that

$$\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$$

is an orthogonal basis for $\mathbb{R}^2$. Dividing each vector by its norm, we obtain an orthonormal basis

$$\left\{\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}, \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}\right\}$$

P2. $B = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$, given that $\chi_B(x) = (x+1)(x-2)$

*Solution.* Observe that $B \neq B^\top$, and so by the Spectral Theorem $B$ is **not** orthogonally diagonalizable. Observe that $B$ has eigenspaces

$$E_{-1} = \text{Span}\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right), E_2 = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

We see that none of the vectors in $E_{-1}$ are orthogonal to the vectors in $E_2$, and so it's not possible to obtain an orthogonal basis of eigenvectors.

P3. $C = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, given that $\chi_C(x) = -(x-4)(x-1)^2$

*Solution.* Observe that $C = C^\top$, and so by the Spectral Theorem $C$ is orthogonally diagonalizable. We see that $C$ has eigenvalues 4 and 1 and we calculate

$$E_4 = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right), E_1 = \text{Span}\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right).$$

We see that the vectors in $E_4$ are orthogonal to every vector in $E_1$, however our choice of basis for $E_1$ is not orthogonal. To find an orthogonal basis for this eigenspace, we can use Gram-Schmidt. Setting $\vec{u}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ we calculate

$$\begin{aligned}
\vec{u}_2 &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \text{proj}_{\vec{u}_1}\left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right) \\
&= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}
\end{aligned}$$

By Gram-Schmidt, we know that $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis for $E_1$. Hence,

$$\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}\right\}$$

is an orthogonal basis for $\mathbb{R}^3$ of eigenvectors for $C$. Dividing through by the norm of each vectors gives the orthonormal basis

$$\left\{\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}\right\}.$$

**Definition 11.19.** Suppose that $A$ is an $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ and orthonormal basis of eigenvectors $\{\vec{v}_1, \dots, \vec{v}_n\}$. We call the equality

$$A = QDQ^\top$$

a SPECTRAL DECOMPOSITION of $A$, where

$$D = \operatorname{diag}(\lambda_1, \dots, \lambda_n), \text{ and } Q = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}.$$

**Lecture Activity 11.8.** Consider the matrix $A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$ from Lecture Activity 11.7.

P1. Find a spectral decomposition for $A$. That is, find an orthogonal matrix $Q$ and diagonal matrix $D$ so that $A = QDQ^\top$.

*Solution.* By P1 of Lecture Activity 11.7 we know that $\mathbb{R}^2$ has an orthonormal basis of eigenvectors for $A$ given by

$$\left\{ \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}, \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \right\}.$$

So, $A$ has spectral decomposition

$$A = \underbrace{\begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}}_{Q^\top}$$

P2. Use Chapter Exercise P11.8 to show that $Q$ is a rotation matrix. Recalling that we can write $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, find the angle $\theta$.

*Solution.* Observe that $\det(Q) = 1$, and so by Chapter Exercise P11.8, $T_Q$ rotates vectors counterclockwise by an angle of $\theta$. So we must have

$$\cos\theta = 1/\sqrt{5} \Rightarrow \theta \approx 63°.$$

P3. Using your work in the previous parts, give a geometric description of how $T_A$ transforms $\mathbb{R}^2$.

Since $T_Q = T_Q \circ T_D \circ T_{Q^\top}$ then from our work above we see that $T_Q$ first rotates vectors $\approx 63°$ counterclockwise about the origin, then dilates by stretching horizontally by 7 and vertically by 2, then rotates vectors *clockwise* about the origin by $\approx 63°$.

**Proposition 11.21.** Let $A$ be an $m \times n$ matrix. Then, there exists an orthonormal basis for $\mathbb{R}^n$ of eigenvectors of $A^\top A$ so that $\{A\vec{v}_1, \dots, A\vec{v}_n\}$ is an orthogonal subset of $\mathbb{R}^m$. Furthermore, if we reindex our basis so that $A\vec{v}_1, \dots, A\vec{v}_r$ are nonzero, and $A\vec{v}_{r+1} = \cdots = A\vec{v}_n = \vec{0}$, then $\{A\vec{v}_1, \dots, A\vec{v}_r\}$ forms an orthogonal basis for $\text{Col}(A)$.

**Complete the proof: show that $A^\top A$ is symmetric.**

We have $(A^\top A)^\top = A^\top (A^\top)^\top = A^\top A$, and so $A$ is symmetric

So, by the Spectral Theorem, there there exists an orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{R}^n$ of eigenvectors of $A^\top A$. Suppose that each $\vec{v}_i$ has eigenvalue $\lambda_i$.

**Complete the proof: show that $(A\vec{v}_i) \cdot (A\vec{v}_j) = 0$ for all $i \neq j$.**

For any $i \neq j$ we have

$$\begin{aligned}(A\vec{v}_i) \cdot (A\vec{v}_j) &= (A\vec{v}_i)^\top (A\vec{v}_j) \\ &= \vec{v}_i^\top A^\top A \vec{v}_j \\ &= \vec{v}_i^\top (\lambda_j \vec{v}_j) \\ &= \vec{v}_i \cdot (\lambda_j \vec{v}_j) \\ &= \lambda_j (\vec{v}_i \cdot \vec{v}_j) \\ &= 0,\end{aligned}$$

where the final equality follows because $\vec{v}_i$ and $\vec{v}_j$ are orthogonal when $i \neq j$. Hence, $\{A\vec{v}_1, \dots, A\vec{v}_m\}$ is an orthogonal subset of $\mathbb{R}^m$.

Next, reindex our basis as in the theorem statement.

**Complete the proof: show that $\text{Col}(A) = \text{Span}(A\vec{v}_1, \dots, A\vec{v}_r)$.**

Noting that $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$, we have $A\vec{y} \in \text{Col}(A)$ if and only if

$$A\vec{y} = A(x_1\vec{v}_1 + \cdots + x_n\vec{v}_n) = x_1 A\vec{v}_1 + \cdots + x_r A\vec{v}_r + \vec{0},$$

and so $\text{Col}(A) = \text{Span}(A\vec{v}_1, \dots, A\vec{v}_r)$.

Since $\{A\vec{v}_1, \dots, A\vec{v}_r\}$ is orthogonal, then this set is linearly independent, and hence is a basis. $\square$

**Lecture Activity 11.9.** Consider the $3 \times 2$ matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & -1 \end{pmatrix}$.

P1. Verify that $A^\top A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$. Then, use your work from Lecture Activity 11.7 to find an orthonormal basis for $\mathbb{R}^2$ of eigenvectors $\{\vec{v}_1, \vec{v}_2\}$ for $A^\top A$.

*Solution.* We have

$$A^\top A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$$

and so by P1 of Lecture Activity 11.7 we know that $\mathbb{R}^2$ has an orthonormal basis of eigenvectors for $A^\top A$ given by $\{\vec{v}_1, \vec{v}_2\}$ where

$$\vec{v}_1 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}.$$

P2. Use Proposition 11.21 to verify that $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$ where

$$\vec{u}_1 = \frac{A\vec{v}_1}{\|A\vec{v}_1\|}, \vec{u}_2 = \frac{A\vec{v}_2}{\|A\vec{v}_2\|}$$

is an orthonormal basis for $\operatorname{im}(T_A)$.

*Solution.* We have

$$A\vec{v}_1 = \begin{pmatrix} 3/\sqrt{5} \\ \sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}, A\vec{v}_2 = \begin{pmatrix} -1/\sqrt{5} \\ 0 \\ -3/\sqrt{5} \end{pmatrix}$$

$$\Rightarrow \vec{u}_1 = \begin{pmatrix} 3/\sqrt{35} \\ \sqrt{35}/7 \\ -1/\sqrt{35} \end{pmatrix}, \vec{u}_2 = \begin{pmatrix} -1/\sqrt{10} \\ 0 \\ -3/\sqrt{10} \end{pmatrix}$$

which is an orthonormal basis for $\operatorname{im}(T_A)$ by Proposition 11.21.

P3. Show that there exists a vector $\vec{u}_3$ so that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

*Solution.* Observe that if we take any $\vec{v}_3$ not in $\operatorname{Span}(\vec{u}_1, \vec{u}_2)$, then $\{\vec{u}_1, \vec{u}_2, \vec{v}_3\}$ is linearly independent, and hence a basis for $\mathbb{R}^3$, and so we can use Gram-Schmidt to obtain a new vector $\vec{u}_3$ which is orthogonal to $\vec{u}_1, \vec{u}_2$. We omit the details of this calculation, and note that such a vector could be found explicitly.

P4. Let's use our work from the previous parts to decompose our transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^3$.

**Step 1: Rotate/reflect.** Let $Q = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \end{pmatrix}$ where $\{\vec{v}_1, \vec{v}_2\}$ is the basis from P1. Observe that $T_{Q^\top}$ rotates the plane by $\theta \approx 63°$ degrees clockwise, and that

$$T_{Q^\top}(\vec{v}_1) = \vec{e}_1 \text{ and } T_{Q^\top}(\vec{v}_2) = \vec{e}_2.$$

*Solution.* By P2 of Lecture Activity 11.8 we know that $T_Q$ rotates vectors counterclockwise by $\approx 63°$. Since $Q$ is orthogonal, we have that $Q^\top = Q^{-1}$ will rotate vectors *clockwise* by the same amount. Furthermore, since $T_Q : \vec{e}_i \mapsto \vec{v}_i$ we have the the inverse function will do the opposite. That is, $T_{Q^\top} : \vec{v}_i \mapsto \vec{e}_i$.

**Step 2: Dilate and embed.** Consider the "*block-diagonal*" matrix

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix}$$

where $\sigma_1 = \|A\vec{v}_1\|$ and $\sigma_2 = \|A\vec{v}_2\|$. Give a geometric description for the matrix-tranformation $T_\Sigma : \mathbb{R}^2 \to \mathbb{R}^3$.

*Solution.* Observe that

$$T_\Sigma(\vec{e}_1) = \begin{pmatrix} \sigma_1 \\ 0 \\ 0 \end{pmatrix}, T_\Sigma(\vec{e}_2) = \begin{pmatrix} 0 \\ \sigma_2 \\ 0 \end{pmatrix}.$$

So, this transformation stretches the $x$-axis by $\sigma_1$ and the $y$-axis by $\sigma_2$, and then "embeds" $\mathbb{R}^2$ into $\mathbb{R}^3$ by placing it on the $xy$-plane.

**Step 3: Rotate/reflect.** Let

$$U = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{pmatrix}$$

where $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is the orthonormal basis for $\mathbb{R}^3$ found in P3. Use geometric reasoning to convince yourself that $T_U$ is either a rotation or reflection transformation. *(Bonus: think about what computations you would need to perform to describe this rotation transformation explicitly).*

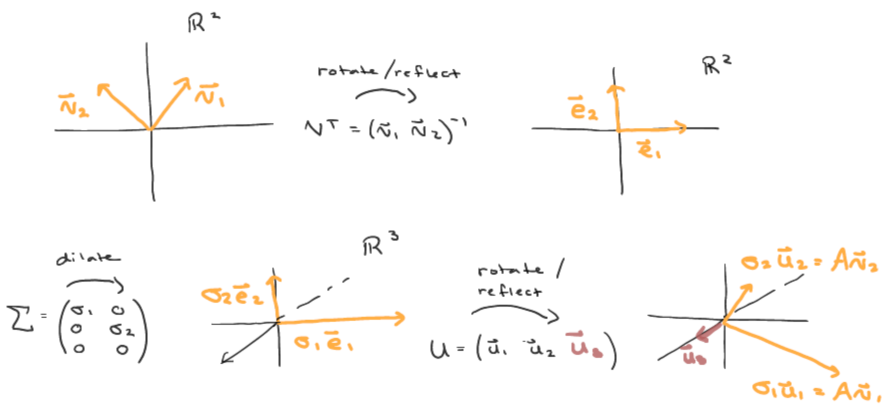
This transformation sends our standard basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ to an orthonormal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$. So, the image of the unit cube remains a unit cube, which can only be obtained by rotation or reflection.

P5. Use geometric reasoning, along with your work from P4, to show that

$$A = U\Sigma Q^\top,$$

where $U$ and $Q$ are orthogonal matrices, and $\Sigma$ is a "block-diagonal" matrix. Discuss how this decomposition describes the transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^3$ as a rotation/reflection, followed by dilation/embedding, followed by another rotation/reflection.

*Solution.* The steps from P4 are demonstrated in the image below




By Proposition 11.21 we see that this composition of transformations is precisely the transformation $T_A$, as demonstrated in the image below

**Definition 11.23.** Let $A$ be an $m \times n$ matrix and $\{\vec{v}_1, \dots, \vec{v}_n\}$ be an orthonormal basis for $\mathbb{R}^n$ of eigenvectors for $A^\top A$, as above. The SINGULAR VALUES of $A$ are ...

the scalars $\sigma_i := \|A\vec{v}_i\|$, for $i = 1, \dots, n$.

**Proposition 11.24.** Let $A$ be an $m \times n$ matrix and $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $A^\top A$. Then, $\lambda_i > 0$ and the singular values of $A$ are given by $\sigma_i = \sqrt{\lambda_i}$.

**Prove Proposition 11.24.**

*Proof.* We have

$$\begin{aligned}
\sigma_i^2 &= \|A\vec{v}_i\|^2 \\
&= (A\vec{v}_i) \cdot (A\vec{v}_i) \\
&= (A\vec{v}_i)^\top (A\vec{v}_i) \\
&= \vec{v}_i^\top A^\top A\vec{v}_i \\
&= \vec{v}_i^\top \lambda_i \vec{v}_i \\
&= \lambda_i \vec{v}_i \cdot \vec{v}_i \\
&= \lambda_i \|\vec{v}_i\|^2 \\
&= \lambda_i,
\end{aligned}$$

where the final equality follows because the $\vec{v}_i$ form an ortho**normal** set. Hence, $\sigma_i^2 = \lambda_i$ and so $\lambda_i > 0$ and $\sigma_i = \sqrt{\lambda_i}$. □

**Lecture Activity 11.10.** Find the singular values of the following matrices. Given your calculations, what can you say about the corresponding transformations?

P1. $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}$

*Solution.* The singular values of $A$ are the square roots of the eigenvalues of

$$A^\top A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}.$$

Since the eigenvalues of $A^\top A$ are 4 and 5, the singular values of $A$ are 2 and $\sqrt{5}$.

By Proposition 11.21, we know that there's an orthonormal basis $\{\vec{v}_1, \vec{v}_2\}$ so that $\{A\vec{v}_1, A\vec{v}_2\}$ is an orthogonal basis for $\text{im}(T_A)$. So, $T_A$ maps a unit square (with sides given by $\vec{v}_1, \vec{v}_2$) to a rectangle of side lengths $2, \sqrt{5}$. We can see that $T_A$ is injective, but not surjective, and that this transformation "stretches out" space (that is, our grid lines get further apart).

P2. $B = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix}$

*Solution.* The singular values of $B$ are the square roots of the eigenvalues of

$$B^\top B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 6 & 3 \\ 3 & 3 & 6 \end{pmatrix}.$$

Now $\chi_{B^\top B}(x) = -x^3 + 14x^2 - 33x = x(x-3)(x-11)$. Since the eigenvalues of $B^\top B$ are 0, 3 and 11, the singular values of $B$ are 0, $\sqrt{3}$ and $\sqrt{11}$.

By Proposition 11.21, we know there's an orthonormal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for $\mathbb{R}^3$ so that $\{B\vec{v}_2, B\vec{v}_3\}$ is a basis for $\text{im}(T_B)$. We see then that $T_B$ sends the unit cube (with sides given by $\vec{v}_1, \vec{v}_2, \vec{v}_3$) to a 2-dimensional rectangle with sides $\sqrt{3}$ and $\sqrt{11}$. So, $T_B$ is not injective nor surjective (since there must be some collapsing). We can see that $T_B$ collapses $\mathbb{R}^3$ onto a 2-dimensional subspaces, which gets "stretched out".

P3. $C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

*Solution.* The singular values of $C$ are the square roots of the eigenvalues of

$$C^\top C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Since $C^\top C$ has only one eigenvalue equal to 2, and so the singular values of $C$ are $\sqrt{2}$.

*Solution.* By Proposition 11.21, we know that $T_C$ sends a unit square to another square with side length $\sqrt{2}$. So, $T_C$ rotates/reflects $\mathbb{R}^2$ and stretches the grid lines out equally in both directions by a factor of $\sqrt{2}$.