CHAPTER 11 ACTIVITY PACKET (PART 2)   NAME:

Instructions. This packet is due **for extra credit** on Quercus no later than **11:59pm on Monday, December 1st**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

Definition 11.15. An $n \times n$ matrix A is called ORTHOGONALLY DIAGONALIZABLE if ...

Theorem 11.17 (The Spectral Theorem)**.** An $n \times n$ matrix A is orthogonally diagonalizable if and only if ...

Lecture Activity 11.7. Determine which of the following matrices are orthogonally diagonalizable. For those that are, find an orthonormal basis of eigenvectors.

P1. $A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$, given that $\chi_A(x) = (x-2)(x-7)$

P2. $B = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$, given that $\chi_B(x) = (x+1)(x-2)$

P3. $C = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, given that $\chi_C(x) = -(x-4)(x-1)^2$

Definition 11.19. Suppose that A is an $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ and orthonormal basis of eigenvectors $\{\vec{v}_1, \dots, \vec{v}_n\}$. We call the equality

a SPECTRAL DECOMPOSITION of A, where

Lecture Activity 11.8. Consider the matrix $A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$ from Lecture Activity 11.7.

P1. Find a spectral decomposition for A. That is, find an orthogonal matrix Q and diagonal matrix D so that $A = QDQ^\top$.

P2. Use Chapter Exercise P11.8 to show that Q is a rotation matrix. Recalling that we can write $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, find the angle θ.

P3. Using your work in the previous parts, give a geometric description of how T_A transforms $\mathbb{R}^2$.

Proposition 11.21. Let A be an $m \times n$ matrix. Then, there exists an orthonormal basis for $\mathbb{R}^n$ of eigenvectors of $A^\top A$ so that $\{A\vec{v}_1, \dots, A\vec{v}_n\}$ is an orthogonal subset of $\mathbb{R}^m$. Furthermore, if we reindex our basis so that $A\vec{v}_1, \dots, A\vec{v}_r$ are nonzero, and $A\vec{v}_{r+1} = \cdots = A\vec{v}_n = \vec{0}$, then $\{A\vec{v}_1, \dots, A\vec{v}_r\}$ forms an orthogonal basis for $\mathrm{Col}(A)$.

Complete the proof: show that $A^\top A$ is symmetric.

So, by the Spectral Theorem, there there exists an orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{R}^n$ of eigenvectors of $A^\top A$. Suppose that each $\vec{v}_i$ has eigenvalue λ_i.

Complete the proof: show that $(A\vec{v}_i) \cdot (A\vec{v}_j) = 0$ for all $i \neq j$.

Next, reindex our basis as in the theorem statement.

Complete the proof: show that $\mathrm{Col}(A) = \mathrm{Span}(A\vec{v}_1, \dots, A\vec{v}_r)$.

Since $\{A\vec{v}_1, \dots, A\vec{v}_r\}$ is orthogonal, then this set is linearly independent, and hence is a basis. □

Lecture Activity 11.9. Consider the 3×2 matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & -1 \end{pmatrix}$.

P1. Verify that $A^\top A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$. Then, use your work from Lecture Activity 11.7 to find an orthonormal basis for $\mathbb{R}^2$ of eigenvectors $\{\vec{v}_1, \vec{v}_2\}$ for $A^\top A$.

P2. Use Proposition 11.21 to verify that $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$ where

$$\vec{u}_1 = \frac{A\vec{v}_1}{\|A\vec{v}_1\|}, \vec{u}_2 = \frac{A\vec{v}_2}{\|A\vec{v}_2\|}$$

is an orthonormal basis for $\text{im}(T_A)$.

P3. Show that there exists a vector $\vec{u}_3$ so that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

P4. Let's use our work from the previous parts to decompose our transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^3$.

Step 1: Rotate/reflect. Let $Q = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \end{pmatrix}$ where $\{\vec{v}_1, \vec{v}_2\}$ is the basis from P1. Observe that $T_{Q^\top}$ rotates the plane by $\theta \approx 63°$ degrees clockwise, and that

$$T_{Q^\top}(\vec{v}_1) = \vec{e}_1 \text{ and } T_{Q^\top}(\vec{v}_2) = \vec{e}_2.$$

Step 2: Dilate and embed. Consider the "*block-diagonal*" matrix

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix}$$

where $\sigma_1 = \|A\vec{v}_1\|$ and $\sigma_2 = \|A\vec{v}_2\|$. Give a geometric description for the matrix-tranformation $T_\Sigma : \mathbb{R}^2 \to \mathbb{R}^3$.

Step 3: Rotate/reflect. Let

$$U = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{pmatrix}$$

where $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is the orthonormal basis for $\mathbb{R}^3$ found in P3. Use geometric reasoning to convince yourself that T_U is either a rotation or reflection transformation. *(Bonus: think about what computations you would need to perform to describe this rotation transformation explicitly).*

P5. Use geometric reasoning, along with your work from P4, to show that

$$A = U\Sigma Q^\top,$$

where U and Q are orthogonal matrices, and Σ is a "block-diagonal" matrix. Discuss how this decomposition describes the transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^3$ as a rotation/reflection, followed by dilation/embedding, followed by another rotation/reflection.

Definition 11.23. Let A be an $m \times n$ matrix and $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be an orthonormal basis for $\mathbb{R}^n$ of eigenvectors for $A^\top A$, as above. The SINGULAR VALUES of A are ...

Proposition 11.24. Let A be an $m \times n$ matrix and $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A^\top A$. Then, $\lambda_i > 0$ and the singular values of A are given by $\sigma_i = \sqrt{\lambda_i}$.

Prove Proposition 11.24.

Lecture Activity 11.10. Find the singular values of the following matrices. Given your calculations, what can you say about the corresponding transformations?

P1. $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}$

P2. $B = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix}$

P3. $C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$