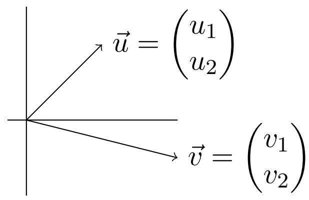
CHAPTER 11 ACTIVITY PACKET (PART 1)   NAME:

Instructions. This packet is due on Quercus no later than **11:59pm on Monday, November 24th**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

Lecture Activity 11.1. Let $\vec{u}$ and $\vec{v}$ be the vectors drawn below.



P1. Find a formula for the length of $\vec{u}$ and $\vec{v}$.

P2. Find a formula for the distance between $\vec{u}$ and $\vec{v}$. That is, find the distance between the points (u_1, u_2) and (v_1, v_2).

Definition 11.1. Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$ with standard-basis coordinates

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

The DOT PRODUCT of $\vec{u}$ and $\vec{v}$ is the scalar

Lemma 11.3. Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^n$ and let $c \in \mathbb{R}$ be a scalar. Then, the dot product satisfies the following properties:

1. *Commutativity:* $\vec{u} \cdot \vec{v} =$
2. *Distributivity with Addition:* $(\vec{u} + \vec{v}) \cdot \vec{w} =$
3. *Distributivity with Scalar Multiplication:* $(c\vec{u}) \cdot \vec{v} =$

Lecture Activity 11.2. Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^2$ and define the following notation

$$\|\vec{u}\| := \sqrt{\vec{u} \cdot \vec{u}}.$$

.

P1. Show that the length of $\vec{u}$ is equal to $\|\vec{u}\|$.

P2. Show that the distance between $\vec{u}$ and $\vec{v}$ is equal to $\|\vec{u} - \vec{v}\|$.

P3. Show that smaller of the two angles between $\vec{u}$ and $\vec{v}$ is equal to

$$\theta = \arccos \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}.$$

(Hint: use law of cosines, along with Lemma 11.3).

P4. Use P3 to show that $\vec{u}$ and $\vec{v}$ are perpendicular if and only if $\vec{u} \cdot \vec{v} = 0$.

Definition 11.4. Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$.

1. The NORM of a vector $\vec{u}$ in $\mathbb{R}^n$ is ...

2. The DISTANCE between vectors $\vec{u}$ and $\vec{v}$ is ...

3. We say that $\vec{u}$ and $\vec{v}$ are ORTHOGONAL if ...

Lecture Activity 11.3. Determine which of the following pairs of vectors $\vec{u}$ and $\vec{v}$ are orthogonal.

P1. $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

P2. $\vec{u} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

P3. $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

Definition 11.5. A basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is ORTHOGONAL if ...

An basis $\mathcal{B}$ is called orthonormal if it's orthogonal and ...

Lecture Activity 11.4. Consider the bases $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ for $\mathbb{R}^2$ given by

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \text{ and } \mathcal{D} = \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right\},$$

P1. Determine which of the bases above are orthogonal and which are orthonormal.

P2. Calculate $\vec{u} \cdot \vec{u}$ given that $[\vec{u}]_{\mathcal{B}} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.

P3. Calculate $\vec{v} \cdot \vec{v}$ given that $[\vec{v}]_{\mathcal{C}} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

P4. Calculate $\vec{w} \cdot \vec{w}$ given that $[\vec{w}]_{\mathcal{D}} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$.

P5. What did you notice in your calculations?

Proposition 11.6. Let $\mathcal{B}$ be an orthonormal basis for $\mathbb{R}^n$ and take any vectors $\vec{x}, \vec{y}$ in $\mathbb{R}^n$. Then

$$[\vec{x}]_{\mathcal{B}} \cdot [\vec{y}]_{\mathcal{B}} = \vec{x} \cdot \vec{y}.$$

In particular, we have $\|\vec{x}\| = \|[\vec{x}]_{\mathcal{B}}\|$.

Proof. Suppose that $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$ and write

$$[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ and } [\vec{y}]_{\mathcal{B}} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

That is,

$$\begin{aligned} \vec{x} &= x_1\vec{v}_1 + \cdots + x_n\vec{v}_n \\ \vec{y} &= y_1\vec{v}_1 + \cdots + y_n\vec{v}_n, \end{aligned}$$

and so we have

$$\vec{x} \cdot \vec{y} = (x_1\vec{v}_1 + \cdots + x_n\vec{v}_n) \cdot (y_1\vec{v}_1 + \cdots + y_n\vec{v}_n). \tag{11.1}$$

Complete the proof: use Proposition 11.3 to show that $\vec{x} \cdot \vec{y} = x_1y_1 + \cdots + x_ny_n$.

Thus, $[\vec{x}]_{\mathcal{B}} \cdot [\vec{y}]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x_1y_1 + \cdots + x_ny_n = \vec{x} \cdot \vec{y}$, as needed. □

Lecture Activity 11.5. Suppose that $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ is an orthonormal basis for $\mathbb{R}^2$, and consider the matrix $Q = \begin{pmatrix} \vec{b}_1 & \vec{b}_2 \end{pmatrix}$.

P1. Show that $Q^\top Q = \begin{pmatrix} \vec{b}_1 \cdot \vec{b}_1 & \vec{b}_1 \cdot \vec{b}_2 \\ \vec{b}_2 \cdot \vec{b}_1 & \vec{b}_2 \cdot \vec{b}_2 \end{pmatrix}$.

P2. Use P1 to show that $Q^\top Q = I_2$.

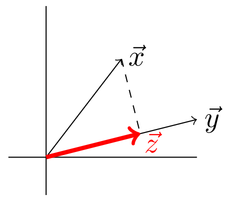
P3. Conclude that Q is invertible with $Q^{-1} = Q^\top$.

Definition 11.8. We call an $n \times n$ matrix Q ORTHOGONAL if ...

Equivalently, Q is called orthogonal if $Q^\top = Q^{-1}$.

Lecture Activity 11.6. Let $\vec{x}$ and $\vec{y}$ be vectors in $\mathbb{R}^2$, and let $\vec{z}$ be the closest point in Span($\vec{y}$) to $\vec{x}$. That is, $\vec{z}$ is the point in Span($\vec{y}$) so that $d(\vec{x}, \vec{z})$ is as small as possible.

P1. Use the picture below to argue that $\vec{y}$ is orthogonal to $\vec{x} - \vec{z}$.



P2. Since $\vec{z}$ is in Span($\vec{y}$), we can write $\vec{z} = c\vec{y}$ for some real number c. Use the previous part to show that

$$c = \frac{\vec{x} \cdot \vec{y}}{\vec{y} \cdot \vec{y}}.$$

P3. Conclude that the closest point on Span($\vec{y}$) to $\vec{x}$ is given by

$$\vec{z} = \frac{\vec{x} \cdot \vec{y}}{\vec{y} \cdot \vec{y}} \vec{y}.$$

Definition 11.10. For vectors $\vec{x}, \vec{y}$ in $\mathbb{R}^n$, the ORTHOGONAL PROJECTION of $\vec{x}$ onto $\vec{y}$ is ...

Theorem 11.11 (The Gram-Schmidt Process)**.** Every vector space has an orthogonal basis. Furthermore, if V is a vector subspace of $\mathbb{R}^n$ with basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$, and we let

then, $\{\vec{u}_1, \ldots, \vec{u}_m\}$ is an orthogonal basis for V. Furthermore,

is an orthonormal basis for V.

Theorem 11.12. Let Q be an $n \times n$ orthogonal matrix. Then, for any $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ we have

$$Q\vec{u} \cdot Q\vec{v} = \vec{u} \cdot \vec{v}.$$

In particular, $\|Q\vec{u}\| = \|\vec{u}\|$ and $\vec{u}$ is orthogonal to $\vec{v}$ if and only if $Q\vec{u}$ is orthogonal to $Q\vec{v}$.

Prove Theorem 11.12.