Instructions. This packet is due on Quercus no later than **11:59pm on Monday, November 17th**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

Definition 10.16. Two $n \times n$ matrices B and C are called *similar* if they represent the same function, but in possibly different bases. That is ...

there is a single linear transformation $F : \mathbb{R}^n \to \mathbb{R}^n$ so that

$$M_{F,\mathcal{B}} = B \text{ and } M_{F,\mathcal{C}} = C,$$

where $\mathcal{B}$ and $\mathcal{C}$ are bases for $\mathbb{R}^n$.

Lecture Activity 10.2. Consider the matrix $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Find a matrix C that's similar to B, but not equal to B.

Solution. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the function with B as its standard defining matrix (that is, $F = T_B$). Let $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$, where

$$\vec{c}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \vec{c}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Recall, by Theorem 10.1 we have

$$M_{F,\mathcal{C}} = \begin{pmatrix} [F(\vec{c}_1)]_{\mathcal{C}} & [F(\vec{c}_2)]_{\mathcal{C}} \end{pmatrix}.$$

We compute

$$F(\vec{c}_1) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}, \text{ and } F(\vec{c}_2) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix},$$

noting that the coordinates above are *standard coordinates*. To write these in $\mathbb{C}$-coordinates we can use our change of basis matrix. By Theorem 9.8, we have

$$M_{\mathcal{E} \leftarrow \mathcal{C}} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix},$$

and so

$$[F(\vec{c}_1)]_{\mathcal{C}} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \text{ and } F(\vec{c}_2) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ -4 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}.$$

Therefore, the matrix

$$C = \begin{pmatrix} 3 & -2 \\ -4 & 2 \end{pmatrix}$$

is similar to B, since by construction $C = M_{F,\mathcal{C}}$ and $B = M_{F,\mathcal{E}}$.

Theorem 10.4. Two $n \times n$ matrices B and C are similar if and only if there exists an invertible $n \times n$ matrix P so that $B = P^{-1}CP$.

Proof. Suppose that B and C are similar $n \times n$ matrices. Then, by definition, there exists a linear transformation $F : \mathbb{R}^n \to \mathbb{R}^n$ and bases $\mathcal{B}$, $\mathcal{C}$ for $\mathbb{R}^n$ so that

$$M_{F,\mathcal{B}} = B \text{ and } M_{F,\mathcal{C}} = C.$$

Complete the proof: show that $B = P^{-1}CP$ where $P = M_{\mathcal{C}\leftarrow\mathcal{B}}$.

Recalling our definition of defining matrices from the previous section, we have

$$[F(\vec{x})]_{\mathcal{C}} = C[\vec{x}]_{\mathcal{C}}.$$

Now, let's use our change of basis matrix: we have $M_{\mathcal{C}\leftarrow\mathcal{B}}[F(\vec{x})]_{\mathcal{B}} = [F(\vec{x})]_{\mathcal{C}}$ and $M_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{C}}$. Replacing this into the equation above yields

$$M_{\mathcal{C}\leftarrow\mathcal{B}}[F(\vec{x})]_{\mathcal{B}} = CM_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}}.$$

Setting $P := M_{\mathcal{C}\leftarrow\mathcal{B}}$ we have

$$P[F(\vec{x})]_{\mathcal{B}} = CP[\vec{x}]_{\mathcal{B}}.$$

Recall that P is invertible, and so we can multiply the left-hand side of the equality above to get

$$[F(\vec{x})]_{\mathcal{B}} = P^{-1}CP[\vec{x}]_{\mathcal{B}}.$$

But then, by definition of defining matrices, $B = P^{-1}CP$.

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Conversely, suppose that $B = P^{-1}CP$ for an invertible matrix P. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation with standard defining matrix $M_F = C$.

Complete the proof: show that $B = M_{F,\mathcal{B}}$ for a basis $\mathcal{B}$ of $\mathbb{R}^n$.

Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be the set of vectors so that

$$P = \begin{pmatrix} [\vec{b}_1]_\mathcal{E} & \cdots & [\vec{b}_n]_\mathcal{E} \end{pmatrix}.$$

Since P is invertible, by Theorem 9.8 we have that $\mathcal{B}$ is a basis for $\mathbb{R}^n$ with $P = M_{\mathcal{E}\leftarrow\mathcal{B}}$ and $P^{-1} = M_{\mathcal{B}\leftarrow\mathcal{E}}$. So,

$$\begin{aligned} B[\vec{x}]_\mathcal{B} &= P^{-1}CP\,[\vec{x}]_\mathcal{B} \\ &= M_{\mathcal{B}\leftarrow\mathcal{E}}\,C\,M_{\mathcal{E}\leftarrow\mathcal{B}}[\vec{x}]_\mathcal{B} \\ &= M_{\mathcal{B}\leftarrow\mathcal{E}}\,C\,[\vec{x}]_\mathcal{E} \\ &= M_{\mathcal{B}\leftarrow\mathcal{E}}[F(\vec{x})]_\mathcal{E} \\ &= [F(\vec{x})]_\mathcal{B}. \end{aligned}$$

Hence, $B = M_{F,\mathcal{B}}$ is the defining matrix of F with respect to the basis $\mathcal{B}$.

So, we've found a single linear transformation $F : \mathbb{R}^n \to \mathbb{R}^n$ so that

$$C = M_{F,\mathcal{E}} \text{ and } B = M_{F,\mathcal{B}}.$$

Therefore, B and C are similar.

$\square$

Lecture Activity 10.3.

P1. Show that if matrices B and C are similar, then B is invertible if and only if C is invertible. *(Hint: use Proposition 10.4 and determinants.)*

Solution. By Proposition 10.4, matrices B and C are similar if and only if there exists an invertible matrix P such that $B = P^{-1}CP$. By Proposition 7.15, we have

$$\begin{aligned} \det(B) &= \det(P^{-1}CP) \\ &= \det(P^{-1})\det(C)\det(P) \\ &= \frac{1}{\det(P)}\det(C)\det(P), \text{ by Chapter Exercise P7.1} \\ &= \det(C). \end{aligned}$$

Hence, $\det(B) = 0$ if and only if $\det(C) = 0$. So, by Theorem 7.16, B is invertible if and only if C is invertible.

P2. Use P1 to show that the matrices

$$B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & -1 & -1 \end{pmatrix}$$

are not similar.

Solution. Note that $\det(B) = 2$ and so B is invertible, while $\det(C) = 0$ and so C is not invertible. So, B and C are not similar by P1.

Lecture Activity 10.4. Let A be an $n \times n$ matrix. Show that if $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a linearly independent set of eigenvectors, then A is similar to a matrix of the form

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Solution. If $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent, then this set forms a basis for $\mathbb{R}^n$. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation with A as its standard defining matrix. Then we have $F(\vec{v}_i) = \lambda_i \vec{v}_i$ for each $i = 1, \dots, n$, for real numbers λ_i. So,

$$\begin{aligned} F(\vec{v}_1) &= \lambda_1 \vec{v}_1 + 0 \cdot \vec{v}_2 + \cdots + 0 \cdot \vec{v}_n \\ F(\vec{v}_2) &= 0 \cdot \vec{v}_1 + \lambda_2 \cdot \vec{v}_2 + \cdots + 0 \cdot \vec{v}_n \\ &\vdots \\ F(\vec{v}_2) &= 0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \cdots + \lambda_n \cdot \vec{v}_n, \end{aligned}$$

which gives

$$[F(\vec{v}_1)]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, [F(\vec{v}_2)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \lambda_2 \\ \vdots \\ 0 \end{pmatrix}, \dots [F(\vec{v}_n)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \lambda_n \end{pmatrix}.$$

So, by Theorem 10.1, we have

$$M_{F,\mathcal{B}} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Definition 10.6. An $n \times n$ matrix D is called DIAGONAL if ...

the only nonzero entries in the matrix appear on the diagonal. That is,

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}.$$

In this case, we write $D = \operatorname{diag}(d_1, d_2, \dots, d_n)$.

Definition 10.7. An $n \times n$ matrix is called DIAGONALIZABLE if ...

> it is similar to a diagonal matrix.

Theorem 10.5 (The Diagonalization Theorem)**.** An $n \times n$ matrix A is diagonalizable if and only if ...

> A has n linearly independent eigenvectors.

In this case, there are linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ for A so that $D = C^{-1}AC$ where ...

$$D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \text{ and } C = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}.$$

Lecture Activity 10.5. Determine which of the following matrices A_i are diagonalizable. For those that are, find an invertible matrix C_i and diagonal matrix D_i so that $D_i = C_i^{-1} A_i C_i$.

P1. $A_1 = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$

Solution. Observe that A_1 has eigenvalues 2 and -1. We can compute

$$E_2 = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) \text{ and } E_{-1} = \operatorname{Span}\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} \right).$$

So, if we set

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } \vec{w} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

we see that $\{\vec{v}, \vec{w}\}$ is a linearly independent set of eigenvectors, and so by the Diagonalization Theorem, the matrix A_1 is diagonalizable. If we let

$$D_1 = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } C_1 = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$$

then by the Diagonalization Theorem we have $D_1 = C_1^{-1} A_1 C_1$.

P2. $A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$

Solution. Observe that A_2 has eigenvalues 2 and 1. We can compute

$$E_2 = \text{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}\right) \text{ and } E_1 = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right).$$

Since any two eigenvectors in E_2 (resp E_1) are linearly dependent, it's not possible to have three linearly independent eigenvectors. So, by the Diagonalization Theorem, we know that A_2 is not diagonalizable.

P3. $A_3 = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

Solution. Observe that A_3 has eigenvalues 2 and 1. We can calculate

$$E_2 = \text{Span}\left(\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}\right) \text{ and } E_1 = \text{Span}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right).$$

Since the vectors

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \text{ and } \vec{v}_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

are linearly independent, then by the Diagonalization theorem A_3 is Diagonalizable. If we let

$$D_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } C_3 = \begin{pmatrix} 3 & 0 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

then we can see that $D_3 = C_3^{-1} A_3 C_3$.

Lecture Activity 10.6. Use Proposition 10.10 to more quickly determine which of the matrices from Lecture Activity 10.5 are diagonalizable.

P1. $A_1 = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$

Solution. As before, we can calculate that A_1 has eigenvalues 2 and -1. We have

$$E_2 = \text{Nul}(A_1 - 2I_2) = \text{Nul}\left(\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\right) = \text{Nul}\left(\begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}\right)$$

$$E_{-1} = \text{Nul}(A_1 + I_2) = \text{Nul}\left(\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\right) = \text{Nul}\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right)$$

and so $\lambda = 2$ has geometric multiplicity 1 and $\lambda = -1$ has geometric multiplicity 1. So, the sum of the geometric multiplicities is $1 + 1 = 2$, and so A (which is 2×2)is diagonalizable by Proposition 10.10.

P2. $A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$

Solution. As before, we can calculate that A_2 has eigenvalues 2 and 1. We have

$$E_2 = \text{Nul}(A_2 - 2I_3) = \text{Nul}\left(\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 2 & -5 & 2 \end{pmatrix}\right) = \text{Nul}\left(\begin{pmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

$$E_1 = \text{Nul}(A_2 - I_3)\,\text{Nul}\left(\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{pmatrix}\right) = \text{Nul}\left(\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

and so both eigenvalues have geometric multiplicity 1. So, the sum of the geometric multiplicities is 2. But A is 3×3 and so A is not diagonalizable by Proposition 10.10.

P3. $A_3 = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

Solution. As before, we can calculate that A_3 has eigenvalues 2 and 1. We have

$$E_2 = \text{Nul}(A_3 - 2I_3) = \text{Nul}\left(\begin{pmatrix} 2 & -3 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & -1 \end{pmatrix}\right) = \text{Nul}\left(\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

$$E_1 = \text{Nul}(A_3 - I_3)\,\text{Nul}\left(\begin{pmatrix} 3 & -3 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{pmatrix}\right) = \text{Nul}\left(\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

and so $\lambda = 2$ has geometric multiplicity 1 and $\lambda = 1$ has geometric multiplicity 2. So, the sum of the geometric multiplicities is 3. Since A_3 is 3×3, we see that A_3 is diagonalizable by Proposition 10.10.

Definition 10.8. Suppose that A is an $n \times n$ diagonalizable matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ and corresponding linearly independent eigenvectors $\vec{v}_1, \dots, \vec{v}_n$. We call the equality ...

$$A = CDC^{-1}$$

the EIGENDECOMPOSITION of the matrix A. By the Diagonalization Theorem, we know that

$$D = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \text{ and } C = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}.$$

Lecture Activity 10.7. Find a 2×2 matrix A so that the linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ stretches every vector in the $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ direction by 2 and in the $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ direction by 3.

Solution. Observe that A has eigenvalues $2, 3$ with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

By the Diagonalization Theorem, A has eigendecomposition

$$A = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}^{-1}$$

and so calculating this product gives

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}.$$

Lecture Activity 10.8. Let

$$A_3 = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

Use your work from P3 of Lecture Activity 10.5 to calculate A_3^{10} by hand (noting that $2^{10} = 1024$).

Solution. From our work in Lecture Activity 10.5 along with Proposition 10.12, we have

$$\begin{aligned}
\begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}^{10} &= \begin{pmatrix} 3 & 0 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2^{10} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 3 & 0 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2^{10} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ -2 & 3 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 3070 & -3069 & 0 \\ 2046 & -2045 & 0 \\ 1023 & -1023 & 1 \end{pmatrix}
\end{aligned}$$