**Instructions.** This packet is due on Quercus no later than **11:59pm on Monday, November 17th**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

**Definition 10.16.** Two $n \times n$ matrices $B$ and $C$ are called *similar* if they represent the same function, but in possibly different bases. That is ...

**Lecture Activity 10.2.** Consider the matrix $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Find a matrix $C$ that's similar to $B$, but not equal to $B$.

**Theorem 10.4.** Two $n \times n$ matrices $B$ and $C$ are similar if and only if there exists an invertible $n \times n$ matrix $P$ so that $B = P^{-1}CP$.

*Proof.* Suppose that $B$ and $C$ are similar $n \times n$ matrices. Then, by definition, there exists a linear transformation $F : \mathbb{R}^n \to \mathbb{R}^n$ and bases $\mathcal{B}$, $\mathcal{C}$ for $\mathbb{R}^n$ so that

$$M_{F,\mathcal{B}} = B \text{ and } M_{F,\mathcal{C}} = C.$$

**Complete the proof: show that $B = P^{-1}CP$ where $P = M_{\mathcal{C}\leftarrow\mathcal{B}}$.**

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Conversely, suppose that $B = P^{-1}CP$ for an invertible matrix $P$. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation with standard defining matrix $M_F = C$.

**Complete the proof: show that $B = M_{F,\mathcal{B}}$ for a basis $\mathcal{B}$ of $\mathbb{R}^n$.**

So, we've found a single linear transformation $F : \mathbb{R}^n \to \mathbb{R}^n$ so that

$$C = M_{F,\mathcal{E}} \text{ and } B = M_{F,\mathcal{B}}.$$

Therefore, $B$ and $C$ are similar.

$\square$

**Lecture Activity 10.3.**

P1. Show that if matrices $B$ and $C$ are similar, then $B$ is invertible if and only if $C$ is invertible. *(Hint: use Proposition 10.4 and determinants.)*

P2. Use P1 to show that the matrices

$$B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & -1 & -1 \end{pmatrix}$$

are not similar.

**Lecture Activity 10.4.** Let $A$ be an $n \times n$ matrix. Show that if $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a linearly independent set of eigenvectors, then $A$ is similar to a matrix of the form

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

**Definition 10.6.** An $n \times n$ matrix $D$ is called DIAGONAL if ...

In this case, we write $D = \operatorname{diag}(d_1, d_2, \dots, d_n)$.

**Definition 10.7.** An $n \times n$ matrix is called DIAGONALIZABLE if ...

**Theorem 10.5** (The Diagonalization Theorem)**.** An $n \times n$ matrix $A$ is diagonalizable if and only if ...

In this case, there are linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ for $A$ so that $D = C^{-1}AC$ where ...

**Lecture Activity 10.5.** Determine which of the following matrices $A_i$ are diagonalizable. For those that are, find an invertible matrix $C_i$ and diagonal matrix $D_i$ so that $D_i = C_i^{-1} A_i C_i$.

P1. $A_1 = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$

P2. $A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$

P3. $A_3 = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$

**Lecture Activity 10.6.** Use Proposition 10.10 to more quickly determine which of the matrices from Lecture Activity 10.5 are diagonalizable.

P1. $A_1 = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$

P2. $A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$

P3. $A_3 = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

**Definition 10.8.** Suppose that $A$ is an $n \times n$ diagonalizable matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$. We call the equality ...

the EIGENDECOMPOSITION of the matrix $A$. By the Diagonalization Theorem, we know that

**Lecture Activity 10.7.** Find a $2 \times 2$ matrix $A$ so that the linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ stretches every vector in the $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ direction by 2 and in the $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ direction by 3.

**Lecture Activity 10.8.** Let

$$A_3 = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

Use your work from P3 of Lecture Activity 10.5 to calculate $A_3^{10}$ by hand (noting that $2^{10} = 1024$).