General Information.

- The Term Test will take place from 5-7pm on Friday, March 7th, unless you have arranged to take an alternate sitting.
- The location of the sitting you are registered for will be released **as a Quercus announcement** by early next week (the week of March 3rd).
- The following pages contains a practice test. **While the problems are different than what will appear on the actual term test, the instructions and format are identical.**
- Please keep in mind that reviewing this practice test **is not enough** to prepare you for the actual test. The practice test is only meant to give you an example of the style and format of the assessment.
- Solutions to the practice test will be posted by the end of the day on Tuesday, March 4th.

Tips for Success

Here is a (non-exhaustive) list of tips that might help you prepare for the term test.

1. Start early, and spread your studying out over several days.
2. Study all definitions and theorems from the list posted on our course website.
3. Complete the attached practice test on your own without help. Only ask for help after you've given yourself time to get stuck and have attempted to solve all problems on your own.
4. Redo material from the course (such as lecture activities, chapter exercises, and Webwork problems). Don't just "review" or "look over" what we've done. Rework problems without the solutions, then use the solutions to check your work.
5. Carefully review the feedback you were given on all Problem Set Quizzes.
6. Don't write anything down that you do not completely understand. Be stubborn. If you can't understand something after giving it a good effort, ask for help.
7. Attend office hours.
8. Form a study group. Write new problems and exchange them with your classmates for extra practice.

* Reminder: the term test will cover Chs 1 - 6 from the course lecture notes

University of Toronto
Faculty of Arts and Sciences
PRACTICE MAT223H1S Term Test

Winter 2025
Duration: 110 minutes
Aids Allowed: None

Name (First then Last): Solutions

University Email Address: ______________________________@mail.utoronto.ca

Student Number: ______________________________

General Instructions:

- Fill out your name, student number, and email address at the top of this page.
- This test contains three sections:
  - **Section A** (8 points available) includes definition statements and theorem proofs.
  - **Section B** (15 points available) includes computational and multiple choice problems.
  - **Section C** (15 points available) includes conceptual and proof-based problems.

  Please read the instructions at the beginning of each section carefully.
- No calculators, notes, or electronics are permitted. Turn off and place all cell phones, smart watches, electronic devices, and unauthorized study materials in your bag under your desk. These devices may not be left in your pockets.
- Place your TCard on your desk so that it can be seen by the invigilators.
- All work must be completed in the space provided. There is additional space at the back of this packet if needed. Do not detach these pages.
- Please ask questions if anything is unclear.
- Once you've finished working, close your test and then raise your hand. We will verify your name against your TCard and collect your exam.
- If you are still working when time is called, promptly close the test packet and wait for an invigilator to come collect your test.

Special Instructions:

- Write legibly and darkly. If we cannot read your work, we will not grade the problem.
- Erase or cross out any work you do not wish to have scored, and clearly indicate if there is work on another page you want scored.
- Fill in your bubbles completely.

Good: ● A ○ B   Bad: ◐ A ⊙ B ⊠ C

Section A.

INSTRUCTIONS:

1. The problems in this section will ask you to **complete a definition** or to **prove a theorem** from the course lecture notes.
2. Definitions must be stated precisely as they are in the course lecture notes (up to rewording). Each definition statement is worth **one point** and no partial credit will be given.
3. Theorem proofs will each be worth **five points**, which will be awarded using our standard rubric (which is available in the Section C instructions).

A1. (1 point) **Complete the following definition:** The SPAN of a set of vectors $\vec{v}_1, \ldots, \vec{v}_n$ in $\mathbb{R}^m$ is ...

either one of these is fine!

the set of all linear combinations of $\vec{v}_1, \cdots, \vec{v}_m$; that is

$$\text{Span}(\vec{v}_1, \ldots, \vec{v}_m) = \{x_1\vec{v}_1 + \cdots + x_n\vec{v}_n : x_1, \ldots, x_n \in \mathbb{R}\}$$

A3. (1 point) **Complete the following definition:** A subset V of $\mathbb{R}^n$ is called a VECTOR SPACE if ...

$\vec{v} + \vec{w} \in V$ and $k\vec{v} \in V$ for all $\vec{v}, \vec{w} \in V$ and $k \in \mathbb{R}$.

A2. (1 point) **Complete the following definition:** The KERNEL of a linear transformation $F : \mathbb{R}^n \to \mathbb{R}^m$ is ...

either of these is fine!

the set of vectors $\vec{x} \in \mathbb{R}^n$ so that $F(\vec{x}) = \vec{0}$, that is

$$\ker(F) = \{\vec{x} \in \mathbb{R}^n : F(\vec{x}) = \vec{0}\}$$

A4. (5 points) **Prove the following theorem** (Theorem 4.10):

The solution set to any homogeneous system of equations is a vector space. Furthermore, if the system has coefficient matrix A, then the solution set is equal to $\text{Nul}(A)$. *(Note that you must also prove* $\text{Nul}(A)$ *is a vector space!)*

Proof. Let A be the coefficient matrix for a homogeneous system. Note that the set of sols to the system with augmented matrix $(A \mid \vec{0})$ is equal to

$$\{A\vec{x} = \vec{0} : \vec{x} \in \mathbb{R}^n\} \quad (\text{spsing } A \text{ is } m \times n),$$

which is precisely $\text{Nul}(A)$.

Next, let's show that $\text{Nul}(A)$ is a vector space.

We have, $A \cdot \vec{0} = \vec{0} \Rightarrow \vec{0} \in \text{Nul}(A)$ and so $\text{Nul}(A)$ is nonempty.

Now, for any $\vec{x}, \vec{y} \in \text{Nul}(A)$ we have

$$\begin{aligned} A(\vec{x} + \vec{y}) &= A\vec{x} + A\vec{y} \\ &= \vec{0} + \vec{0}, \quad \text{since } \vec{x}, \vec{y} \in \text{Nul}(A) \\ &= \vec{0} \Rightarrow \vec{x} + \vec{y} \in \text{Nul}(A) \end{aligned}$$

Finally, for any $\vec{x} \in \text{Nul}(A)$ and $k \in \mathbb{R}$ we have

$$\begin{aligned} A(k\vec{x}) &= kA\vec{x} \\ &= k \cdot \vec{0}, \quad \text{since } \vec{x} \in \text{Nul}(A) \\ &= \vec{0} \Rightarrow k\vec{x} \in \text{Nul}(A), \text{ as needed} \end{aligned}$$

$\square$

Section B.

Instructions:

1. Each problem in this section is worth **three points**.
2. Problems with multiple parts will be worth one point each. Otherwise, no partial credit will be given.
3. You do not need to show your work or provide justification on any problem in Section B.
4. **Your answer must be placed in the answer box provided.**
5. We have provided extra space for your scratch work on each problem, but nothing outside of the answer box will be considered toward your score on the Section B problems.

B1. (3 points) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation given by

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \\ x_1 + x_4 \end{pmatrix}.$$

Find the rank and nullity of T.

Answer: rank(T) = 3   nullity(T) = 1

$$A_T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \underset{\sim}{R_4 - R_1} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix} \underset{\sim}{R_4 - R_3} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$$

$$\underset{\sim}{R_4 + R_2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{\sim}{R_2 - R_3} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{\sim}{R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

B2. (3 points) Find a basis for the vector space

$$V = \operatorname{Span}\left(\begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\-4\\4\end{pmatrix}, \cancel{\begin{pmatrix}2\\-3\\5\end{pmatrix}}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right).$$

Answer: $\left\{ \begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\-4\\4\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$

$$\begin{pmatrix}-1 & 1 & 2 & 1\\-1 & -4 & -3 & 0\\-1 & 4 & 5 & 1\end{pmatrix}$$

$$\underset{R_2-R_1 \text{ \& } R_3-R_1}{\sim} \begin{pmatrix}-1 & 1 & 2 & 1\\0 & -5 & -5 & 1\\0 & 3 & 3 & 0\end{pmatrix}$$

$$\underset{\frac{1}{3}R_3}{\sim} \begin{pmatrix}-1 & 1 & 2 & 1\\0 & -5 & -5 & 1\\0 & 1 & 1 & 0\end{pmatrix}$$

$$\underset{R_2+5R_3}{\sim} \begin{pmatrix}-1 & 1 & 2 & 1\\0 & 0 & 0 & 1\\0 & 1 & 1 & 0\end{pmatrix}$$

$$\underset{R_2 \leftrightarrow R_3}{\sim} \begin{pmatrix}-1 & 1 & 2 & 1\\0 & 1 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}$$

$$\underset{R_1-R_2}{\sim} \begin{pmatrix}-1 & 0 & 1 & 1\\0 & 1 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}$$

$$\underset{R_1-R_3}{\sim} \begin{pmatrix}-1 & 0 & 1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}$$

$$\underset{(-1)R_1}{\sim} \begin{pmatrix}1 & 0 & -1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}$$

B3. (3 points) Let $\vec{v}$ and $\vec{w}$ be linearly independent vectors in $\mathbb{R}^3$. Determine which of the following sets are linearly independent and which are linearly dependent. If there's not enough information to determine, select "not enough information".

a) $\{\vec{v}, \vec{v}+\vec{w}\}$

● linearly independent	○ linearly dependent	○ not enough information.

$$\vec{0} = x\vec{v} + y(\vec{v}+\vec{w})$$
$$= (x+y)\vec{v} + y\vec{w} \Rightarrow \begin{cases} x+y=0 \\ y=0 \end{cases} \Rightarrow x=y=0$$

b) $\{\vec{v}, \vec{w}, \vec{v}+\vec{w}\}$

○ linearly independent	● linearly dependent	○ not enough information.

$$(1)(\vec{v}+\vec{w}) + (-1)(\vec{v}+\vec{w}) = \vec{0}$$

c) $\{\vec{v}, \vec{w}, \vec{0}\}$

○ linearly independent	● linearly dependent	○ not enough information.

$$0\vec{v} + 0\vec{w} + \underline{\underline{2}}\,\vec{0} = \vec{0}$$

B4. Determine which of the following matrices are invertible. If there is not enough information to determine whether the matrix is invertible, select "not enough information".

a) $A + B$ where A and B are invertible matrices

○ is invertible ○ is not invertible ● not enough information

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \leftarrow \text{invertible}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \leftarrow \text{not invertible}$$

$\Rightarrow B$ not invertible

b) AB where A and B are two $n \times n$ matrices and the columns of B are linearly dependent.

○ is invertible ● is not invertible ○ not enough information

B not inv $\Rightarrow \ker T_B \neq \{\vec{0}\}$

$\Rightarrow$ nonzero $\vec{x}$ s.t. $B\vec{x} = \vec{0}$

$\Rightarrow AB\vec{x} = \vec{0}$

$\Rightarrow \ker T_{AB} \neq \{\vec{0}\} \Rightarrow AB$ not inv

c) The matrix $A = \begin{pmatrix} T(\vec{v}_1) & \cdots & T(\vec{v}_n) \end{pmatrix}$ where $T : \mathbb{R}^n \to \mathbb{R}^n$ is an injective linear transformation and $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$

● is invertible ○ is not invertible ○ not enough information.

$x_1 T(\vec{v}_1) + \cdots + x_n T(\vec{v}_n) = \vec{0}$

$\Rightarrow T(x_1\vec{v}_1 + \cdots + x_n\vec{v}_n) = \vec{0}$

$\Rightarrow x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$, since T inj

$\Rightarrow x_1 = \cdots = x_n = 0$, since $\{\vec{v}_i\}$ basis

$\Rightarrow \{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ lin'ly ind

$\Rightarrow A$ has pivot in all cols, & since $n \times n$ also has pivot in all rows $\Rightarrow A \sim I_n$

B5. For each of the following augmented matrices, determine how many solutions the corresponding system of linear equations has.

a) $A = \left(\begin{array}{ccc|c} 2 & 3 & 4 & 2 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right) \overset{R_3-R_2}{\sim} \left(\begin{array}{cccc} 2 & 3 & 4 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \end{array}\right) \overset{R_1-2R_2}{\sim} \left(\begin{array}{cccc} 0 & -1 & -2 & 0 \\ 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \end{array}\right) \overset{R_3+R_1}{\sim} \left(\begin{array}{cccc} 0 & -1 & -2 & 0 \\ 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Answer: The system of linear equations represented by A has

● No solutions ○ Exactly one solution ● Infinitely many solutions

$\overset{R_1 \leftrightarrow R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 4 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right)$

Col w/o pivot + no pivot in last col

b) $B = \left(\begin{array}{cccc|c} 1/2 & -1 & 0 & 2 & 1 \\ 1 & 3 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{3} \\ 1 & 1 & 1 & 1 & 1 \end{array}\right)$

Answer: The system of linear equations represented by B has

● No solutions ○ Exactly one solution ○ Infinitely many solutions

c) $C = \left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right)$, where $*$ denotes unknown real numbers.

Answer: The system of linear equations represented by C must have

○ No solutions ● Exactly one solution ○ Infinitely many solutions

Section C.

INSTRUCTIONS:

1. Each problem in this section is worth **5 points**.
2. You must provide justification for all of your answers in Section C.
3. Points will be awarded based on the rubric below. Note that half points may be awarded, and further rubric items may be added to cover potential cases not outlined below.

Points	**Rubric**
5	Solution is presented with clear justification that is logically complete and correct. May include minor typos and computational errors if they do not majorly impact the argument. No important steps are missing or assumed. All assumptions and special cases have been covered. All suggestions for improvement come under the category of "improvements for clarity" rather than "correcting logical errors". Omission of details will be judged depending on context of the material, with simpler steps being acceptable for omission when covering more advanced topics.
4	Solution is close to full and complete, but contains either a computational error or an error in reasoning that majorly impacts the argument. This score is also appropriate for solutions that are mathematically sound but confusingly written.
3	Solution is incorrect, but understanding of the problem was demonstrated and student provided a clear outline of a potential approach with information about where they got stuck **-or-** solution is correct but justification is insufficient or so confusingly written that it cannot be followed with a reasonable amount of effort.
2	Solution is incorrect, but student demonstrated understanding of the problem **-or-** solution is correct and student did not provide justification for their answer.
1	Solution is incorrect and student did not demonstrate understanding of the problem, but did demonstrate some knowledge of relevant material.
0	Solution is incorrect or incomplete, and there was no demonstration of knowledge of relevant material.

C1. (5 points) Suppose that $\vec{u}_1, \vec{u}_2$ and $\vec{v}_1, \vec{v}_2$ are two bases for $\mathbb{R}^2$. Show that we can write

$$\vec{u}_1 = a\vec{v}_1 + b\vec{v}_2, \quad \vec{u}_2 = c\vec{v}_1 + d\vec{v}_2$$

with $ad - bc \neq 0$.

Proof. Since $\{\vec{v}_1, \vec{v}_2\}$ spans $\mathbb{R}^2$, and $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$ there exists $a, b, c, d \in \mathbb{R}$ s.t.

$$\vec{u}_1 = a\vec{v}_1 + b\vec{v}_2$$
$$\& \quad \vec{u}_2 = c\vec{v}_1 + d\vec{v}_2$$

$\Downarrow$ $\Downarrow$

$$\begin{array}{rl} & d\vec{u}_1 = ad\vec{v}_1 + bd\vec{v}_2 \\ - & b\vec{u}_2 = bc\vec{v}_1 + bd\vec{v}_2 \\ \hline \end{array} \quad \& \quad \begin{array}{rl} & c\vec{u}_1 = ac\vec{v}_1 + bc\vec{v}_2 \\ - & a\vec{u}_2 = ac v_1 + ad\vec{v}_2 \\ \hline \end{array}$$

$$d\vec{u}_1 - b\vec{u}_2 = (ad - bc)\vec{v}_1 \quad \& \quad c\vec{u}_1 - a\vec{u}_2 = (bc - ad)\vec{v}_2.$$

So, if $ad - bc = 0$, then by above get

$$d\vec{u}_1 - b\vec{u}_2 = \vec{0}$$
$$\& \quad c\vec{u}_1 - a\vec{u}_2 = \vec{0}$$

$\Rightarrow a = b = c = d = 0$, since $\{\vec{u}_1, \vec{u}_2\}$ lin'ly ind.

But, by (+) this gives

$$\vec{u}_1 = \vec{u}_2 = \vec{0}$$

a contradiction since $\{\vec{u}_1, \vec{u}_2\}$ basis for $\mathbb{R}^2$.

So, must have $ad - bc \neq 0$, as needed $\boxtimes$

C2. (5 points) True or False: If A is a 3×4 matrix, then there are infinitely many solutions to the vector equation $A\vec{x} = \vec{0}$. **If true provide a proof. If false, provide a counterexample, and justify why this is a counterexample.**

● True ○ False

Proof or Counterexample:

Since A has 3 rows, rref(A) can have at most 3 pivots.

& since A has 4 cols, rref(A) has column w/o pivot.

We know $A\vec{v} = \vec{0}$ is consistent (since $\vec{v} = 0$ is a sol),

So by Rouché-Capelli there are ∞ sols to $A\vec{v} = \vec{0}$.

C3. (5 points) Let V be a vector subspace of $\mathbb{R}^n$ of dimension $1 \leq m < n$ and suppose that B is a basis for V. Show that for any vector $\vec{v}$ in $\mathbb{R}^n$ that is **not** an element of V, the set $B \cup \{\vec{v}\}$ is linearly independent.

Proof. Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_m\}$ be a basis for $V \subseteq \mathbb{R}^n$ where $m < n$, and take $\vec{v} \in \mathbb{R}^n$ that's not in V.

Consider the vector eq

$$(*) \quad x_1\vec{b}_1 + \cdots + x_m\vec{b}_m + x_{m+1}\vec{v} = \vec{0}.$$

If $x_{m+1} \neq 0$, then we could write

$$\vec{v} = \frac{-x_1}{x_{m+1}}\vec{b}_1 + \cdots + \frac{-x_m}{x_{m+1}}\vec{b}_m$$

$$\in \operatorname{Span}(\vec{b}_1, \ldots, \vec{b}_m) = V$$

$\Rightarrow \vec{v} \in V$, a contradiction.

So, we must have $x_{m+1} = 0$. Plugging this into $(*)$ gives

$$x_1\vec{b}_1 + \cdots + x_m\vec{b}_m = \vec{0}$$

& since $\mathcal{B}$ is a basis, it's a lin'ly ind set, & so

$$x_1 = \cdots = x_m = 0.$$

Hence, the only sol to $(*)$ is $x_1 = \cdots = x_m = x_{m+1} = 0$ & so $\{\vec{b}_1, \ldots, \vec{b}_m, \vec{v}\}$ is lin'ly ind $\boxtimes$

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

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