**University of Toronto**
**Faculty of Arts and Sciences**
**PRACTICE MAT223H1S Final Exam**

**Winter 2025**
**Duration: 180 minutes**
**Aids Allowed: None**

Name (First then Last): ____________________________________________

University Email Address: ______________________________@mail.utoronto.ca

Student Number: ____________________________________________

General Instructions:

- Fill out your name, student number, and email address at the top of this page.
- This test contains three sections:
  - **Section A** (10 points available) includes definition statements and theorem proofs.
  - **Section B** (24 points available) includes computational and multiple choice problems.
  - **Section C** (20 points available) includes conceptual and proof-based problems.

  Please read the instructions at the beginning of each section carefully.
- No calculators, notes, or electronics are permitted. Turn off and place all cell phones, smart watches, electronic devices, and unauthorized study materials in your bag under your desk. These devices may not be left in your pockets.
- Place your TCard on your desk so that it can be seen by the invigilators.
- All work must be completed in the space provided. There is additional space at the back of this packet if needed. Do not detach these pages.
- Please ask questions if anything is unclear.
- Once you've finished working, close your exam and then raise your hand. We will verify your name against your TCard and collect your exam.
- If you are still working when time is called, promptly close the test packet and wait for an invigilator to come collect your test.

Special Instructions:

- Write legibly and darkly. If we cannot read your work, we will not grade the problem.
- Erase or cross out any work you do not wish to have scored, and clearly indicate if there is work on another page you want scored.
- Fill in your bubbles completely.

Good: ● A ○ B    Bad: ◐ A ⊙ B ⦸ C

# Section A.

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INSTRUCTIONS:

1. The problems in this section will ask you to **complete a definition** or to **prove a theorem** from the course lecture notes.
2. Definitions must be stated precisely as they are in the course lecture notes (up to rewording). Each definition statement is worth **one point** and no partial credit will be given.
3. Theorem proofs will each be worth **five points**, which will be awarded using our standard rubric (which is available in the Section C instructions).

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A1. (1 point) **Complete the following definition:** The SPAN of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ in $\mathbb{R}^n$ is ...

A2. (1 point) **Complete the following definition:** A system of linear equations is called CONSISTENT if ...

A3. (1 point) **Complete the following definition:** The DIMENSION of a vector space is ...

A4. (1 point) **Complete the following definition:** A non-zero vector $\vec{x}$ is called an EIGENVECTOR of a matrix $A$ if ...

A5. (1 point) **Complete the following definition:** The SINGULAR VALUES of a matrix are ...

A7. (5 points) **Prove the following Proposition** (Proposition 11.10):

Let $\mathcal{B}$ be an orthonormal basis for $\mathbb{R}^n$ and take any vectors $\vec{x}, \vec{y}$ in $\mathbb{R}^n$. Then, $[\vec{x}]_{\mathcal{B}} \cdot [\vec{y}]_{\mathcal{B}} = \vec{x} \cdot \vec{y}$.

*Proof.*

# Section B.

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INSTRUCTIONS:

1. Each problem in this section is worth **three points**.
2. Problems with multiple parts will be worth one point each. Otherwise, no partial credit will be given.
3. You do not need to show your work or provide justification on any problem in Section B.
4. **Your answer must be placed in the answer box provided.**
5. We have provided extra space for your scratch work on each problem, but nothing outside of the answer box will be considered toward your score on the Section B problems.

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B1. (3 points) For the systems of linear equations described below, determine whether the system has no solution, exactly one solution, or infinitely many solutions.

a) A system whose coefficient matrix is invertible.

**Answer:** The system of linear equations has

○ No solutions ○ Exactly one solution ○ Infinitely many solutions

b) A system whose augmented matrix is invertible.

**Answer:** The system of linear equations has

○ No solutions ○ Exactly one solution ○ Infinitely many solutions

c) The system with augmented matrix $A^T$, where $A$ is the augmented matrix representing the system in part (a).

**Answer:** The system of linear equations must have

○ No solutions ○ Exactly one solution ○ Infinitely many solutions

B2. (3 points) For each linear transformation defined below, determine whether the reduced row echelon form of its defining matrix has a pivot in every row, every column, both, or neither.

a) $F : \mathbb{R}^4 \to \mathbb{R}^3$ satisfying that $\{F(\vec{e}_1), F(\vec{e}_3), F(\vec{e}_4)\}$ is linearly independent.

**Answer:**

- ○ Pivots in every row and every column
- ○ Pivots in every row but not every column
- ○ Pivots in every column but not every row
- ○ None of the above

b) $G = T_Q$, where $Q$ is an $n \times n$ orthogonal matrix.

**Answer:**

- ○ Pivots in every row and every column
- ○ Pivots in every row but not every column
- ○ Pivots in every column but not every row
- ○ None of the above

c) $H : \mathbb{R}^2 \to \mathbb{R}^4$ with $H(\vec{e}_1) \neq H(\vec{e}_2) \neq \vec{0}$.

**Answer:**

- ○ Pivots in every row and every column
- ○ Pivots in every row but not every column
- ○ Pivots in every column but not every row
- ○ None of the above

B3. (3 points) Calculate the following determinants.

a) $\det(AB)$ where $A = \begin{pmatrix} 3 & 40 & -1 \\ 0 & 7 & 3 \\ 0 & 0 & -3 \end{pmatrix}$ and $B = A^T$, the transpose of $A$.

$\det(A) =$

b) $\det(F)$, where $F$ is the inverse of the function $G$ where $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which stretches vectors in the $\vec{e}_1 + \vec{e}_2$ direction by -3 and leaves the $\vec{e}_1 - \vec{e}_2$ direction unchanged.

$\det(F) =$

c) Let $C$ be standard defining matrix of F from part (b). Is it possible that $C$ similar to the matrix $AB$ from part (a)?

○ Yes, it is possible ○ No, it is not possible

B4. (3 points) Determine which of the following matrices are invertible. If there is not enough information to determine whether the matrix is invertible or not invertible, select "could be either".

a) A $3 \times 3$ matrix $N$ satisfying that $N^3$ is the zero matrix.

○ Is invertible ○ Is not invertible ○ Could be either

b) A symmetric matrix.

○ Is invertible ○ Is not invertible ○ Could be either

c) The defining matrix of the linear transformation in $\mathbb{R}^3$ that rotates around the $z$-axis by an angle of $\frac{\pi}{4}$.

○ Is invertible ○ Is not invertible ○ Could be either

B5. (3 points) Chapter 10, part B problem (Kevin)

Let $A$ be a $3 \times 3$ matrix with eigenvalues $0, 1, 2$. Find the eigenvalues of the following matrices:

a) The matrix $A^2$.

b) The matrix $A - I_3$.

c) The matrix $3(A^T)^2$.

B6. (3 points) Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^3$. Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

a) Find $[\vec{v}]_{\mathcal{E}}$ given that $[\vec{v}]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

$$[\vec{v}]_{\mathcal{E}} = \begin{pmatrix} \square \\ \square \\ \square \end{pmatrix}$$

b) Find the change of basis matrix from $\mathcal{B}$ to $\mathcal{E}$.

$$M_{\mathcal{E} \leftarrow \mathcal{B}} = \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}$$

c) Find the change of basis matrix from $\mathcal{E}$ to $\mathcal{B}$.

$$M_{\mathcal{B} \leftarrow \mathcal{E}} = \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}$$

B7. (3 points) Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

.

a) Find the characteristic polynomial of $A$.

**Answer:** $\chi_A(x) =$

b) Find the dimension of the 1-eigenspace $E_1$.

**Answer:** $\dim E_1 =$

c) Find an invertible matrix $C$ so that $C^{-1}AC$ is a diagonal matrix.

$$C = \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}$$

B8. (3 points) Determine which of the following statements are always true and which are always false. If there's not enough information to determine whether a statement is always true or always false, select "could be true or false".

a) If $\vec{v}$ and $\vec{w}$ are two vectors in $\mathbb{R}^n$ such that $\vec{v} \cdot \vec{w} = 0$, then $\{\vec{v}, \vec{w}\}$ is a linearly independent set.

○ Always true ○ Always false ○ Could be true or false

b) If a linear transformation $F : \mathbb{R}^3 \to \mathbb{R}^3$ preserves the angle between every pair of vectors, then its defining matrix $A$ is orthogonal.

○ Always true ○ Always false ○ Could be true or false

c) Let $V$ be a vector subspace of $\mathbb{R}^n$. Every basis of $V$ has $n$ elements.

○ Always true ○ Always false ○ Could be true or false

# Section C.

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INSTRUCTIONS:

1. Each problem in this section is worth **5 points**.
2. You must provide justification for all of your answers in Section C.
3. Points will be awarded based on the rubric below. Note that half points may be awarded, and further rubric items may be added to cover potential cases not outlined below.

| Points | Rubric |
|---|---|
| 5 | Solution is presented with clear justification that is logically complete and correct. May include minor typos and computational errors if they do not majorly impact the argument. No important steps are missing or assumed. All assumptions and special cases have been covered. All suggestions for improvement come under the category of "improvements for clarity" rather than "correcting logical errors". Omission of details will be judged depending on context of the material, with simpler steps being acceptable for omission when covering more advanced topics. |
| 4 | Solution is close to full and complete, but contains either a computational error or an error in reasoning that majorly impacts the argument. This score is also appropriate for solutions that are mathematically sound but confusingly written. |
| 3 | Solution is incorrect, but understanding of the problem was demonstrated and student provided a clear outline of a potential approach with information about where they got stuck **-or-** solution is correct but justification is insufficient or so confusingly written that it cannot be followed with a reasonable amount of effort. |
| 2 | Solution is incorrect, but student demonstrated understanding of the problem **-or-** solution is correct and student did not provide justification for their answer. |
| 1 | Solution is incorrect and student did not demonstrate understanding of the problem, but did demonstrate some knowledge of relevant material. |
| 0 | Solution is incorrect or incomplete, and there was no demonstration of knowledge of relevant material. |

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C1. (5 points) Let $A$ be an $m \times (n+1)$ matrix, and suppose that the system of linear equations in $n$ variables with augmented matrix $A$ has at least one solution. Show that the homogeneous system of linear equations in $n+1$ variables with coefficient matrix $A$ has infinitely many solutions

*Proof.*

C2. (5 points) Let $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_6\}$ be a set of vectors in $\mathbb{R}^4$. Then $B$ cannot be a basis of $\mathbb{R}^4$. **If true, provide a proof. If false, provide a counterexample, and justify why this is one.**

○ True ○ False

*Proof or Counterexample:*

C3. (5 points) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and $\mathcal{B}$ a basis for $\mathbb{R}^n$. Show that if the defining matrix $A_F$ is invertible, then $A_{F,\mathcal{B}}$ is also invertible.

*Proof.*

C4. (5 points) Let $A$ be an $6 \times 7$ matrix. Is it possible that the nullity of $A$ equals the nullity of its transpose $A^T$? **If yes, find an example and prove that it is an example. If no, prove it.**

*Proof or Example.*

**YOU MUST SUBMIT THIS PAGE.**

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

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