Vassiliev and Quantum Invariants of Braids

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ABSTRACT. We prove that braid invariants coming from quantum $gl(N)$ separate braids, by recalling that these invariants (properly decomposed) are all Vassiliev invariants, showing that all Vassiliev invariants of braids arise in this way, and reproving that Vassiliev invariants separate braids. We discuss some corollaries of this result and of our method of proof.

CONTENTS

1. Introduction

1.1. The result. Recall [**28**, **29**] that given a list $R_1, \ldots, R_n$ of representations of the lie algebra $gl(N)$ (or in fact, any other semi-simple Lie algebra) one can construct an n-component tangle invariant (the *Reshetikhin-Turaev invariant*), and in particular, an invariant $J_{R_1, \ldots, R_n}$ of n-strand pure braids, with values in $\text{End}(R_1 \otimes \cdots \otimes R_n)[[\hbar]]$,

the ring of formal power series[1] in the variable $\hbar$ with coefficients in $\text{End}(R_1 \otimes \cdots \otimes R_n)$. The main goal of this paper is to prove the following theorem

THEOREM 1. *These invariants, coming from $gl(N)$ and all of its representations, separate pure braids.*

See remark 4.2 for a comment about (not necessarily pure) braids.

The main tools we will use in proving theorem 1 are Vassiliev invariants, chord diagrams, and weight systems. So before sketching the proof of theorem 1 in section 1.3, let us briefly recall these important notions.

1.2. Vassiliev invariants. Although originally defined only for knots, the notion of 'a Vassiliev invariant' ([**12, 13, 30, 31**]; see also [**2, 6, 8, 19**]) can be easily generalized to many other classes of 'knot-like' objects, such as braids, pure braids, tangles, links, string links, knotted graphs, knots in a 3-manifold, etc. The idea is always the same. Let $\mathcal{K}$ be a class of knot-like objects — a class of embeddings of oriented 1-dimensional objects in some 3-dimensional oriented space, perhaps satisfying some boundary conditions, considered modulo some reasonable notion of 'isotopy'. Let $V : \mathcal{K} \to \mathbf{A}$ be an invariant (under 'isotopy') with values in some Abelian group $\mathbf{A}$. It is always possible to extend V to 'knot-like objects with self-intersections' ('singular knot-like objects') by the formula

$$V\left(\times\!\!\!\!\times\right) = V\left(\times\!\!\!\!\times\right) - V\left(\times\!\!\!\!\times\right). \tag{1}$$

(We say that the double point is 'resolved into an overcrossing minus an undercrossing'). V is called *a Vassiliev invariant of type m* if its natural extension vanishes whenever it is evaluated on an object with more than m self-intersections:

$$V\left(\underbrace{\times\!\!\!\!\times \;\; \times\!\!\!\!\times \; \cdots \; \times\!\!\!\!\times}_{>m}\right) = 0.$$

The relevance of Vassiliev invariants to our issue stems from the following two facts:

FACT 1. *(Lin [**22**], see also [**2, 6**]) The coefficient $J_{R_1,\ldots,R_n,m}$ of $\hbar^m$ in $J_{R_1,\ldots,R_n}$ is a Vassiliev invariant of type m.*

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[1]The Laurent polynomials in the formal parameter q of [**28, 29**] become formal power series in $\hbar$ upon the substitution $q = e^{\hbar}$.

FACT 2. *(Bar-Natan* [**3**]*, see also Kohno* [**18**]*) Vassiliev invariants of (pure) braids separate (pure) braids.*

For the convenience of the reader, we will sketch a proof of fact 2 in section 3.

Differences are in many ways similar to derivatives, and as $V\left(\times\right)$ is defined to be a difference, one can think of Vassiliev invariant of type m as invariants whose higher-than-m derivatives vanish, or, as 'polynomials of degree at most m'. Just as in the case of degree m polynomials, the mth order 'derivatives' of a type m Vassiliev invariant are 'constant'. More precisely, if our knot-like object K has precisely m self-intersections, then by (1) and the definition of Vassiliev invariants, one can replace overcrossings in K by undercrossings (and vice versa) freely, without changing the values of any type m Vassiliev invariant V. This means (at least in the case where the ambient space is simply connected) that V doesn't really depend on the topology of K, but rather it depends only on the combinatorial object defined by the parameter space S of K together with the pairs of points in S that map into each of the self-intersections of K. Such pairs of points on S are usually signified by drawing a 'chord' connecting them, and the resulting combinatorial object is called *a chord diagram.*

The simplest and best known (see e.g. [**2, 6, 8, 19**]) example is that of oriented knots in oriented space. In that case, the parameter space is an oriented circle (conventionally oriented counterclockwise when drawn), and an example for a chord diagrams is in figure 1.
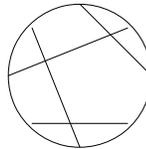

Figure 1. A chord diagram of degree 4.

The example that will be of interest for us is that of $\mathcal{K} = \{n$-strand pure braids$\}$. In that case, (degree m) chord diagrams are diagrams made of n vertical directed lines ('strands') and m horizontal lines ('chords') connecting them, as in figure 2.

Thus, to every $\mathbf{A}$-valued Vassiliev invariant V of type m of pure braids, corresponds a map $W_m(V) : \mathcal{G}_m\mathcal{D}^{pb} \to \mathbf{A}$ defined on the $\mathbf{Z}$-module $\mathcal{G}_m\mathcal{D}^{pb}$ freely generated by all degree m (pure braid) chord diagrams. Notice that by stacking chord diagrams vertically, $\mathcal{D}^{pb} \overset{\text{def}}{=} \bigoplus_m \mathcal{G}_m\mathcal{D}^{pb}$ becomes a (non-commutative) graded algebra. As an algebra $\mathcal{D}^{pb}$ is generated by the degree 1 chord diagrams t^{ij}, $(1 \leq i \neq j \leq n)$, given by

Figure 2. A singular 3-strand pure braid with 4 self-intersections, and the corresponding chord diagram.
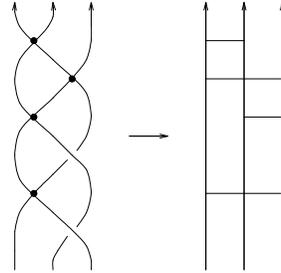
$$t^{ij} = t^{ji} = \left| \ldots \left| \overset{\overset{i}{\uparrow} \quad \overset{j}{\uparrow}}{\vphantom{|}} \right| \ldots \right| \ldots \left| \right. .$$

FACT 3. $W_m(V)$ *vanishes on the degree* m *piece of the double-sided ideal* $\mathcal{I}$ *of* $\mathcal{D}^{pb}$ *generated by the relations*

$$t^{ij}t^{kl} = t^{kl}t^{ij} \quad when \quad |\{i,j,k,l\}| = 4, \tag{2}$$

$$\left[t^{ik} + t^{jk}, t^{ij}\right] = 0 \quad when \quad |\{i,j,k\}| = 3. \tag{3}$$

Indeed, relation 2 just says that double points 'can be moved across each other', as in figure 3, while relation 3 is just the $4T$ relation of [**2**], in a slightly disguised form. Its proof is sketched in figure 4.

Figure 3. The two singular braids (which may be parts of 'bigger' pure braids) displayed here are equivalent, implying relation 2.
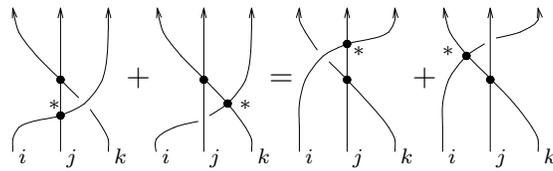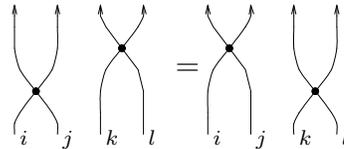
Figure 4. The chord diagrams corresponding to the four singular braids displayed here are the four terms of (3). When the double points marked by a $*$ is resolved into overcrossings and undercrossings as in 1, the resulting 8 singular braids cancel in pairs.

We set $\mathcal{A}^{pb} = \mathcal{D}^{pb}/\mathcal{I}$, and then fact 3 just says that to every **A**-valued Vassiliev invariant V of pure braids corresponds a map (a *weight*

system) $W_m(V) : \mathcal{G}_m\mathcal{A}^{pb} \to \mathbf{A}$. Over the rationals, the converse is also true:

FACT 4. *Every weight system comes from an invariant. More precisely, if $\mathbf{A}$ is a $\mathbf{Q}$-module and $W : \mathcal{G}_m\mathcal{A}^{pb} \to \mathbf{A}$ is arbitrary, then there exists a $\mathbf{A}$-valued Vassiliev invariant V of pure braids for which $W = W_m(V)$. The invariant V is determined uniquely up to invariants of lower type.*

Over the real numbers, fact 4 can be proven using the 'Kontsevich integral formula' of [19] (see also [2]). Over the rationals it can be deduced from Drinfel'd's work [10, 11] on quasi-Hopf algebras. See e.g. [1, 4, 9, 15, 21, 23, 27].

1.3. Sketch of the proof. Fact 2 (see section 3) implies that in order to prove theorem 1, it is enough to prove that all Vassiliev invariants of pure braids come (as in fact 1) from the $gl(N)$ invariants $J_{R_1,\dots,R_n,m}$. Facts 3 and 4 imply that it is enough to do that on the level of weight systems; that is, it is enough to prove that the weight systems corresponding to (various traces of) the $J_{R_1,\dots,R_n,m}$'s span the space $\mathcal{W}^{pb} = (\mathcal{A}^{pb})^*$ of all weight systems. This is exactly corollary 2.6 of section 2.5. In section 4 we will prove some corollaries of theorem 1 and discuss some related questions.

REMARK 1.1. The technique used in this paper appears to be a part of a pattern — statements about knot polynomial or quantum group invariants of knots become simpler when restated in terms of Vassiliev invariants, weight systems, and chord diagrams. Chord diagrams (which are just abstract graphs) are much more manageable objects than knots (whose embedding into space matters), and so complicated facts about knots become provable once stated in this simpler language. Perhaps an even better example for this principle is the proof of the Melvin-Morton-Rozansky conjecture in [5].

REMARK 1.2. The two main problems in the theory of Vassiliev invariants are:

- Do Vassiliev invariants separate knotted objects (in some class $\mathcal{K}$)?
- And do they all come from lie algebras?

In the case of $\mathcal{K} = \{\text{knots}\}$, we know (see e.g. [20]) that the answer cannot be "yes" for both questions, though we don't know any of the answers. Fact 2 says that the answer to the first question is "yes" in the case of $\mathcal{K} = \{\text{pure braids}\}$, and the main point in section 2 is to show that the answer to the second question is also "yes" in that case.

Another case in which both questions have an affirmative answer is $\mathcal{K} = \{\text{homotopy string links}\}$. See [**3**].

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2. All Vassiliev invariants of pure braids come from $gl(N)$

In this section we will prove (as promised in section 1.3) that the weight systems corresponding to the $gl(N)$ invariants $J_{R_1,\dots,R_n,m}$ span the space of all pure braid weight systems. Denote the weight system of $J_{R_1,\dots,R_n,m}$ by $W_{R_1,\dots,R_n}$ (suppressing the degree m, which anyway can be read from the degree of the chord diagrams being fed into $W_{R_1,\dots,R_n}$). Our first step is to better understand the $\text{End}(R_1 \otimes \cdots \otimes R_n)$-valued weight system $W_{R_1,\dots,R_n}$, for some fixed list $R_1,\dots,R_n$ of representations of $gl(N)$.

2.1. The weight system $W_{R_1,\dots,R_n}$. Let R be the defining (N-dimensional) representation of $gl(N)$, and let $\{M_a\}_{a=1}^{N^2}$ be a basis of $gl(N)$, orthonormal with respect to the invariant metric $\langle M_a, M_b \rangle = \text{tr}_R(M_a M_b)$. Define a map $\mathcal{D}^{pb} \to \text{End}(R_1 \otimes \cdots \otimes R_n)$ by extending the map

$$t^{ij} = \left| \ \dots \ \begin{matrix} {}^{\uparrow i} & {}^{\uparrow j} \\ \overline{} & \\ \dots & \end{matrix} \ \dots \ \right|$$

$$\mapsto \sum_{a=1}^{N^2} 1 \otimes \cdots \otimes 1 \otimes \underset{i}{M_a} \otimes 1 \cdots 1 \otimes \underset{j}{M_a} \otimes 1 \otimes \cdots \otimes 1 \quad (4)$$

to be an algebra morphism.

FACT 5.   1. *(Kohno* [**17**]*, see also* [**2**]*, proposition 2.11]) The above defined map* $\mathcal{D}^{pb} \to \text{End}(\otimes_i R_i)$ *descends to a well defined map* $\mathcal{A}^{pb} \to \text{End}(\otimes_i R_i)$.
   2. *(Piunikhin* [**26**]*, see also* [**2**]*, remark 4.8]) The resulting map* $\mathcal{A}^{pb} \to \text{End}(\otimes_i R_i)$ *is the weight system* $W_{R_1,\dots,R_n}$.

2.2. The defining representation of $gl(N)$. Let us now specialize to the case when $R_{1,\dots,n} = R$, the defining (N-dimensional) representation of $gl(N)$.

FACT 6. *(See e.g.* [**2**]*, exercise 6.36]) The tensor* $t = \sum_a M_a \otimes M_a \in \text{End}(R \otimes R)$ *is the 'crossed identity' tensor given by the formula*

$$t(v \otimes w) = w \otimes v.$$

It is consistent with the notation in (4) to denote the identity operator by a directed line, and then fact 6 becomes the statement

$$t = \;\raisebox{-0.5em}{\includegraphics[height=2em]{crossing}}\;,$$

and (4) becomes

$$W_{R,\dots,R} : \left|\;\dots\;\overset{i}{\underset{}{\left|}}\!\cdots\!\overset{j}{\left|}\;\dots\;\right| \;\mapsto\; \left|\;\dots\;\overset{i}{\underset{}{\left|}}\!\!\cdots\!\!\overset{j}{\left|}\;\dots\;\right|.$$

We can get numerical valued weight systems out of $W_{R,\dots,R}$ by composing it with various traces $\mathrm{End}(R^{\otimes n}) \to \mathbf{C}$. I.e., if $\sigma \in \mathcal{S}_n$ is a permutation of $1,\dots,n$, it defines in a natural way a permutation operator (also denoted by the letter σ) in $\mathrm{End}(R^{\otimes n})$, and then one can define $\mathrm{tr}_\sigma : \mathrm{End}(R^{\otimes n}) \to \mathbf{C}$ by $\mathrm{tr}_\sigma E = \mathrm{tr}\,\sigma \circ E$. Let W_σ denote the numerical valued weight system $\mathrm{tr}_\sigma \circ W_{R,\dots,R}$.

PROPOSITION 2.1. *Let $D \in \mathcal{D}^{pb}$ be a chord diagram. Then $W_\sigma(D)$ can be computed in three steps as below (and as in figure 5):*

1. *Append the permutation σ to the top of D, and 'close around'.*
2. *Replace all chords in D by 'crossings'.*
3. *Count the number c of components in the resulting diagram. $W_\sigma(D)$ is then N^c.*
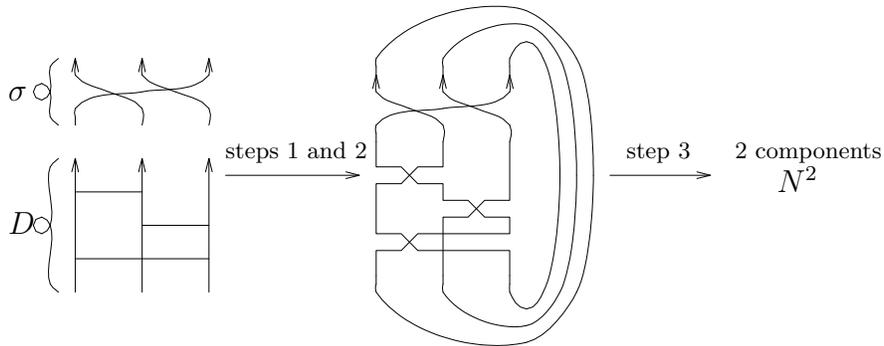
Figure 5. The three steps of computing $W_\sigma(D)$.

2.3. Tensor powers of the defining representation. The following definition and fact allow us to compute $W_{R_1,\dots,R_n}(D)$ whenever D is a chord diagram and $\{R_i = R^{\otimes k_i}\}$ are tensor powers of the defining representation.

DEFINITION 2.2. Let $k = (k_1, \ldots, k_n)$ be a sequence of non negative integers, let $|k| = \sum_i k_i$ be their sum, and let $l_i = \sum_{\alpha < i} k_\alpha$. Define an algebra morphism $\Delta^k : \mathcal{D}_n^{pb} \to \mathcal{D}_{|k|}^{pb}$ by setting

$$\Delta^k(t^{ij}) = \sum_{i'=l_i+1}^{l_{i+1}} \sum_{j'=l_j+1}^{l_{j+1}} t^{i'j'}.$$

This map descends to a morphism (called by the same name) $\Delta^k :$ $\mathcal{A}_n^{pb} \to \mathcal{A}_{|k|}^{pb}$.

Δ^k can be thought of as replacing the ith strand in D by a bundle of k_i strands (for each i) and summing over all possible ways of 'lifting' the chords from their original strand to the bundle that replaced it. For example, $\Delta^{(22)}(t^{12}) = t^{13} + t^{14} + t^{23} + t^{24}$, or

$$\Delta^{(22)} \left(\left| \vdash\!\!\!\dashv \right| \right) = \left|\!\right|\!\vdash\!\!\!\dashv\!\left|\!\right| + \left|\!\right|\!\vdash\!\!\!\dashv\!\left|\!\right| + \left|\!\right|\!\vdash\!\!\!\dashv\!\left|\!\right| + \left|\!\right|\!\vdash\!\!\!\dashv\!\left|\!\right|.$$

FACT 7. *(See proposition 6.18 of [2]) The following diagram commutes:*

$$
\begin{array}{ccc}
\mathcal{A}_n^{pb} & \xrightarrow{\Delta^k} & \mathcal{A}_{|k|}^{pb} \\
& & \\
W_{R^{\otimes k_1},\ldots,R^{\otimes k_n}} \searrow & & \swarrow W_{R,\ldots,R \ (|k| \ times)} \\
& \mathrm{End}(R^{\otimes|k|}) &
\end{array}
$$

Given a list $k = (k_1, \ldots, k_n)$ and a permutation $\sigma \in \mathcal{S}_{|k|}$ of the integers 1 through $|k|$, we consider the numerical valued weight system

$$W_{k,\sigma} = \mathrm{tr}_\sigma \circ W_{R^{\otimes k_1},\ldots,R^{\otimes k_n}}.$$

Fact 7 and proposition 2.1 show that we can compute $W_{k,\sigma}(D)$ for any chord diagram D following the steps below:

1. For each i, replace the ith strand in D by a bundle of k_i strands and consider all possible ways $\{D_\alpha\}$ of lifting all chords.
2. For each α, append the permutation σ at the top of D_α, close around, replace all chords by crossings, and call the result D'_α.
3. $W_{k,\sigma}(D)$ is $\sum_\alpha N^{c_\alpha}$, where c_α is the number of components in D'_α.

2.4. Paths. It is easy to see that $W_{k,\sigma}(D)$ does not change if $\sigma \in$ $\mathcal{S}_{|k|}$ is conjugated by a permutation in $\mathcal{S}_{k_1} \times \cdots \times \mathcal{S}_{k_n} \subset \mathcal{S}_{|k|}$, as this just corresponds to permuting the strands within each bundle. A *path*, defined below, is basically a pair (k, σ), with the redundancy in σ removed.

DEFINITION 2.3. A connected path is a word in the n letters S_1 through S_n. A path is an unordered list (possibly with multiplicities) of connected paths.

Figure 2.4 explains by an example how a pair (k, σ) as above determines a path P (up to cyclically permuting the letters of each word in P), and how a path P determines a pair (k, σ) (up to conjugating σ by a permutation in $\mathcal{S}_{k_1} \times \cdots \times \mathcal{S}_{k_n}$).
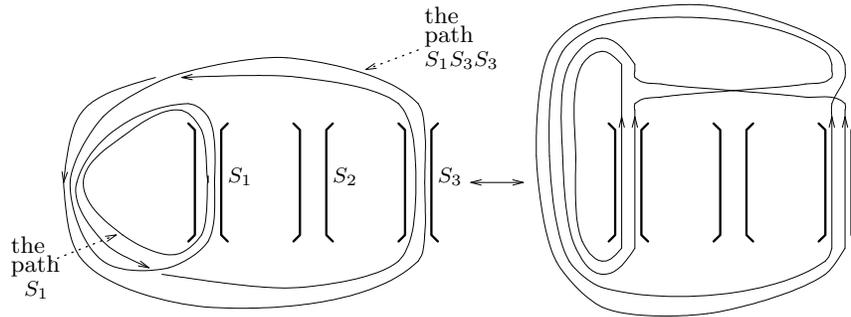

Figure 6. The path $P = (S_1, S_1 S_3 S_3)$ and the pair $(k = (2, 0, 2),\ \sigma = (1)(234))$ correspond.

We thus find that to every path P corresponds a weight system $W_P = W_{k,\sigma}$, where (k, σ) correspond to P as in figure 2.4. The algorithm in section 2.3 becomes the following algorithm for computing $W_P(D)$, where D is a chord diagram of degree m:

1. For each connected component of P write a long interval, subdivided into shorter subintervals corresponding to the letters making up that component. Mark sites corresponding to the integers $1, \ldots, m$ in order along each of the subintervals. For example, if $m = 2$, the path $P = (S_1, S_1 S_3 S_3)$ becomes:

$$\underbrace{\overset{1 \quad 2}{}}_{S_1} \qquad \underbrace{\overset{1 \quad 2}{}}_{S_1} \underbrace{\overset{1 \quad 2}{}}_{S_3} \underbrace{\overset{1 \quad 2}{}}_{S_3} \tag{5}$$

2. Consider all liftings $\{D_\alpha\}$ of D to the picture just drawn, where each end of each chord is lifted to one of the sites in the picture, so that if the (say) 7th chord in D is t^{23}, then its ends are lifted to sites on marked by the integer 7 on subintervals corresponding to the letters S_2 and S_3. For example, the chord diagram $D = t^{13} t^{13}$ has 16 possible liftings to the path $(S_1, S_1 S_3 S_3)$, as the first t^{13} in D can be lifted to become either one of the 4 chords in the first

figure below, and the second t^{13} in D can be lifted to become either one of the 4 chords in the second figure:
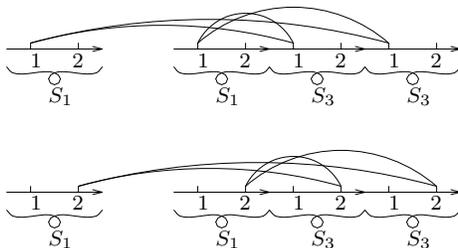
As a second example, notice that the chord diagram $t^{13}t^{23}$ has no liftings to the path $(S_1, S_1 S_3 S_3)$, as that path does not pass through the strand S_2 at all, and there's nowhere to lift t^{23} to.

3. Replace all chords in each D_α by 'bridges' (notice that the 'cross-ings' of section 2.3 become 'bridges' in the current picture), erase all markings, close all intervals into loops, and call the result D'_α:
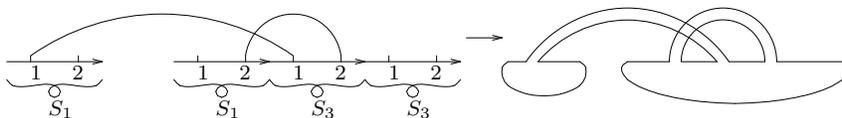
4. $W_P(D)$ is $\sum_\alpha N^{c_\alpha}$, where c_α is the number of components in D'_α.

2.5. The conclusion of the proof. Define the *order* of a chord t^{ij} to be $\max(i, j)$. We say that a chord diagram is *non-decreasing* if its chords appear in non-decreasing order. We say that it is *flat* if all the chords in it are of the same order. For example, $t^{13}t^{23}t^{13}$ is flat and non-decreasing, $t^{12}t^{23}t^{13}$ is non-decreasing but not flat, and $t^{13}t^{12}$ is neither flat nor non-decreasing. The following fact is a trivial consequence of the relations (3) and (2) defining $\mathcal{A}^{pb}$.

FACT 8. *(Drinfel'd [11]) $\mathcal{A}^{pb}$ is generated by non-decreasing chord diagrams.*

The following two propositions make the key technical observation of this paper.

PROPOSITION 2.4. *The weight systems W_P separate flat chord diagrams. More precisely, if $D_\nu(i_1, \dots, i_m) = t^{i_1\nu} \cdots t^{i_m\nu}$ is a flat chord diagram of degree m (that is, $i_\alpha < \nu$ for all α) and $P_\nu(j_1, \dots, j_m)$ (with $j_\alpha < \nu$ for all α) is the connected path $S_{j_m} S_{j_{m-1}} \dots S_{j_1} S_\nu$, and if $C_{N,p}$ is the operator that maps a polynomial in N to the coefficient of N^p in it, then*

$$C_{N,m+1} W_{P_\nu(j_1,\dots,j_m)}(D_\nu(i_1,\dots,i_m)) = \begin{cases} 1 & \text{if } i_\alpha = j_\alpha \text{ for all } \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Consider the diagrams D_α and the corresponding numbers c_α that appear when $W_{P_\nu(j_1,\dots,j_m)}(D_\nu(i_1,\dots,i_m))$ is computed using the algorithm of section 2.4. It is easy to show that the maximal possible value for the c_α's is $m+1$, and that this maximum is attained iff the corresponding diagram D_α has no chord intersections:

 (maximal)

 (not maximal)
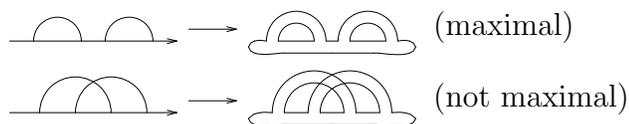
Therefore, we only care about those D_α's in which there are no chord intersections. Notice that the right ends of the chords in $D_\nu(i_1,\dots,i_m)$ can only be lifted (in order) to the last interval (S_ν) in $P_\nu(j_1,\dots,j_m)$ — they simply have nowhere else to go. The left ends of these chords should be lifted to one of the earlier intervals in $P_\nu(j_1,\dots,j_m)$, and so far the picture is:
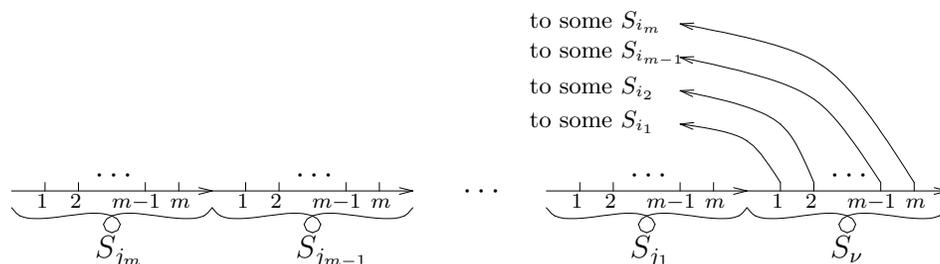
It is only possible to connect the left ends of the chords above subject to the restrictions and without introducing chord intersections if $i_\alpha = j_\alpha$ for all $1 \leq \alpha \leq m$, and this is then possible in a unique way. $\square$

In order to show that the W_P's separate non-decreasing chord diagrams, we have to work just a little bit harder. By mapping $D = D_\nu(i_1,\dots,i_m)$ to $P(D) = P_\nu(i_1,\dots,i_m)$ we see that to every flat chord diagram D corresponds a connected path $P(D)$. This map can be extended to non-decreasing chord diagrams — simply write any non-decreasing chord diagram D as an increasing product of flat chord diagrams, and let $P(D)$ be the list of connected paths corresponding to the flat parts of D (including a connected path S_ν for each $1 \leq \nu \leq n$ for which the order ν part of D is empty). Let the profile of a diagram D be the sequence $(l_n, l_{n-1}, \dots, l_1)$, where l_ν is the number of chords in D which are of order ν. Let us order all degree m chord diagrams using some order $\prec$ that refines the lexicographic order on their profiles. The following proposition is a generalization of proposition 2.4, and its proof is very similar.

PROPOSITION 2.5. *The W_P's separate non-decreasing chord diagrams. More precisely, if D_1 and D_2 are chord diagrams of degree m, then*

$$C_{N,m+n}W_{P(D_1)}(D_2) = \begin{cases} 1 & \text{if } D_1 = D_2, \\ 0 & \text{if } D_1 \prec D_2. \end{cases}$$

PROOF. Consider the diagrams D_α and the numbers c_α that appear when $W_{P(D_1)}(D_2)$ is computed using the algorithm of section 2.4. It is easy to show that the maximal possible value for the c_α's is $m+n$, and that this maximum is attained iff the corresponding diagram D_α has no chord intersections, and no chords connecting different components of $P(D_1)$. Therefore, we only care about those D_α's in which there are no chord intersections and no such connecting chords. If the profile of D_1 is lexicographically smaller than the profile of D_2, there is simply no room to fit a lifting of D_2 on $P(D_1)$ without chord intersections or connecting chords. If the profiles of D_1 and D_2 are the same, the flat components of D_2 must be lifted to the corresponding connected paths in $P(D_1)$, and the same argument as in the proof of proposition 2.4 shows that such a lifting (having no chord intersections or connecting chords) exists iff $D_1 = D_2$, and in that case it is unique.   □

COROLLARY 2.6. *The weight systems corresponding to (traces of) the $gl(N)$ invariants $J_{R_1,\ldots,R_n,m}$ span the space $\mathcal{W}^{pb} = (\mathcal{A}^{pb})^*$ of all weight systems.*   □

## 3. Vassiliev invariants separate pure braids

Just for the amusement of the reader, we include here the 'moral reason' for why Vassiliev invariants separate pure braids. As proofs of this fact are available elsewhere (see e.g. [**3, 18**]), we will leave the details of the proof presented here as an exercise to the reader.

FACT 9.   1. *Every pure braid has a unique presentation as a 'combed braid', as in figure 7. Equivalently, the group K_n of pure braids on n strands is a semi-direct product of free groups:*

$$K_n = F_1 \ltimes F_2 \ltimes \cdots \ltimes F_{n-1}. \tag{6}$$

2. *(Kohno [**17**]) The non-decreasing chord diagrams form a basis of $\mathcal{A}^{pb}$. Equivalently, the algebra $\mathcal{A}^{pb}_n$ of chord diagrams on n strands is a semi-direct product of free associative algebras:*

$$\mathcal{A}^{pb}_n = FA_1 \ltimes FA_2 \ltimes \cdots \ltimes FA_{n-1}. \tag{7}$$
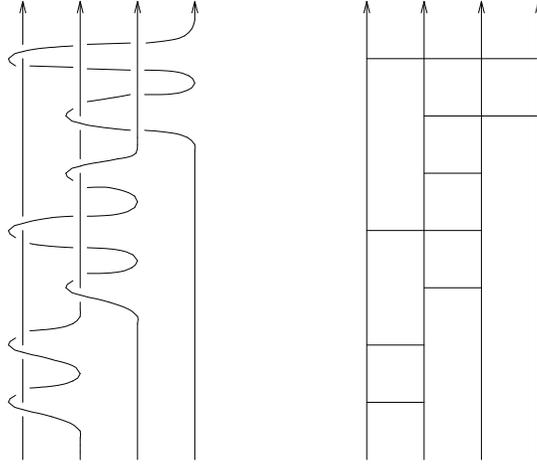
Figure 7. Left: In a combed braid, the first strand is always straight. The second twists around the first some number of times, and then becomes straight. The third waits patiently until the second finishes twisting around the first, tangles a bit between the first and the second, and then becomes straight. The fourth waits for the third to be done, tangles, etc. Right: a non-decreasing chord diagram.

PROOF.   1. Every braid can be 'combed' by induction. Assume the first four strands are already combed as in figure 7, and that a fifth strand tangles between them. Think of the first four strands as made of copper wires and grease them very well. Think of the fifth strand as made of soft spaghetti, and use a fan to blow strong wind from the bottom up. The spaghetti wants to fly up, and the copper wires are very smooth (and sloped upwards) so they can't stop it from doing so. When all the spaghetti (except the very beginning, which is tied to the bottom plane) reaches the top of the figure, freeze it in place. It is now a path in the the plane minus four points (the four other strands). Present it as a product of generators, replace it by a copper wire, and you are ready to deal with the sixth strand. For a formal proof, see e.g. [**7**].

2. Immediate from fact 8 (that non-decreasing chord diagrams generate $\mathcal{A}^{pb}$) and from the main technical observation of this paper, proposition 2.5.

$\square$

EXERCISE 3.1. The (formal) Knizhnik-Zamolodchikov connection [**16, 17, 19, 2**] defines a 'holonomy map' $Z : K_n \to \mathcal{A}_n^{pb}$. Use simple properties of Z (or of any of the other 'universal Vassiliev invariants' constructed in [**1, 4, 9, 15, 21, 27**]) and the obvious similarity between equation 6 and equation 7 to show that Z is injective. Composing with linear functionals on $\mathcal{A}_n^{pb}$, one gets exactly the Vassiliev invariants.

4. Corollaries

4.1. The HOMFLY polynomial and braids. It is well known (see e.g. [**28, 29**] that $gl(N)$ in its defining representation corresponds to the HOMFLY polynomial (in some parametrization), and that the higher representations of $gl(N)$ correspond to various cabling operations applied to the HOMFLY polynomial (for a similar situation, see [**25**]).

COROLLARY 4.1. *A pure braid is determined by the HOMFLY polynomials of all closures into links of all of its cablings. (We allow an arbitrary permutation of the strands before closing).*

REMARK 4.2. The above corollary remains true even if the words 'pure braid' are replaced by the word 'braid'. Indeed, it is easy to read the number of components of a link from its HOMFLY polynomial, and knowing this number for all possible closures of a braid (using all possible permutations of the strands) determines the permutation σ underlying that braid, as that number is maximal only if the closure is done using the permutation σ^{-1}.

4.2. Braids and string links. The Reshetikhin-Turaev $gl(N)$ invariants $J_{R_1,\ldots,R_n}$ were originally defined as tangle invariants, and, in particular, they extend to n-component string links (for a definition, check [**14**] or [**3**]). Clearly, the same holds for the various traces of the $J_{R_1,\ldots,R_n}$'s, and as these span the space of all Vassiliev invariants of pure braids, it follows that

COROLLARY 4.3. *A Vassiliev invariant of pure braids can always be extended to become a Vassiliev invariant (of the same type) of string links.*

On the level of chord diagrams (which are dual to invariants), this corollary implies that chord diagrams of pure braids inject into chord diagrams of string links. Recall that the latter space, $\mathcal{A}^{sl}$, discussed in

more detail in [**3**], is the space of all diagrams of the form
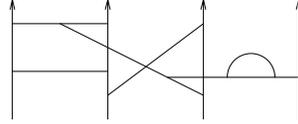
(non-horizontal chords and oriented internal trivalent vertices are allowed, apparent quadrivalent vertices in the planar projection are not true vertices), modulo the STU relation,
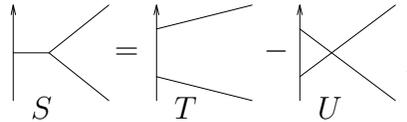
COROLLARY 4.4. *The obvious map $\mathcal{A}^{pb} \to \mathcal{A}^{sl}$ is an injection.*

In fact, curiosity whether corollary 4.4 holds is what lead me to the investigations described in this paper.

References

[1] D. Altschuler & L. Freidal, *On universal Vassiliev invariants*, February 1994, preprint.

[2] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** 423–472 (1995).

[3] _____ , *Vassiliev homotopy string link invariants*, Jour. of Knot Theory and its Ramifications, to appear.

[4] _____ , *Non-associative tangles*, to appear in the Georgia Inter. Topology Conf. proceedings.

[5] _____ & S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture*, July 1994, preprint.

[6] J. S. Birman, *New points of view in knot theory*, Bull. Amer. Math. Soc. **28** (1993) 253–287.

[7] J. S. Birman, *Braids, links, and mapping class groups*, Princeton Univ. Press, Princeton, 1975.

[8] _____ & X-S. Lin, *Knot polynomials and Vassiliev's invariants*, Inv. Math. **111** (1993) 225–270.

[9] P. Cartier, *Construction combinatoire des invariants de Vassiliev-Kontsevich des nœuds*, C. R. Acad. Sci. Paris **316** Série I (1993) 1205–1210.

[10] V. G. Drinfel'd, *Quasi-Hopf algebras*, Leningrad Math. J. **1** (1990) 1419–1457.

[11] _____ , *On quasitriangular Quasi-Hopf algebras and a group closely connected with Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)*, Leningrad Math. J. **2** (1991) 829–860.

[12] M. Gusarov, *A new form of the Conway-Jones polynomial of oriented links*, in *Topology of manifolds and varieties* (O. Viro, editor), Amer. Math. Soc., Providence 1994, 167–172.

[13] _____ , *On n-equivalence of knots and invariants of finite degree*, in *Topology of manifolds and varieties* (O. Viro, editor), Amer. Math. Soc., Providence 1994, 173–192.

[14] N. Habegger & X-S. Lin, *The Classification of links up to link-homotopy*, J. Amer. Math. Soc. **3** (1990) 389–419.

[15] C. Kassel, *Quantum groups*, Springer-Verlag GTM **155**, Heidelberg 1994.

[16] V. G. Knizhnik and A. B. Zamolodchikov, *Current algebra and Wess-Zumino model in two dimensions*, Nucl. Phys. **B247** (1984) 83–103.

[17] T. Kohno, *Monodromy representations of braid groups and Yang-Baxter equations*, Ann. Inst. Fourier **37** (1987) 139–160.

[18] _____, *Vassiliev invariants and de-Rham complex on the space of knots*, Tokyo Univeristy preprint, 1994.

[19] M. Kontsevich, *Vassiliev's knot invariants*, Adv. in Sov. Math., **16(2)** (1993), 137–150.

[20] G. Kuperberg, *Detecting knot invertibility*, University of Chicago preprint, February 1995.

[21] T. Q. T. Le & J. Murakami, *The universal Vassiliev-Kontsevich invariant for framed oriented links*, Max-Planck-Institut Bonn preprint, 1993.

[22] X-S. lin, *Vertex models, quantum groups and Vassiliev's knot invariants*, Columbia Univ. preprint, 1991.

[23] _____, *Knot invariants and iterated integrals*, Columbia Univ. preprint, 1994.

[24] _____, *Braid algebras, trace modules and Vassiliev invariants*, Columbia Univ. preprint, 1994.

[25] H. R. Morton and P. Strickland, *Jones polynomial invariants of knots and satellites*, Math. Proc. Camb. Phil. Soc. **109** (1991) 83–103.

[26] S. Piunikhin, *Weights of Feynman diagrams, link polynomials and Vassiliev knot invariants*, Moscow State Univ. preprint, 1992.

[27] _____, *Combinatorial expression for universal Vassiliev link invariant*, to appear in Commun. Math. Physics.

[28] N. Yu. Reshetikhin, *Quasitriangle Hopf algebras and invariants of tangles*, Leningrad Math. J. **1** (1990) 491–513.

[29] _____ & V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Commun. Math. Phys. **127** (1990) 1–26.

[30] V. A. Vassiliev, *Cohomology of knot spaces*, Theory of Singularities and its Applications (Providence) (V. I. Arnold, ed.), Amer. Math. Soc., Providence, 1990.

[31] _____, *Complements of discriminants of smooth maps: topology and applications*, Trans. of Math. Mono. **98**, Amer. Math. Soc., Providence, 1992.

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