FUNDAMENTAL CONCEPTS IN DIFFERENTIAL GEOMETRY
FALL 2000
HANDOUT # 3

1. Exercises for the Proper Course

1. For $p \in M^n$ let $\mathcal{C}$ be the collection of of triples (U, φ, α) where U is a neighborhood of p, φ is a chart carrying p to u, and α is a vector in $\mathbb{R}^n$. Define an equivalence relation on $\mathcal{C}$ by defining $(U, \varphi, \alpha) \sim (V, \psi, \beta)$ if

$$\beta = (\psi\varphi^{-1})'|_u(\alpha)$$

where $\psi\varphi^{-1}$ is defined on an appropriate neighborhoodof u.

Show that $\mathcal{C}/\sim$ has a natural structure of a vector space. Show that there is a natural isomorphism of vector spaces $\mathcal{C}/\sim \cong T_pM$.

2. Show that if U is an open subset of a smooth manifold M, and if $p \in U$, then $T_pU = T_pM$. Make this statement precise !

3. (a) Make a precise sense of the following statement.
The tangent space to the sphere $S^2 = \{x \in \mathbb{R}^3 : ||x|| = 1\}$ at the point u, consists of all vectors in $\mathbb{R}^3$ perpendicular to u.

(b) Consider Euler's parameterization of the sphere

$$\Psi : (\theta, \phi) \mapsto (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

where $-\pi < \theta < \pi$ and $0 < \phi < \pi$. Compute $\Psi_*(\frac{\partial}{\partial\theta})$ and $\Psi_*(\frac{\partial}{\partial\phi})$. Show that these are indeed vectors in the tangent space to the sphere.

Date: 14 Nov., 2000.