Dror Bar-Natan: Talks: UofT-GS-090930:

Kolmogorov's Solution of Hilbert's 13th Problem — A Guided Ascent

http://www.math.toronto.edu/~drorbn/Talks/UofT-GS-090930/

Fix once and for all some irrational number $\lambda \in (0,1)$. It may as well be the most irrational number there is, $(\sqrt{5}-1)/2$.

Step 1. If $\epsilon > 0$ and $f : [0,1] \times [0,1] \to \mathbf{R}$ is continuous, then there exists a continuous function $\phi : [0,1] \to [0,1]$ and a continuous function $g : [0, 1+\lambda] \to \mathbf{R}$ so that $|f(x,y) - g(\phi(x) + \lambda\phi(y))| < \epsilon$ on at least 98% of the area of $[0,1] \times [0,1]$.

Step 2. There exists a continuous function $\phi : [0,1] \to [0,1]$ so that for every $\epsilon > 0$ and every continuous function $f : [0,1] \times [0,1] \to \mathbf{R}$ there exists a continuous function $g : [0, 1+\lambda] \to \mathbf{R}$ so that $|f(x,y) - g(\phi(x) + \lambda\phi(y))| < \epsilon$ on a set of area at least $1 - \epsilon$ in $[0,1] \times [0,1]$. (Notice the different order of the quantifiers!).

Step 3. There exists 5 continuous functions $\phi_i : [0,1] \to [0,1]$ $(1 \leq i \leq 5)$ so that for every $\epsilon > 0$ and every continuous function $f : [0,1] \times [0,1] \to \mathbf{R}$ there exists a continuous function $g : [0, 1+\lambda] \to \mathbf{R}$ so that

$$|f(x,y) - \sum_{i=1}^{5} g(\phi_i(x) + \lambda\phi_i(y))| < \left(\frac{2}{3} + \epsilon\right) ||f||$$

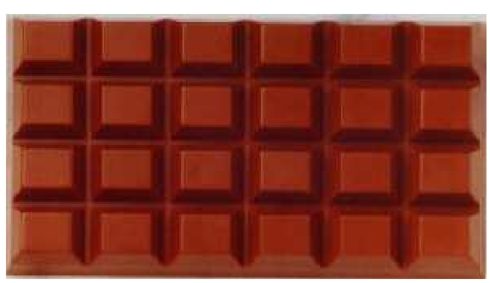
for every $x, y \in [0,1]$.

Step 4. There exists 5 continuous functions $\phi_i : [0,1] \to [0,1]$ $(1 \leq i \leq 5)$ so that for every continuous function $f : [0,1] \times [0,1] \to \mathbf{R}$ there exists a continuous function $g : [0, 1+\lambda] \to \mathbf{R}$ so that

$$f(x,y) = \sum_{i=1}^{5} g(\phi_i(x) + \lambda\phi_i(y))$$

for every $x, y \in [0,1]$.

Hints: Chocolate tablets, nested chocolate tablets, shifted chocolate tablets and Tietze.



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