

Kolmogorov's Solution of Hilbert's 13th Problem — A Guided Ascent

UofT October 10, 2002

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See <http://www.math.toronto.edu/~drorbn/Talks/UofT-021010/>

Base Camp. If $\lambda > 0$ is irrational, $\epsilon > 0$, and $f : [0, 1] \times [0, 1] \rightarrow \mathbf{R}$ is continuous, then there exists a continuous function $\phi : [0, 1] \rightarrow [0, 1]$ and a continuous function $g : [0, 1 + \lambda] \rightarrow \mathbf{R}$ so that $|f(x, y) - g(\phi(x) + \lambda\phi(y))| < \epsilon$ on at least 98% of the area of $[0, 1] \times [0, 1]$.

Camp I. If $\lambda > 0$ is irrational then there exists a continuous function $\phi : [0, 1] \rightarrow [0, 1]$ so that for every $\epsilon > 0$ and every continuous function $f : [0, 1] \times [0, 1] \rightarrow \mathbf{R}$ there exists a continuous function $g : [0, 1 + \lambda] \rightarrow \mathbf{R}$ so that $|f(x, y) - g(\phi(x) + \lambda\phi(y))| < \epsilon$ on a set of area at least $1 - \epsilon$ in $[0, 1] \times [0, 1]$. (Notice the different order of the quantifiers!).

Camp II. If $\lambda > 0$ is irrational and $\epsilon > 0$ then there exists 5 continuous functions $\phi_i : [0, 1] \rightarrow [0, 1]$ ($1 \leq i \leq 5$) so that for every continuous function $f : [0, 1] \times [0, 1] \rightarrow \mathbf{R}$ there exists a continuous function $g : [0, 1 + \lambda] \rightarrow \mathbf{R}$ so that

$$|f(x, y) - \sum_{i=1}^5 g(\phi_i(x) + \lambda\phi_i(y))| < \left(\frac{2}{3} + \epsilon\right) \|f\|$$

for every $x, y \in [0, 1]$.

Summit. If $\lambda > 0$ is irrational then there exists 5 continuous functions $\phi_i : [0, 1] \rightarrow [0, 1]$ ($1 \leq i \leq 5$) so that for every continuous function $f : [0, 1] \times [0, 1] \rightarrow \mathbf{R}$ there exists a continuous function $g : [0, 1 + \lambda] \rightarrow \mathbf{R}$ so that

$$f(x, y) = \sum_{i=1}^5 g(\phi_i(x) + \lambda\phi_i(y))$$

for every $x, y \in [0, 1]$.

Hints: Chocolate tablets, nested chocolate tablets, shifted chocolate tablets and Tietze.

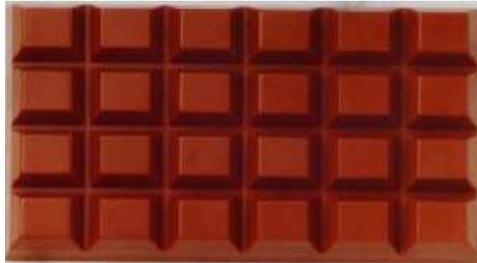


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