### Cosmic Coincidences and Several Other Stories, 2

"Low Algebra" and universal formulae in Lie algebras. 

\[ [x,y] = xy - yx, \quad [x,[y,z]] = [x,y]z - [y,[x,z]], \quad \{x,y\} = \frac{1}{2}([x,y] + [y,x]) \]

More precisely, let \( \mathfrak{g} = (X_a, \{X_a, X_c\}) \) be a Lie algebra with an orthonormal basis, and let \( R = \{v_\alpha\} \) be a representation.

Set

\[ f_{abc} := \{X_a, X_c\}, \quad X_a v_\beta = \sum_\gamma t_{a\gamma}^\beta v_\gamma \]

and then

\[ W_{\mathfrak{g}, R} : \begin{array}{cc}
\alpha & \beta \\
\gamma & \delta
\end{array} \rightarrow \sum_{a\beta\gamma\delta} f_{abc} t_{\gamma\delta}^\beta r_{a\delta}^\alpha \]

\( W_{\mathfrak{g}, R} \circ Z \) is often interesting:

- \( \mathfrak{g} = sl(2) \) \quad \rightarrow \quad The Jones polynomial
- \( \mathfrak{g} = sl(N) \) \quad \rightarrow \quad The HOMFLYPT polynomial
- \( \mathfrak{g} = so(N) \) \quad \rightarrow \quad The Kauffman polynomial

Knots are the wrong objects to study in knot theory! They are not finitely generated and they carry no interesting operations.

The Miller Institute knot

### Chern-Simons-Witten theory and Feynman diagrams.

\[ \int D A \ hol_K(A) \ \exp \left[ \frac{ik}{4\pi} \int \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right] \rightarrow \sum_D W_{\mathfrak{g}}(D) \sum E(D) \rightarrow \sum_D D \int E(D) \]

**Definition.** \( V \) is finite type (Vassiliev, Goussarov) if it vanishes on sufficiently large alternations as on the right

**Theorem.** All knot polynomials (Conway, Jones, etc.) are of finite type.

**Conjecture.** (Taylor’s theorem) Finite type invariants separate knots.

**Theorem.** \( Z(K) \) is a universal finite type invariant! (sketch: to dance in many parties, you need many feet).

### The u→v→w & p Stories

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<th>Topology</th>
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<td>v-Knotted</td>
<td>Arrow diagrams and v-Jacobi diagrams, modulo 6T and various “directed” STUs and IHHs, etc.</td>
<td>Finite dimensional Lie bi-algebras, representations, and associated spaces.</td>
<td>Likely, quantum groups and the Etingof-Kazhdan theory of quantization of Lie bi-algebras.</td>
<td>No clue.</td>
<td>No clue.</td>
<td>No clue.</td>
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<tr>
<td>w-Knotted</td>
<td>Like v, but also with “tails commute”. Only “two in one out” internal vertices.</td>
<td>Finite dimensional co-commutative Lie bi-algebras (( g \bowtie g' )), representations, and associated spaces.</td>
<td>The Kashihara-Vergne-Alekseev-Torossian theory of convolutions on Lie groups / algebras.</td>
<td>No clue.</td>
<td>Probably related to 4D BF theory.</td>
<td>Studied.</td>
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<td>P-Objects</td>
<td>“Acrobat towers” with 2-in many-out vertices.</td>
<td>Deformation quantization of poisson manifolds.</td>
<td>Configuration space integrals are key, but they don’t reduce to counting.</td>
<td>Work of Cattaneo.</td>
<td>Hyperbolic geometry?</td>
<td>Studied.</td>
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