

The Taylor Remainder Formulas. Let f be a smooth function, let $P_{n,a}(x)$ be the n th order Taylor polynomial of f around a and evaluated at x , so with $a_k = f^{(k)}(a)/k!$,

$$P_{n,a}(x) := \sum_{k=0}^n a_k(x-a)^k,$$

and let $R_{n,a}(x) := f(x) - P_{n,a}(x)$ be the “mistake” or “remainder term”. Then

$$R_{n,a}(x) = \int_a^x dt \frac{f^{(n+1)}(t)}{n!} (x-t)^n, \quad (1)$$

or alternatively, for some t between a and x ,

$$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{(n+1)!} (x-a)^{n+1}. \quad (2)$$

(In particular, the Taylor expansions of \sin , \cos , \exp , and of several other lovely functions converges to these functions *everywhere*, no matter the odds.)

Proof of (1) (for adults; I learned it from my son Itai). The fundamental theorem of calculus says that if $g(a) = 0$ then $g(x) = \int_a^x dx_1 g(x_1)$. By design, $R_{n,a}^{(k)}(a) = 0$ for $0 \leq k \leq n$. Therefore

$$\begin{aligned} R_{n,a}(x) &= \int_a^x dx_1 R'_{n,a}(x_1) \\ &= \int_a^x dx_1 \int_a^{x_1} dx_2 R''_{n,a}(x_2) \\ &= \dots = \int_a^x dx_1 \int_a^{x_1} dx_2 \dots \int_a^{x_n} dx_n \int_a^t dt R^{(n+1)}_{n,a}(t) \\ &= \int_a^x dx_1 \int_a^{x_1} dx_2 \dots \int_a^{x_n} dx_n \int_a^t dt f^{(n+1)}(t), \end{aligned}$$

when $x > a$, and with similar logic when $x < a$,

$$\begin{aligned} &= \int_{a \leq t \leq x_n \leq \dots \leq x_1 \leq x} f^{(n+1)}(t) = \int_a^t dt f^{(n+1)}(t) \int_{t \leq x_n \leq \dots \leq x_1 \leq x} 1 \\ &= \int_a^t dt \frac{f^{(n+1)}(t)}{n!} \int_{(x_1, \dots, x_n) \in [t, x]^n} 1 = \int_a^x dt \frac{f^{(n+1)}(t)}{n!} (x-t)^n. \end{aligned}$$

de-Fubini (obfuscation in the name of simplicity). Prematurely aborting the above chain of equalities, we find that for any $1 \leq k \leq n+1$,

$$R(x) = \int_a^x dt R^{(k)}(t) \frac{(x-t)^{k-1}}{(k-1)!}.$$

But these are easy to prove by induction using integration by parts, and there's no need to invoke Fubini.



Brook Taylor

Partial Derivatives Commute.

Make Fubini Smile Again!

If $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is C^2 near $a \in \mathbb{R}^2$, then $f_{12}(a) = f_{21}(a)$.

Proof. Let $x \in \mathbb{R}^2$ be small, and let $R := [a_1, a_1+x_1] \times [a_2, a_2+x_2]$.

$$f_{12}(a) \sim \int f_{12} = \sum f = \int f_{21} \sim f_{21}(a)$$

$$\begin{aligned} f_{12}(a) &\sim \frac{1}{|R|} \int_R f_{12} = \frac{1}{|R|} \int_{a_1}^{a_1+x_1} dt_1 (f_1(t_1, a_2+x_2) - f_1(t_1, a_2)) \\ &= \frac{1}{|R|} \left(f(a_1+x_1, a_2+x_2) - f(a_1+x_1, a_2) - f(a_1, a_2+x_2) + f(a_1, a_2) \right). \end{aligned}$$

But the answer here is the same as in

$$\begin{aligned} f_{21}(a) &\sim \frac{1}{|R|} \int_R f_{21} = \frac{1}{|R|} \int_{a_2}^{a_2+x_2} dt_2 (f_2(a_1+x_1, t_2) - f_2(a_1, t_2)) \\ &= \frac{1}{|R|} \left(f(a_1+x_1, a_2+x_2) - f(a_1, a_2+x_2) - f(a_1+x_1, a_2) + f(a_1, a_2) \right), \end{aligned}$$

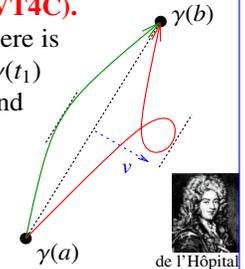
and both of these approximations get better and better as $x \rightarrow 0$. \square

The Mean Value Theorem for Curves (MVT4C).

If $\gamma: [a, b] \rightarrow \mathbb{R}^2$ is a smooth curve, then there is some $t_1 \in (a, b)$ for which $\gamma(b) - \gamma(a)$ and $\dot{\gamma}(t_1)$ are linearly dependent. If also $\gamma(a) = 0$, and

$\gamma = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ and $\eta \neq 0 \neq \dot{\eta}$ on (a, b) , then

$$\frac{\xi(b)}{\eta(b)} = \frac{\xi(t_1)}{\dot{\eta}(t_1)} \quad \left(\text{when lucky, } = \frac{\ddot{\xi}(t_2)}{\ddot{\eta}(t_2)} \dots \right).$$



Proof of (2). Iterate the lucky MVT4C as follows:

$$\frac{R_{n,a}(x)}{(x-a)^{n+1}} = \frac{R'_{n,a}(t_1)}{(n+1)(t_1-a)^n} = \dots = \frac{R^{(n+1)}_{n,a}(t_{n+1})}{(n+1)!} = \frac{f^{(n+1)}(t)}{(n+1)!}.$$

π is Irrational following Ivan Niven, Bull. Amer. Math. Soc. (1947) pp. 509:

Theorem: π is irrational.

Proof: Assume $\pi = a/b$ and consider the polynomial $p(x) = \frac{x^n(a-bx)^n}{n!}$ For n quite large. Clearly $p(x)$ is positive yet small, hence $I = \int_0^\pi p(x) \sin x dx < 1$. On the other hand, integration by parts shows that $I = (\text{boundary terms}) \pm \int p^{(n+1)}(x) \cos x dx$. The second term is 0 because p is a polynomial of degree $2n$, and the first term is an integer for clearly $p^{(k)}(0)$ is always an integer, for $p(\pi-x) = p(x)$ hence same is true for $p^{(k)}(\pi)$ and for \sin & \cos of 0 & π are all integers. Ergo I is an integer between 0 and 1, and these are rare indeed. \square

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