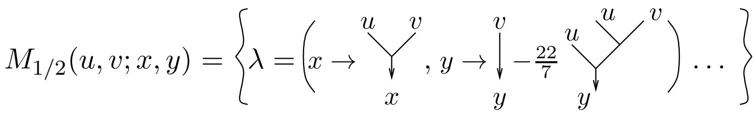
Trees and Wheels and Balloons and Hoops

Dror Bar-Natan, Zurich, September 2013
ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/Zurich-130919

15 Minutes on Algebra

Let T be a finite set of "tail labels" and H a finite set of "head labels". Set
$$M_{1/2}(T;H) := FL(T)^H,$$
"H-labeled lists of elements of the degree-completed free Lie algebra generated by T".

$$FL(T) = \left\{2t_2 - \frac{1}{2}[t_1,[t_1,t_2]] + \dots\right\} \Big/ \begin{pmatrix}\text{anti-symmetry}\\ \text{Jacobi}\end{pmatrix}$$
... with the obvious bracket.



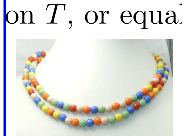
Operations $M_{1/2} \to M_{1/2}$. newspeak!

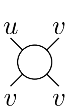
Tail Multiply tm^{uv}_w is $\lambda \mapsto \lambda /\!/ (u,v \to w)$, satisfies "meta-associativity", $tm^{uv}_u /\!/ tm^{uw}_u = tm^{vw}_v /\!/ tm^{uv}_u$.

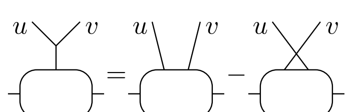
Head Multiply hm^{xy}_z is $\lambda \mapsto (\lambda\setminus\{x,y\}) \cup (z \to \mathrm{bch}(\lambda_x,\lambda_y))$, where
$$\mathrm{bch}(\alpha,\beta) := \log(e^\alpha e^\beta) = \alpha+\beta+\frac{[\alpha,\beta]}{2}+\frac{[\alpha,[\alpha,\beta]]+[[\alpha,\beta],\beta]}{12}+\dots$$
satisfies $\mathrm{bch}(\mathrm{bch}(\alpha,\beta),\gamma) = \log(e^\alpha e^\beta e^\gamma) = \mathrm{bch}(\alpha,\mathrm{bch}(\beta,\gamma))$ and hence meta-associativity, $hm^{xy}_x /\!/ hm^{xz}_x = hm^{yz}_y /\!/ hm^{xy}_x$.

Tail by Head Action tha^{ux} is $\lambda \mapsto \lambda /\!/ RC_u^{\lambda_x}$, where $C_u^{-\gamma}\colon FL \to FL$ is the substitution $u \to e^{-\gamma}ue^{\gamma}$, or more precisely,
$$C_u^{-\gamma}\colon u \to e^{-\operatorname{ad}\gamma}(u) = u - [\gamma,u] + \frac{1}{2}[\gamma,[\gamma,u]] - \dots,$$
and $RC_u^\gamma = (C_u^{-\gamma})^{-1}$. Then $C_u^{\mathrm{bch}(\alpha,\beta)} = C_u^{\alpha/\!/RC_u^{-\beta}} /\!/ C_u^\beta$ hence $RC_u^{\mathrm{bch}(\alpha,\beta)} = RC_u^\alpha /\!/ RC_u^{\beta/\!/RC_u^\alpha}$ hence "meta $u^{xy} = (u^x)^y$",
$$hm^{xy}_z /\!/ tha^{uz} = tha^{ux} /\!/ tha^{uy} /\!/ hm^{xy}_z,$$
and $tm^{uv}_w /\!/ C_w^{\gamma/\!/tm^{uv}_w} = C_u^{\gamma/\!/RC_v^{-\gamma}} /\!/ C_v^\gamma /\!/ tm^{uv}_w$ and hence "meta $(uv)^x = u^xv^x$", $tm^{uv}_w /\!/ tha^{wx} = tha^{ux} /\!/ tha^{vx} /\!/ tm^{uv}_w$.

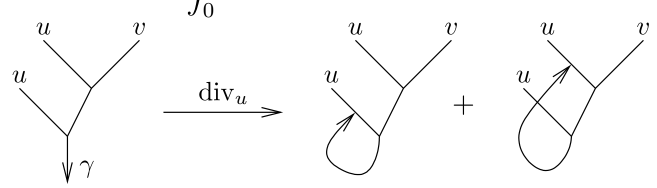
Wheels. Let $M(T;H) := M_{1/2}(T;H) \times CW(T)$, where $CW(T)$ is the (completed graded) vector space of cyclic words on T, or equaly well, on $FL(T)$:


ωεβ/antiq-ave





Operations. On $M(T;H)$, define tm^{uv}_w and hm^{xy}_z as before, and tha^{ux} by adding some J-spice:
$$(\lambda;\omega) \mapsto (\lambda, \omega + J_u(\lambda_x)) /\!/ RC_u^{\lambda_x},$$
where $J_u(\gamma) := \int_0^1 ds\ \mathrm{div}_u(\gamma /\!/ RC_u^{s\gamma}) /\!/ C_u^{-s\gamma}$, and




Alekseev

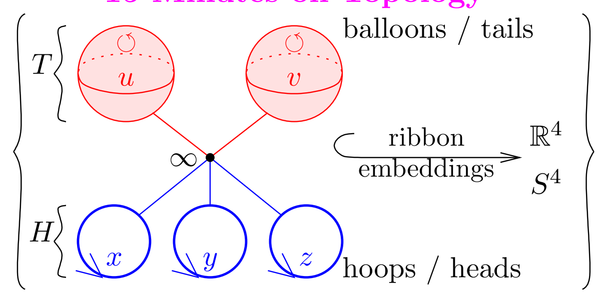

Torossian

Theorem Blue. All blue identities still hold.

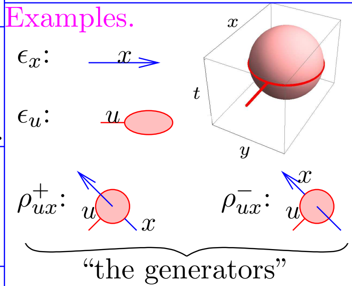
Merge Operation. $(\lambda_1;\omega_1)*(\lambda_2;\omega_2) := (\lambda_1 \cup \lambda_2; \omega_1+\omega_2)$.

$\mathcal{K}^{bh}(T;H)$. "Ribbon-knotted balloons and hoops"

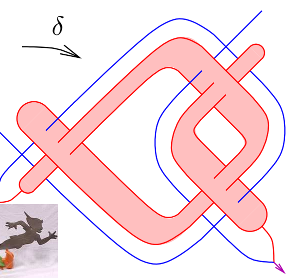
15 Minutes on Topology

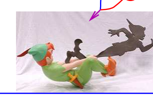


Examples.

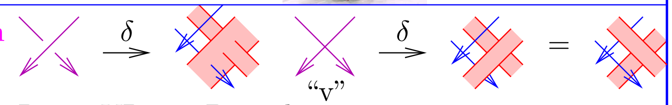



Shin Satoh

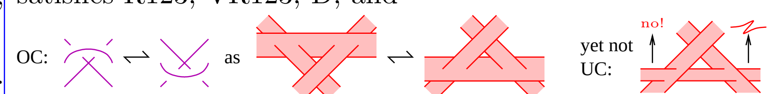




More on δ

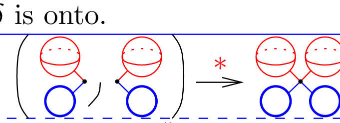


satisfies R123, VR123, D, and

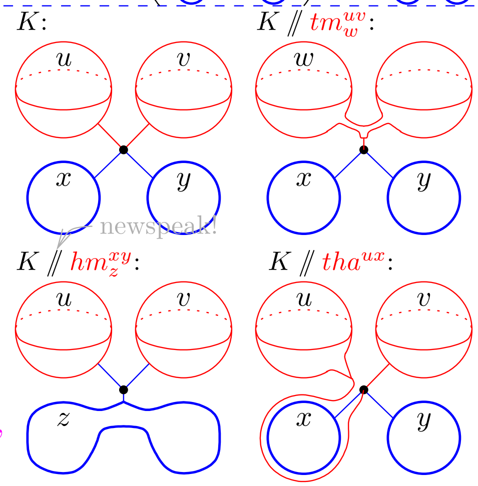


- δ injects u-knots into $\mathcal{K}^{bh}$ (likely u-tangles too).
- δ maps v-tangles to $\mathcal{K}^{bh}$; the kernel contains the above and conjecturally (Satoh), that's all.
- Allowing punctures and cuts, δ is onto.

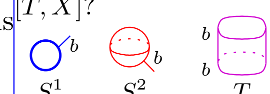
Operations Punctures & Cuts | Connected Sums.



If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2.



Riddle. People often study $\pi_1(X) = [S^1,X]$ and $\pi_2(X) = [S^2,X]$. Why not $\pi_T(X) := [T,X]$?

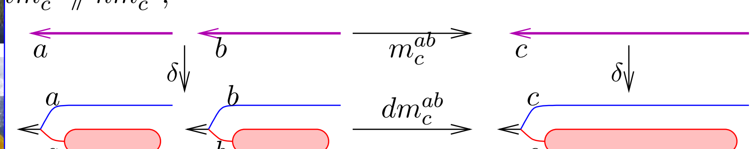


"Meta-Group-Action"

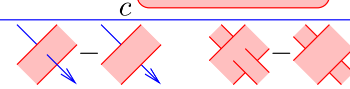
Properties.

- Associativities: $m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$, for $m = tm, hm$.
- "$(uv)^x = u^xv^x$": $tm^{uv}_w /\!/ tha^{wx} = tha^{ux} /\!/ tha^{vx} /\!/ tm^{uv}_w$,
- "$u^{(xy)} = (u^x)^y$": $hm^{xy}_z /\!/ tha^{uz} = tha^{ux} /\!/ tha^{uy} /\!/ hm^{xy}_z$.

Tangle concatenations $\to$ $\pi_1 \ltimes \pi_2$. With $dm^{ab}_c := tha^{ab} /\!/ tm^{ab}_c /\!/ hm^{ab}_c$,



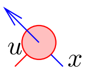
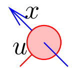
Finite type invariants make sense in the usual way, and "algebra" is (the primitive part of) "gr" of "topology".



Trees and Wheels and Balloons and Hoops: Why I Care

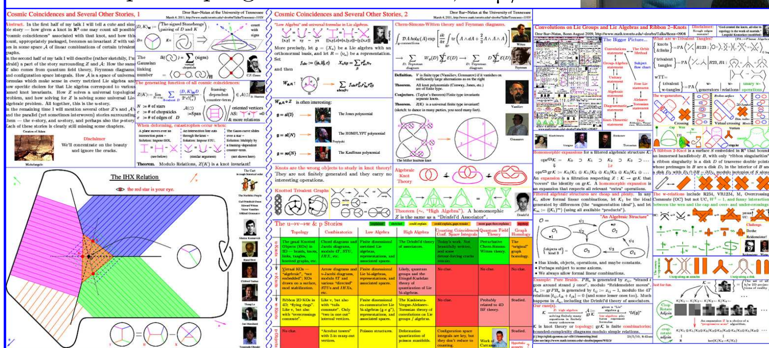
Moral. To construct an M-valued invariant ζ of (v-)tangles, and nearly an invariant on $\mathcal{K}^{bh}$, it is enough to declare ζ on the generators, and verify the relations that δ satisfies.

The Invariant ζ. Set $\zeta(\epsilon_x) = (x \to 0; 0)$, $\zeta(\epsilon_u) = ((); 0)$, and

ζ:  $\longmapsto \left(\big\downarrow^{u}_{x} \; ; 0 \right)$  $\longmapsto \left(- \big\downarrow^{u}_{x} \; ; 0 \right)$

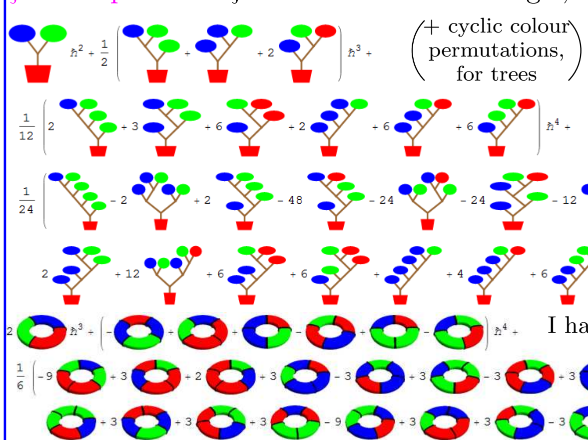
Theorem. ζ is (log of) the unique homomorphic universal finite type invariant on $\mathcal{K}^{bh}$.
(… and is the tip of an iceberg)

Paper in progress with Dancso, ωεβ/wko

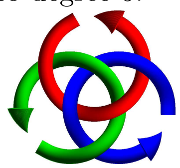


See also ωεβ/tenn, ωεβ/bonn, ωεβ/swiss, ωεβ/portfolio

ζ is computable! ζ of the Borromean tangle, to degree 5:



(+ cyclic colour permutations, for trees)



I have a nice free-Lie calculator!

Tensorial Interpretation. Let $\mathfrak{g}$ be a finite dimensional Lie algebra (any!). Then there's $\tau : FL(T) \to \mathrm{Fun}(\oplus_T \mathfrak{g} \to \mathfrak{g})$ and $\tau : CW(T) \to \mathrm{Fun}(\oplus_T \mathfrak{g})$. Together, $\tau : M(T; H) \to \mathrm{Fun}(\oplus_T \mathfrak{g} \to \oplus_H \mathfrak{g})$, and hence

$$e^\tau : M(T; H) \to \mathrm{Fun}(\oplus_T \mathfrak{g} \to \mathcal{U}^{\otimes H}(\mathfrak{g})).$$

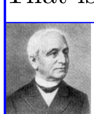
ζ and BF Theory. (See Cattaneo-Rossi, arXiv:math-ph/0210037) Let A denote a $\mathfrak{g}$-connection on S^4 with curvature F_A, and B a $\mathfrak{g}^*$-valued 2-form on S^4. For a hoop γ_x, let $\mathrm{hol}_{\gamma_x}(A) \in \mathcal{U}(\mathfrak{g})$ be the holonomy of A along γ_x. For a ball γ_u, let $\mathcal{O}_{\gamma_u}(B) \in \mathfrak{g}^*$ be (roughly) the integral of B (transported via A to ∞) on γ_u.

Cattaneo

Loose Conjecture. For $\gamma \in \mathcal{K}(T; H)$,

$$\int \mathcal{D}A\mathcal{D}B e^{\int B \wedge F_A} \prod_u e^{\mathcal{O}_{\gamma_u}(B)} \bigotimes_x \mathrm{hol}_{\gamma_x}(A) = e^\tau(\zeta(\gamma)).$$

That is, ζ is a complete evaluation of the BF TQFT.

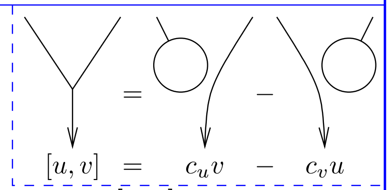


"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org The Knot Atlas Anyone Can Edit

The β quotient is M divided by all relations that universally hold when when $\mathfrak{g}$ is the 2D non-Abelian Lie algebra. Let $R = \mathbb{Q}[\![\{c_u\}_{u \in T}]\!]$ and



$L_\beta := R \otimes T$ with central R and with $[u, v] = c_u v - c_v u$ for $u, v \in T$. Then $FL \to L_\beta$ and $CW \to R$. Under this,

$$\mu \to ((\lambda_x); \omega) \quad \text{with } \lambda_x = \sum_{u \in T} \lambda_{ux} ux, \quad \lambda_{ux}, \omega \in R,$$

$$\mathrm{bch}(u, v) \to \frac{c_u + c_v}{e^{c_u + c_v} - 1} \left(\frac{e^{c_u} - 1}{c_u} u + e^{c_u} \frac{e^{c_v} - 1}{c_v} v \right),$$

if $\gamma = \sum \gamma_v v$ then with $c_\gamma := \sum \gamma_v c_v$,

$$u /\!/ RC_u^\gamma = \left(1 + c_u \gamma_u \frac{e^{c_\gamma} - 1}{c_\gamma} \right)^{-1} \left(e^{c_\gamma} u - c_u \frac{e^{c_\gamma} - 1}{c_\gamma} \sum_{v \neq u} \gamma_v v \right),$$

$\mathrm{div}_u \gamma = c_u \gamma_u$, and $J_u(\gamma) = \log\left(1 + \frac{e^{c_\gamma} - 1}{c_\gamma} c_u \gamma_u\right)$, so ζ is formula-computable to all orders! **Can we simplify?**

Repackaging. Given $((x \to \lambda_{ux}); \omega)$, set $c_x := \sum_v c_v \lambda_{vx}$, replace $\lambda_{ux} \to \alpha_{ux} := c_u \lambda_{ux} \frac{e^{c_x} - 1}{c_x}$ and $\omega \to e^\omega$, use $t_u = e^{c_u}$, and write α_{ux} as a matrix. Get **"β calculus"**.

β Calculus. Let $\beta(T; H)$ be

$$\left\{ \begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline u & \alpha_{ux} & \alpha_{uy} & \cdot \\ v & \alpha_{vx} & \alpha_{vy} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \;\middle|\; \begin{array}{l} \omega \text{ and the } \alpha_{ux}\text{'s are} \\ \text{rational functions in} \\ \text{variables } t_u, \text{ one for} \\ \text{each } u \in T. \end{array} \right\},$$

With Selmani, ωεβ/meta

$$tm^{uv}_w : \begin{array}{c|c} \omega & \cdots \\ \hline u & \alpha \\ v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline w & \alpha + \beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} * \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & \alpha_1 & 0 \\ T_2 & 0 & \alpha_2 \end{array},$$

$$hm^{xy}_z : \begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & z & \cdots \\ \hline \vdots & \alpha + \beta + \langle\alpha\rangle\beta & \gamma \end{array},$$

$$tha^{ux} : \begin{array}{c|cc} \omega & x & \cdots \\ \hline u & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & x & \cdots \\ \hline u & \alpha(1 + \langle\gamma\rangle/\epsilon) & \beta(1 + \langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta - \gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1 + \alpha$, $\langle\alpha\rangle := \sum_v \alpha_v$, and $\langle\gamma\rangle := \sum_{v \neq u} \gamma_v$, and let

$$R^+_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u - 1 \end{array} \qquad R^-_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u^{-1} - 1 \end{array}.$$

On long knots, ω is the Alexander polynomial!

Why happy? An ultimate Alexander invariant: Manifestly polynomial (time and size) extension of the (multivariable) Alexander polynomial to tangles. Every step of the computation is the computation of the invariant of some topological thing (no fishy Gaussian elimination). *If there should be an Alexander invariant with a computable algebraic categorification, it is this one!*

See also ωεβ/regina, ωεβ/caen, ωεβ/newton.

May class: ωεβ/aarhus **Class next year:** ωεβ/1350
Paper: ωεβ/kbh

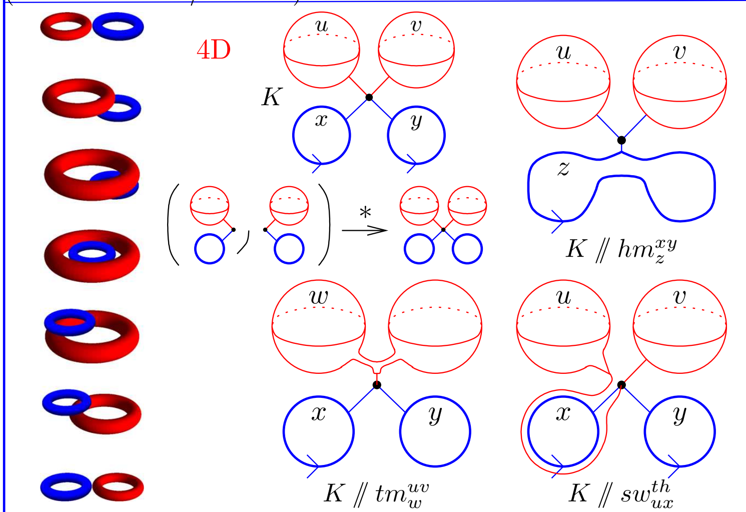
Meta−Groups, Meta−Bicrossed−Products, and the Alexander Polynomial, 1

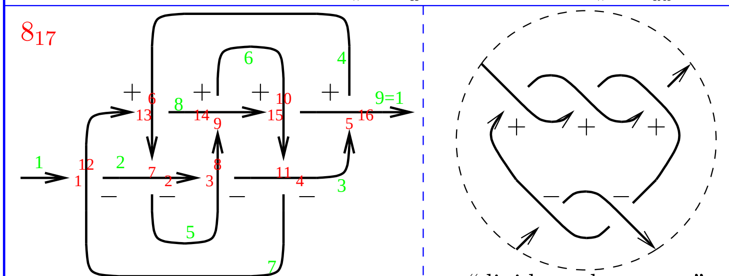
Dror Bar−Natan at Sheffield, February 2013.

http://www.math.toronto.edu/~drorbn/Talks/Sheffield-130206/

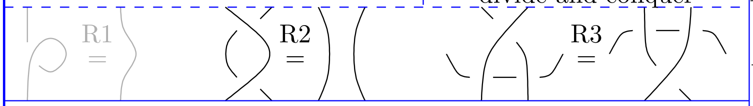
Abstract. I will define "meta-groups" and explain how one specific meta-group, which in itself is a "meta-bicrossed-product", gives rise to an "ultimate Alexander invariant" of tangles, that contains the Alexander polynomial (multivariable, if you wish), has extremely good composition properties, is evaluated in a topologically meaningful way, and is least-wasteful in a computational sense. If you believe in categorification, that's a wonderful playground.

This work is closely related to work by Le Dimet (Comment. Math. Helv. **67** (1992) 306–315), Kirk, Livingston and Wang (arXiv:math/9806035) and Cimasoni and Turaev (arXiv:math.GT/0406269).



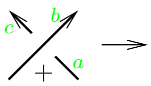


"divide and conquer"

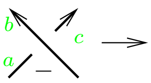


A Standard Alexander Formula.

Label the arcs 1 through $(n+1)=1$, make an $n\times n$ matrix as below, delete one row and one column, and compute the determinant:



	a	b	c
c	-1	$1-X$	X



	a	b	c
c	$-X$	$X-1$	1

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & X-1 & 0 & -X \\ -1 & X & 0 & 0 & 0 & 0 & 1-X & 0 \\ 0 & -1 & X & 0 & 1-X & 0 & 0 & 0 \\ X-1 & 0 & -X & 1 & 0 & 0 & 0 & 0 \\ 0 & 1-X & 0 & -1 & X & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -X & 1 & 0 & X-1 \\ 0 & 0 & 1-X & 0 & 0 & -1 & X & 0 \\ 0 & 0 & 0 & X-1 & 0 & 0 & -X & 1 \end{pmatrix}$$

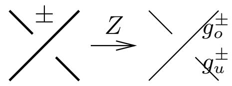
```
[[1 ;; 7, 1 ;; 7]] // Det
```

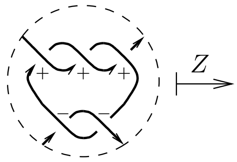
$-1+4X-8X^2+11X^3-8X^4+4X^5-X^6$

Alexander Issues.

- Quick to compute, but computation departs from topology.
- Extends to tangles, but at an exponential cost.
- Hard to categorify.

Idea. Given a group G and two "YB" pairs $R^\pm=(g_o^\pm,g_u^\pm)\in G^2$, map them to xings and "multiply along", so that

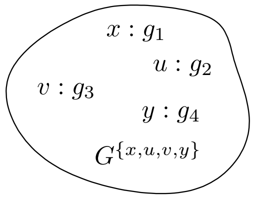




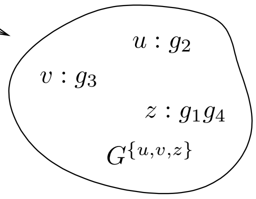
$$\xrightarrow{Z}\begin{pmatrix} g_o^+g_u^+g_o^+g_u^-g_o^-g_u^+g_o^+g_u^+ \\ g_u^-g_o^- \end{pmatrix}$$

This Fails! R2 implies that $g_o^\pm g_o^\mp=e=g_u^\pm g_u^\mp$ and then R3 implies that g_o^+ and g_u^+ commute, so the result is a simple counting invariant.

A Group Computer. Given G, can store group elements and perform operations on them:



$\xrightarrow{m^{xy}_z}$ ...so that $m^{xy}_u /\!\!/ m^{uz}_v = m^{yz}_u /\!\!/ m^{xu}_v$ (or $m^{uz}_v\circ m^{xy}_u = m^{xu}_v\circ m^{yz}_u$, in old-speak).



Also has S_x for inversion, e_x for unit insertion, d_x for register deletion, Δ^z_{xy} for element cloning, ρ^x_y for renamings, and $(D_1,D_2)\mapsto D_1\cup D_2$ for merging, and many obvious composition axioms relating those. $P=\{x:g_1,y:g_2\}\Rightarrow P=\{d_yP\}\cup\{d_xP\}$

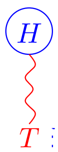
A Meta-Group. Is a similar "computer", only its internal structure is unknown to us. Namely it is a collection of sets $\{G_\gamma\}$ indexed by all finite sets γ, and a collection of operations m^{xy}_z, S_x, e_x, d_x, Δ^z_{xy} (sometimes), ρ^x_y, and $\cup$, satisfying the exact same *linear* properties.

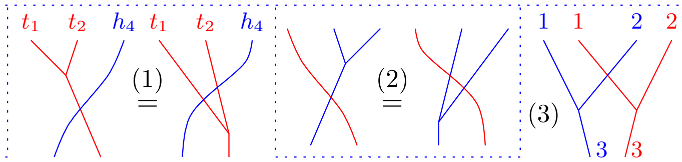
Example 0. The non-meta example, $G_\gamma:=G^\gamma$.

Example 1. $G_\gamma:=M_{\gamma\times\gamma}(\mathbb{Z})$, with simultaneous row and column operations, and "block diagonal" merges. Here if $P=\begin{pmatrix} x: & a & b \\ y: & c & d\end{pmatrix}$ then $d_yP=(x:a)$ and $d_xP=(y:d)$ so $\{d_yP\}\cup\{d_xP\}=\begin{pmatrix} x: & a & 0 \\ y: & 0 & d\end{pmatrix}\neq P$. So this G is truly meta.

Claim. From a meta-group G and YB elements $R^\pm\in G_2$ we can construct a knot/tangle invariant.

Bicrossed Products. If $G=HT$ is a group presented as a product of two of its subgroups, with $H\cap T=\{e\}$, then also $G=TH$ and G is determined by H, T, and the "swap" map $sw^{th}:(t,h)\mapsto(h',t')$ defined by $th=h't'$. The map sw satisfies (1) and (2) below; conversely, if $sw:T\times H\to H\times T$ satisfies (1) and (2) (+ lesser conditions), then (3) defines a group structure on $H\times T$, the "bicrossed product".





$tm^{12}_1 /\!\!/ sw_{14} = sw_{24} /\!\!/ sw_{14} /\!\!/ tm^{12}_1$

$gm^{12}_3 := sw_{12} /\!\!/ tm^{12}_3 /\!\!/ hm^{12}_3$

$h_1t_1h_2t_2=h_1(h_2't_1')t_2=(h_1h_2')(t_1't_2)=h_3t_3$

Meta−Groups, Meta−Bicrossed−Products, and the Alexander Polynomial, 2

A Meta-Bicrossed-Product is a collection of sets $\beta(\eta,\tau)$ and operations tm^{uv}_w, hm^{xy}_z and sw^{th}_{ux} (and lesser ones), such that tm and hm are "associative" and (1) and (2) hold (+ lesser conditions). A meta-bicrossed-product defines a meta-group with $G_\gamma := \beta(\gamma,\gamma)$ and gm as in (3).

Example. Take $\beta(\eta,\tau) = M_{\tau\times\eta}(\mathbb{Z})$ with row operations for the tails, column operations for the heads, and a trivial swap.

β Calculus. Let $\beta(\eta,\tau)$ be

$$\left\{\begin{array}{c|cccc} \omega & h_1 & h_2 & \cdots \\ \hline t_1 & \alpha_{11} & \alpha_{12} & \cdot \\ t_2 & \alpha_{21} & \alpha_{22} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \;\middle|\; \begin{array}{l} h_j\in\eta,\ t_i\in\tau, \text{ and } \omega \text{ and} \\ \text{the } \alpha_{ij} \text{ are rational func-} \\ \text{tions in a variable } X \end{array}\right\},$$

$$tm^{uv}_w : \begin{array}{c|c} \omega & \cdots \\ \hline t_u & \alpha \\ t_v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline t_w & \alpha+\beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & \eta_1 \\ \hline \tau_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_2 & \eta_2 \\ \hline \tau_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & \eta_1 & \eta_2 \\ \hline \tau_1 & \alpha_1 & 0 \\ \tau_2 & 0 & \alpha_2 \end{array},$$

$$hm^{xy}_z : \begin{array}{c|cccc} \omega & h_x & h_y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & h_z & \cdots \\ \hline \vdots & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array},$$

$$sw^{th}_{ux} : \begin{array}{c|cc} \omega & h_x & \cdots \\ \hline t_u & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & h_x & \cdots \\ \hline t_u & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta-\gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1+\alpha$ and $\langle c\rangle := \sum_i c_i$, and let

$$R^p_{ab} := \begin{array}{c|cc} 1 & h_a & h_b \\ \hline t_a & 0 & X-1 \\ t_b & 0 & 0 \end{array} \qquad R^m_{ab} := \begin{array}{c|cc} 1 & h_a & h_b \\ \hline t_a & 0 & X^{-1}-1 \\ t_b & 0 & 0 \end{array}.$$

Theorem. Z^β is a tangle invariant (and more). Restricted to knots, the ω part is the Alexander polynomial. On braids, it is equivalent to the Burau representation. A variant for links contains the multivariable Alexander polynomial.

Why Happy? • Applications to w-knots.

• Everything that I know about the Alexander polynomial can be expressed cleanly in this language (even if without proof), except HF, but including genus, ribbonness, cabling, v-knots, knotted graphs, etc., and there's potential for vast generalizations.

• The least wasteful "Alexander for tangles" I'm aware of.

• Every step along the computation is the invariant of something.

• Fits on one sheet, including implementation & propaganda.

Further meta-monoids. Π (and variants), $\mathcal{A}$ (and quotients), vT, ...

Further meta-bicrossed-products. Π (and variants), $\overrightarrow{\mathcal{A}}$ (and quotients), M_0, M, $\mathcal{K}^{bh}$, $\mathcal{K}^{rbh}$, ...

Meta-Lie-algebras. $\mathcal{A}$ (and quotients), $\mathcal{S}$, ...

Meta-Lie-bialgebras. $\overrightarrow{\mathcal{A}}$ (and quotients), ...

I don't understand the relationship between gr and H, as it appears, for example, in braid theory.

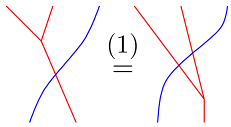
I mean business!

```
βSimp = Factor; SetAttributes[βCollect, Listable];
βCollect[B[ω_, Λ_]] := B[βSimp[ω],
   Collect[Λ, h_, Collect[#, t_, βSimp] &]];
βForm[B[ω_, Λ_]] := Module[{ts, hs, M},
  ts = Union[Cases[B[ω, Λ], t_u_ :> u, Infinity]];
  hs = Union[Cases[B[ω, Λ], h_x_ :> x, Infinity]];
  M = Outer[βSimp[Coefficient[Λ, h_#1 t_#2]] &, hs, ts];
  PrependTo[M, t_# & /@ ts];
  M = Prepend[Transpose[M], Prepend[h_# & /@ hs, ω]];
  MatrixForm[M]];
βForm[else_] := else /. β_B :> βForm[β];
Format[β_B, StandardForm] := βForm[β];
```

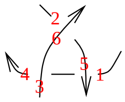
```
⟨μ_⟩ := μ /. t_ → 1;
tm_{u_,v_→w_}[β_] := βCollect[β /. t_{u|v} → t_w];
hm_{x_,y_→z_}[B[ω_, Λ_]] := Module[
   {α = D[Λ, h_x], β = D[Λ, h_y], γ = Λ /. h_{x|y} → 0},
   B[ω, (α + (1 + ⟨α⟩) β) h_z + γ] // βCollect];
sw_{u_,x_}[B[ω_, Λ_]] := Module[{α, β, γ, δ, ϵ},
   α = Coefficient[Λ, h_x t_u]; β = D[Λ, t_u] /. h_x → 0;
   γ = D[Λ, h_x] /. t_u → 0;   δ = Λ /. h_x | t_u → 0;
   ϵ = 1 + α;
   B[ω * ϵ, α (1 + ⟨γ⟩ / ϵ) h_x t_u + β (1 + ⟨γ⟩ / ϵ) t_u
      + γ / ϵ h_x              + δ - γ * β / ϵ
   ] // βCollect];
gm_{a_,b_→c_}[β_] := β // sw_ab // hm_{ab→c} // tm_{ab→c};
B /: B[ω1_, Λ1_] B[ω2_, Λ2_] := B[ω1 * ω2, Λ1 + Λ2];
Rp_{a_,b_} := B[1, (X - 1) t_a h_b];
Rm_{a_,b_} := B[1, (X^-1 - 1) t_a h_b];
```

```
{β = B[ω, Sum[α_{10 i+j} t_i h_j, {i, {1, 2, 3}}, {j, {4, 5}}]],
  (β // tm_{12→1} // sw_14) == (β // sw_24 // sw_14 // tm_{12→1})}
```

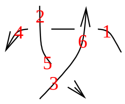
$$\left\{\begin{pmatrix} \omega & h_4 & h_5 \\ t_1 & \alpha_{14} & \alpha_{15} \\ t_2 & \alpha_{24} & \alpha_{25} \\ t_3 & \alpha_{34} & \alpha_{35} \end{pmatrix}, \text{True}\right\}$$



Some testing




```
{Rm_51 Rm_62 Rp_34 // gm_{14→1} // gm_{25→2} // gm_{36→3},
  Rp_61 Rm_24 Rm_35 // gm_{14→1} // gm_{25→2} // gm_{36→3}}
```



$$\left\{\begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+X}{X} & 0 \\ t_3 & \frac{-1+X}{X} & -\frac{-1+X}{X} \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+X}{X} & 0 \\ t_3 & \frac{-1+X}{X} & -\frac{-1+X}{X} \end{pmatrix}\right\}$$

... divide and conquer!

```
β = Rm_{12,1} Rm_27 Rm_83 Rm_{4,11} Rp_{16,5} Rp_{6,13} Rp_{14,9} Rp_{10,15}
```

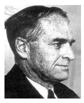
8_{17}

$$\begin{pmatrix} 1 & h_1 & h_3 & h_5 & h_7 & h_9 & h_{11} & h_{13} & h_{15} \\ t_2 & 0 & 0 & 0 & -\frac{-1+X}{X} & 0 & 0 & 0 & 0 \\ t_4 & 0 & 0 & 0 & 0 & 0 & -\frac{-1+X}{X} & 0 & 0 \\ t_6 & 0 & 0 & 0 & 0 & 0 & 0 & -1+X & 0 \\ t_8 & 0 & -\frac{-1+X}{X} & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+X \\ t_{12} & -\frac{-1+X}{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{14} & 0 & 0 & 0 & 0 & -1+X & 0 & 0 & 0 \\ t_{16} & 0 & 0 & -1+X & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
Do[β = β // gm_{1k→1}, {k, 2, 10}]; β
```

8_{17}, cont.

$$\begin{pmatrix} \frac{1}{X} & h_1 & h_{11} & h_{13} & h_{15} \\ t_1 & -\frac{(-1+X)(1+X)}{X} & -(-1+X)(1-X+X^2) & (-1+X)(1-X+X^2) & -1+X \\ t_{12} & -\frac{-1+X}{X} & 0 & 0 & 0 \\ t_{14} & -1+X & \frac{(-1+X)^2(1-X+X^2)}{X} & -\frac{(-1+X)^2(1-X+X^2)}{X} & 0 \\ t_{16} & \frac{-1+X}{X} & (-1+X)^2 & -\frac{(-1+X)^3}{X} & 0 \end{pmatrix}$$



James Waddell Alexander

```
Do[β = β // gm_{1k→1}, {k, 11, 16}]; β
```

$$\left(-\frac{1-4X+8X^2-11X^3+8X^4-4X^5+X^6}{X^3}\right)$$

A Partial To Do List. 1. Where does it *more simply* come from?

2. Remove all the denominators.

3. How do determinants arise in this context?

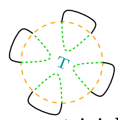
4. Understand links ("meta-conjugacy classes").

5. Find the "reality condition".

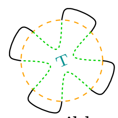
6. Do some "Algebraic Knot Theory".

7. Categorify.

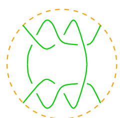
8. Do the same in other natural quotients of the v/w-story.



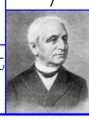
trivial



ribbon



example



"God created the knots, all else in topology is the work of mortals."

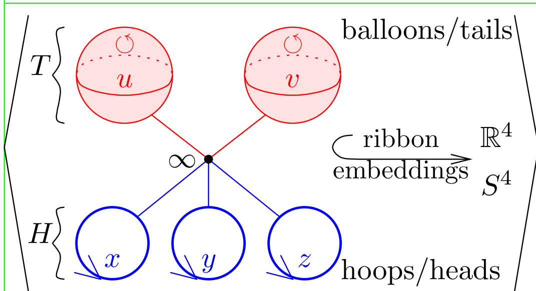
Leopold Kronecker (modified)

www.katlas.org

The Knot Atlas

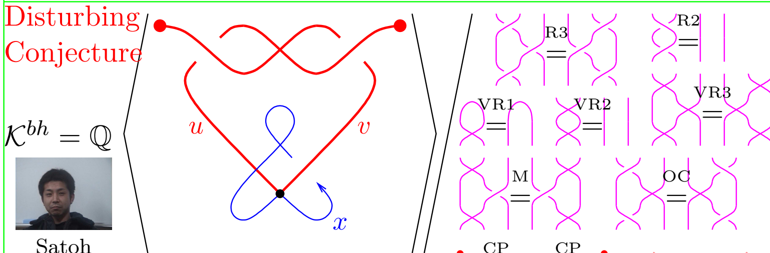
Finite Type Invariants of Ribbon Knotted Balloons and Hoops

Abstract. On my September 17 Geneva talk (ω/sep) I described a certain trees-and-wheels-valued invariant ζ of ribbon knotted loops and 2-spheres in 4-space, and my October 8 Geneva talk (ω/oct) describes its reduction to the Alexander polynomial. Today I will explain how that same invariant arises completely naturally within the theory of finite type invariants of ribbon knotted loops and 2-spheres in 4-space.

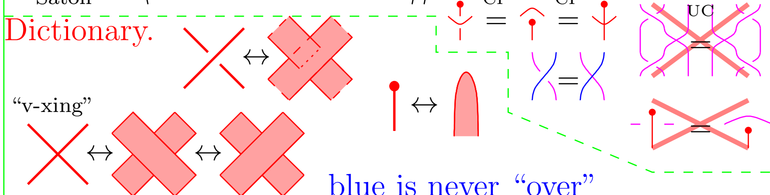


My goal is to tell you why such an invariant is expected, yet not to derive the computable formulas.

Disturbing Conjecture



Dictionary.



Expansions the semi-virtual

Let $\mathcal{I}^n := \langle$pictures with $\geq n$ semi-virts$\rangle \subset \mathcal{K}^{bh}$.
We seek an "expansion"

$$Z\colon \mathcal{K}^{bh} \to \operatorname{gr}\mathcal{K}^{bh} = \widehat{\bigoplus} \mathcal{I}^n/\mathcal{I}^{n+1} =: \mathcal{A}^{bh}$$

satisfying "property U": if $\gamma \in \mathcal{I}^n$, then

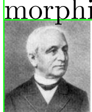
$$Z(\gamma) = (0,\dots,0,\gamma/\mathcal{I}^{n+1},*,*,\dots).$$


X.-S. Lin

Why? • Just because, and this is vastly more general.
• $\left(\mathcal{K}^{bh}/\mathcal{I}^{n+1}\right)^\star$ is "finite-type/polynomial invariants".
• The Taylor example: Take $\mathcal{K} = C^\infty(\mathbb{R}^n)$, $\mathcal{I} = \{f \in \mathcal{K}\colon f(0)=0\}$. Then $\mathcal{I}^n = \{f\colon f$ vanishes like $|x|^n\}$ so $\mathcal{I}^n/\mathcal{I}^{n+1}$ is homogeneous polynomials of degree n and Z is a "Taylor expansion"! (So Taylor expansions are vastly more general than you'd think).

Plan. We'll construct a graded $\tilde{\mathcal{A}}^{bh}$, a surjective graded $\pi\colon \tilde{\mathcal{A}}^{bh} \to \mathcal{A}^{bh}$, and a filtered $\tilde{Z}\colon \mathcal{K}^{bh} \to \mathcal{A}^{bh}$ so that $\pi \mathbin{/\!\!/} \operatorname{gr}\tilde{Z} = Id$ (property U: if $\deg D = n$, $\tilde{Z}(\pi(D)) = \pi(D) + (\deg \geq n)$). Hence • π is an isomorphism. • $Z := \tilde{Z} \mathbin{/\!\!/} \pi$ is an expansion.

$$\begin{tikzcd} & \tilde{\mathcal{A}}^{bh} \arrow[d, shift left, "\operatorname{gr}\tilde{Z}"] \\ \mathcal{K}^{bh} \arrow[ur, "\tilde{Z}"] \arrow[r, "Z"'] & \mathcal{A}^{bh} \arrow[u, shift left, "\pi"] \end{tikzcd}$$



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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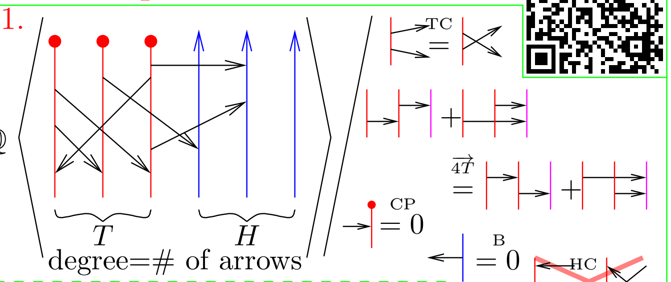
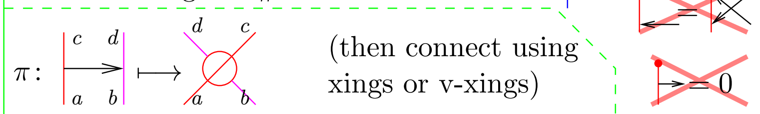
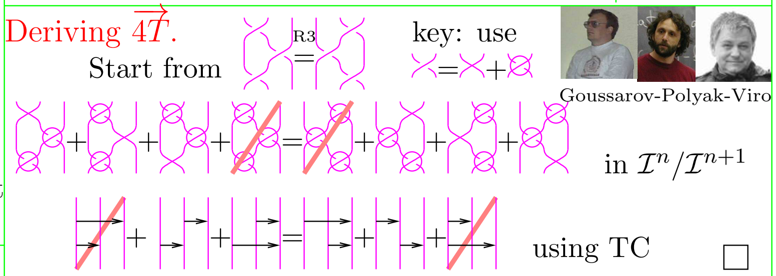
Action 1.

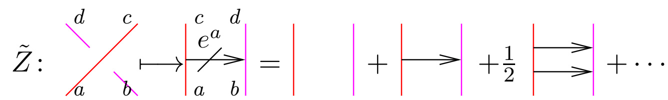




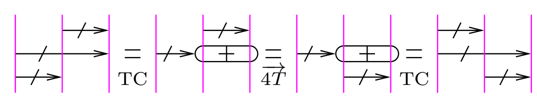
Deriving $\overrightarrow{4T}$. Start from



Action 2.

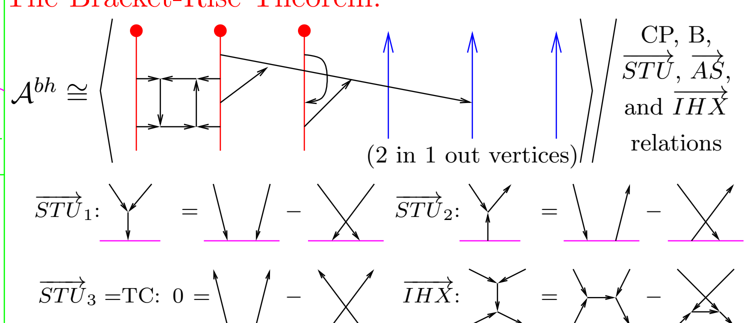


R3.

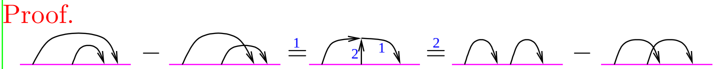


Exercise. Prove property U.

The Bracket-Rise Theorem.



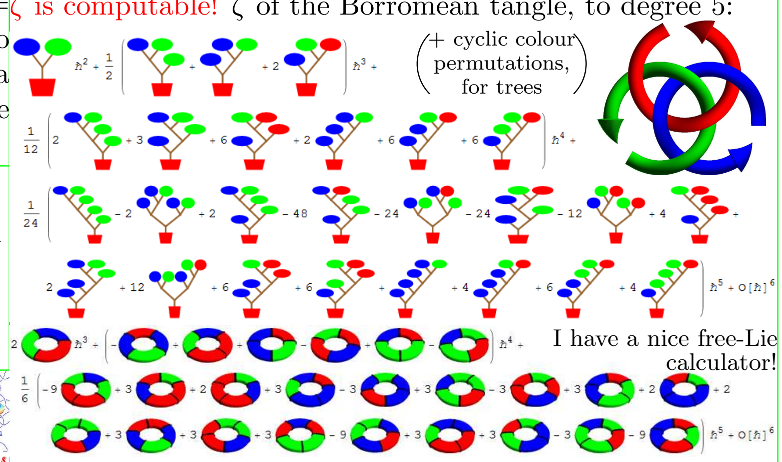
Proof.



Corollaries. (1) Related to Lie algebras! (2) Only trees and wheels persist.

Theorem. $\mathcal{A}^{bh}$ is a bi-algebra. The space of its primitives is $FL(T)^H \times CW(T)$, and $\zeta = \log Z$.

ζ is computable! ζ of the Borromean tangle, to degree 5:

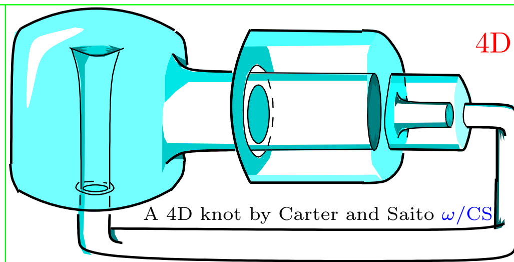


The Kashiwara-Vergne Problem and Topology

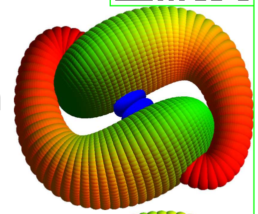
Handout, video, and links at $\omega/$

Abstract. I will describe the general "expansions" machine whose inputs are topics in topology (and more) and whose outputs are problems in algebra. There are many inputs the machine can take, and many outputs it produces, but I will concentrate on just one input/output pair. When fed with a certain class of knotted 2-dimensional objects in 4-dimensional space, it outputs the Kashiwara-Vergne Problem (1978 ω/KV, solved Alekseev-Meinrenken 2006 ω/AM, elucidated Alekseev-Torossian 2008-2012 ω/AT), a problem about convolutions on Lie groups and Lie algebras.

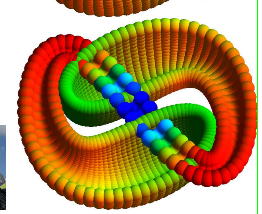
4D Knots.



A 4D knot by Carter and Saito ω/CS



ω/F



Satoh

Dalvit ω/Dal

The Kashiwara-Vergne Conjecture.

There exist two series F and G in the completed free Lie algebra FL in generators x and y so that

$$x+y-\log e^y e^x = (1-e^{-\operatorname{ad} x})F + (e^{\operatorname{ad} y}-1)G \quad \text{in } FL$$

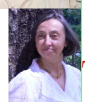
and so that with $z=\log e^x e^y$,

$$\operatorname{tr}(\operatorname{ad} x)\partial_x F + \operatorname{tr}(\operatorname{ad} y)\partial_y G \quad \text{in cyclic words}$$

$$= \frac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1} + \frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1} - \frac{\operatorname{ad} z}{e^{\operatorname{ad} z}-1} - 1\right)$$

Implies the loosely-stated convolutions statement: Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra.

Kashiwara



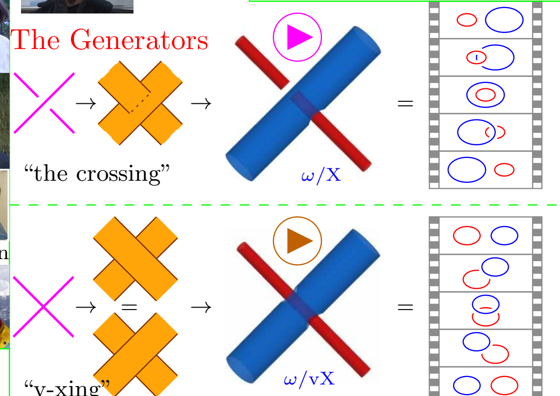
Vergne

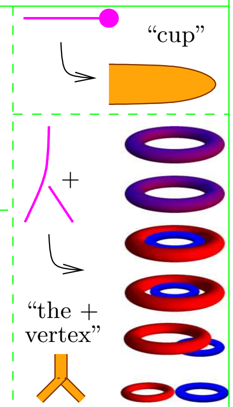
Alekseev

Meinrenken

Torossian

The Generators





The Machine.

Let G be a group, $\mathcal{K}=\mathbb{Q}G=\{\sum a_i g_i\colon a_i\in\mathbb{Q},\ g_i\in G\}$ its group-ring, $\mathcal{I}=\{\sum a_i g_i\colon \sum a_i=0\}\subset\mathcal{K}$ its augmentation ideal. Let

$$\mathcal{A}=\operatorname{gr}\mathcal{K} := \widehat{\bigoplus_{m\geq 0}}\mathcal{I}^m/\mathcal{I}^{m+1}.$$

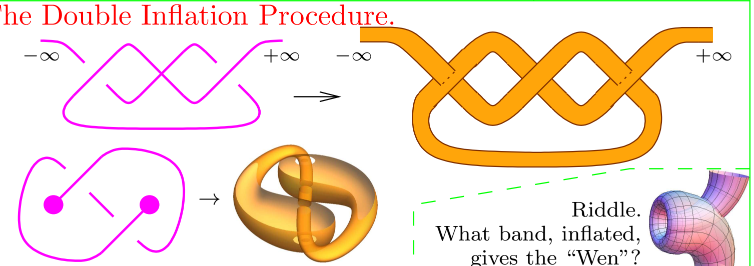
P.S. $(\mathcal{K}/\mathcal{I}^{m+1})^*$ is Vassiliev / finite-type / polynomial invariants.

Note that $\mathcal{A}$ inherits a product from G.

Definition. A linear $Z\colon\mathcal{K}\to\mathcal{A}$ is an "expansion" if for any $\gamma\in\mathcal{I}^m$, $Z(\gamma)=(0,\ldots,0,\gamma/\mathcal{I}^{m+1},*,\ldots)$, and a "homomorphic expansion" if in addition it preserves the product.

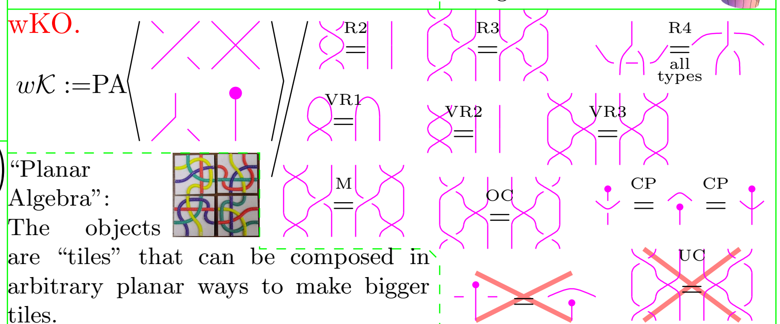
Example. Let $\mathcal{K}=C^\infty(\mathbb{R}^n)$ and $\mathcal{I}=\{f\colon f(0)=0\}$. Then $\mathcal{I}^m=\{f\colon f \text{ vanishes like } |x|^m\}$ so $\mathcal{I}^m/\mathcal{I}^{m+1}$ is degree m homogeneous polynomials and $\mathcal{A}=\{\text{power series}\}$. The Taylor series is a homomorphic expansion!

The Double Inflation Procedure.

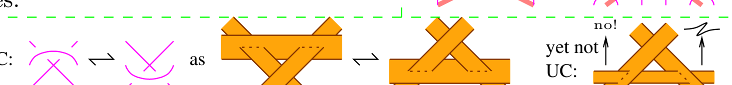


Riddle. What band, inflated, gives the "Wen"?

wKO.



"Planar Algebra": The objects are "tiles" that can be composed in arbitrary planar ways to make bigger tiles.



Just for fun.

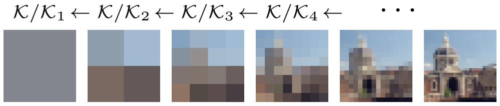
$\mathcal{K}=$

$=\left(\text{The set of all 2D projections of reality } (=\mathbb{Q}^{3\mathbb{R}^2})\right)$

(Wikimedia Commons image, ω/WM)

$\mathcal{K}/\mathcal{K}_1 \leftarrow \mathcal{K}/\mathcal{K}_2 \leftarrow \mathcal{K}/\mathcal{K}_3 \leftarrow \mathcal{K}/\mathcal{K}_4 \leftarrow \quad \cdots$

Rotate
Colour Correct
Adjoin



. . .

An expansion Z is a choice of a "progressive scan" algorithm.

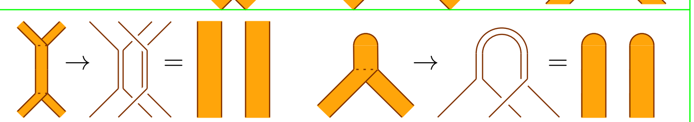
Rotate
Colour Correct
Adjoin

$\mathcal{K}/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \mathcal{K}_4/\mathcal{K}_5 \oplus \mathcal{K}_5/\mathcal{K}_6 \oplus \quad \cdots$

$\mathcal{K}/\mathcal{K}_1 = \mathbb{Q}^3$, $\quad \mathcal{K}_3/\mathcal{K}_4 = \ker(\mathcal{K}/\mathcal{K}_4 \to \mathcal{K}/\mathcal{K}_3)$

In the finitely presented case, finding Z amounts to solving a system of equations in a graded space.

The unary w–operations



Theorem

(with Zsuzsanna Dancso, ω/WKO). There is a bijection between the set of homomorphic expansions for $w\mathcal{K}$ and the set of solutions of the Kashiwara-Vergne problem. This is the tip of a major iceberg!

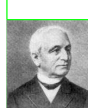
Dancso, ω/ZD

The Machine generalizes to arbitrary algebraic structures!

Why care?

$$\begin{array}{ccc} u\mathcal{K} & \xrightarrow{Z:\ \text{the Kontsevich integral}} & \mathcal{A}^u \\ v\mathcal{K} & \xrightarrow[\text{Etingof-Kazhdan}]{Z:\ \text{hard work}} & \mathcal{A}^v \\ w\mathcal{K} & \xrightarrow[\text{work}]{Z:\ \text{today's}} & \mathcal{A}^w \end{array}$$

$\ker_{\mathfrak{g}}$ and more! (given $\mathfrak{g}$) Around $\operatorname{Rep}(\mathfrak{g})$



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

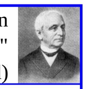
www.katlas.org

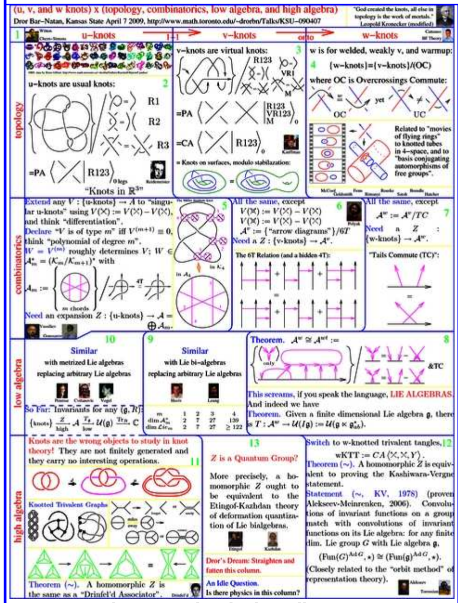
Convolutions on Lie Groups and Lie Algebras and Ribbon 2−Knots

Dror Bar−Natan, Bonn August 2009, http://www.math.toronto.edu/~drorbn/Talks/Bonn−0908

Disclaimer: Rough edges remain!

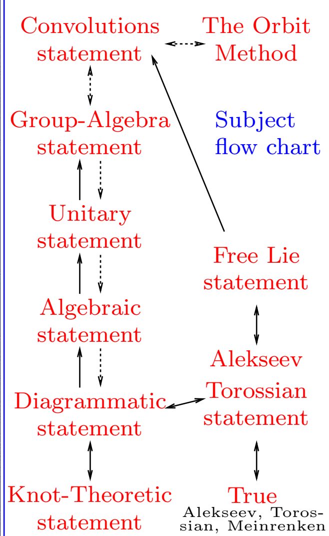
"God created the knots, all else in topology is the work of mortals." Leopold Kronecker (modified)

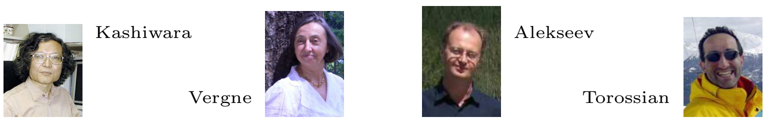




www.math.toronto.edu/~drorbn/Talks/KSU−090407

The Bigger Picture...





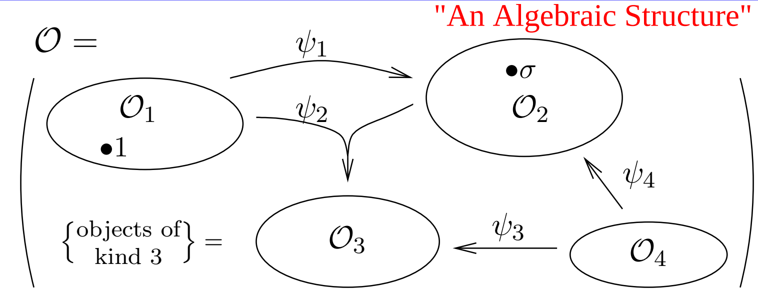
Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

$$\begin{array}{ccccccccccc} \text{ops} \circlearrowleft \mathcal{K} & = & \mathcal{K}_0 & \supset & \mathcal{K}_1 & \supset & \mathcal{K}_2 & \supset & \mathcal{K}_3 & \supset & \dots \\ \Downarrow & & & & & & \downarrow Z & & & & \\ \text{ops} \circlearrowleft \operatorname{gr}\mathcal{K} & := & \mathcal{K}_0/\mathcal{K}_1 & \oplus & \mathcal{K}_1/\mathcal{K}_2 & \oplus & \mathcal{K}_2/\mathcal{K}_3 & \oplus & \mathcal{K}_3/\mathcal{K}_4 & \oplus & \dots \end{array}$$

An expansion is a filtration respecting $Z : \mathcal{K} \to \operatorname{gr}\mathcal{K}$ that "covers" the identity on $\operatorname{gr}\mathcal{K}$. A homomorphic expansion is an expansion that respects all relevant "extra" operations.

Filtered algebraic structures are cheap and plenty. In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle(\mathcal{K}_1)^m\rangle$ (using all available "products").

"An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Example: Pure Braids. PB_n is generated by x_{ij}, "strand i goes around strand j once", modulo "Reidemeister moves". $A_n := \operatorname{gr} PB_n$ is generated by $t_{ij} := x_{ij} - 1$, modulo the $4T$ relations $[t_{ij}, t_{ik} + t_{jk}] = 0$ (and some lesser ones too). Much happens in A_n, including the Drinfel'd theory of associators.

Our case(s).

$$\mathcal{K} \xrightarrow[\text{solving finitely many equations in finitely many unknowns}]{Z\text{: high algebra}} \mathcal{A} := \operatorname{gr}\mathcal{K} \xrightarrow[\text{low algebra: pictures represent formulas}]{\text{given a "Lie" algebra } \mathfrak{g}} \text{"}\mathcal{U}(\mathfrak{g})\text{"}$$

$\mathcal{K}$ is knot theory or topology; $\operatorname{gr}\mathcal{K}$ is finite combinatorics: bounded-complexity diagrams modulo simple relations.

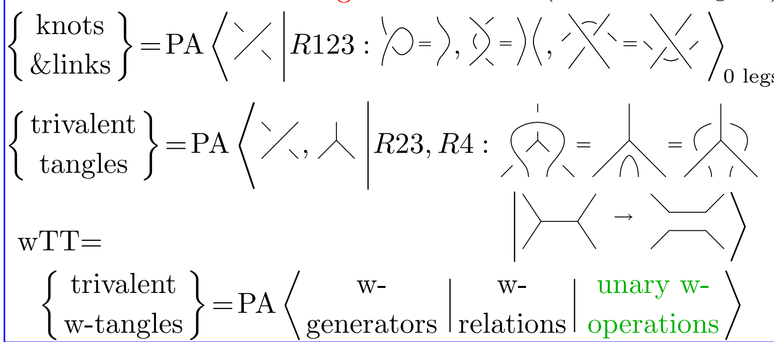
[1] http://qlink.queensu.ca/~4lb11/interesting.html

29/5/10, 8:42am

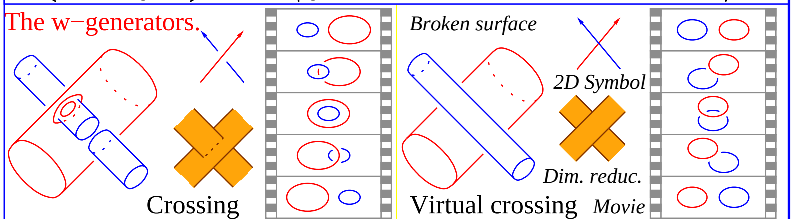
Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

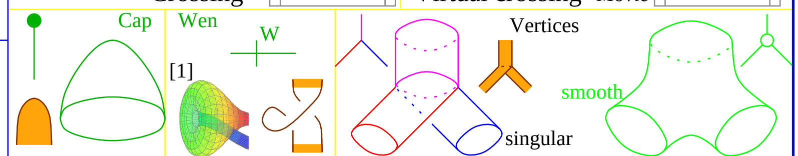
What are w-Trivalent Tangles?

(PA :=Planar Algebra)

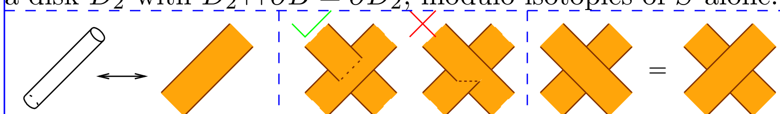


The w−generators.

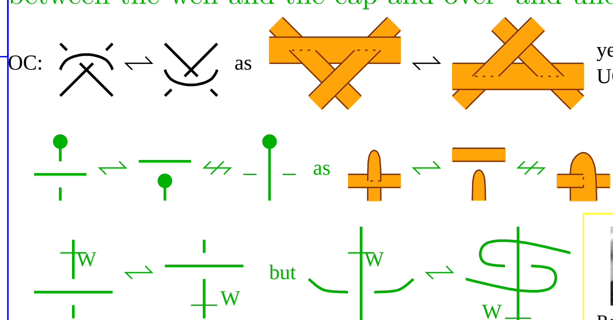




A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.

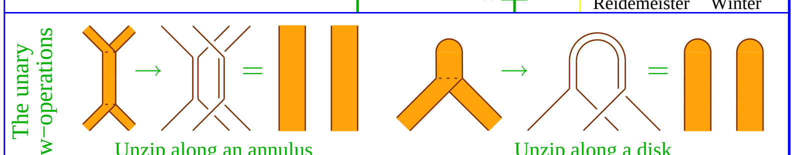


The w-relations include R234, VR1234, M, Overcrossings Commute (OC) but not UC, $W^2 = 1$, and funny interactions between the wen and the cap and over- and under-crossings:

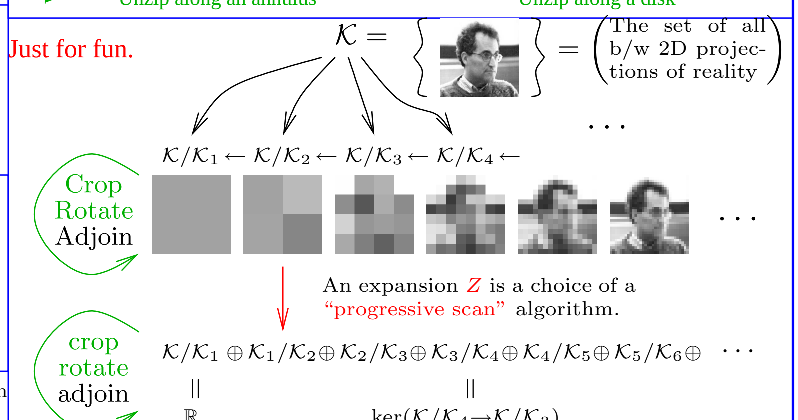


Challenge.

Do the Reidemeister!

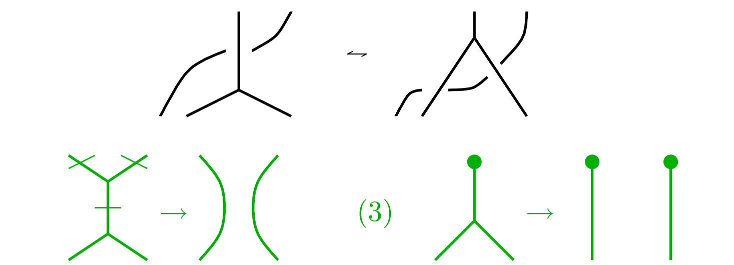


Just for fun.



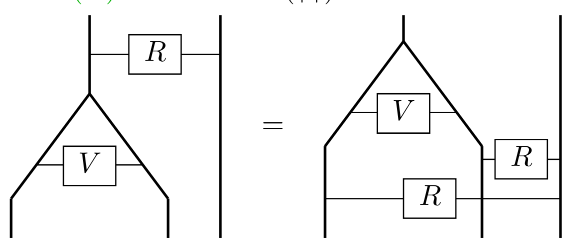
Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:

(1) (2) (3)

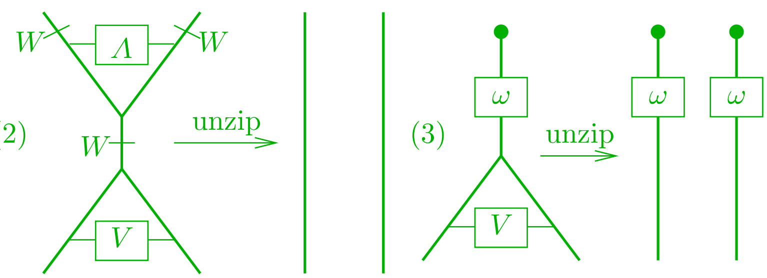


Diagrammatic statement. Let $R = \exp \mapsto \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that

(1)



(2) (3)



Algebraic statement. With $I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g}$, with $c : \hat{\mathcal{U}}(I\mathfrak{g}) \to \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{S}}(\mathfrak{g}^*)$ the obvious projection, with S the antipode of $\hat{\mathcal{U}}(I\mathfrak{g})$, with W the automorphism of $\hat{\mathcal{U}}(I\mathfrak{g})$ induced by flipping the sign of $\mathfrak{g}^*$, with $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{\mathcal{U}}(I\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$ there exist $\omega \in \hat{\mathcal{S}}(\mathfrak{g}^*)$ and $V \in \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that

(1) $V(\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2} \otimes \hat{\mathcal{U}}(\mathfrak{g})$

(2) $V \cdot SWV = 1$ (3) $(c \otimes c)(V\Delta(\omega)) = \omega \otimes \omega$

Unitary statement. There exists $\omega \in \mathrm{Fun}(\mathfrak{g})^G$ and an (infinite order) tangential differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that

(1) $V\widehat{e^{x+y}} = \widehat{e^x}\widehat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)

(2) $VV^* = I$ (3) $V\omega_{x+y} = \omega_x\omega_y$

Group-Algebra statement. There exists $\omega^2 \in \mathrm{Fun}(\mathfrak{g})^G$ so that for every $\phi, \psi \in \mathrm{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $\omega^2 = j^{1/2}$)

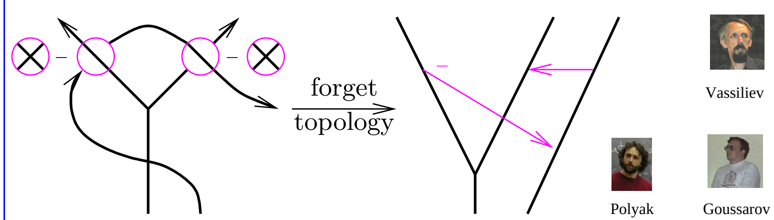
$$\iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2_{x+y}e^{x+y} = \iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2_x\omega^2_y e^x e^y.$$

(shhh, this is Duflo)

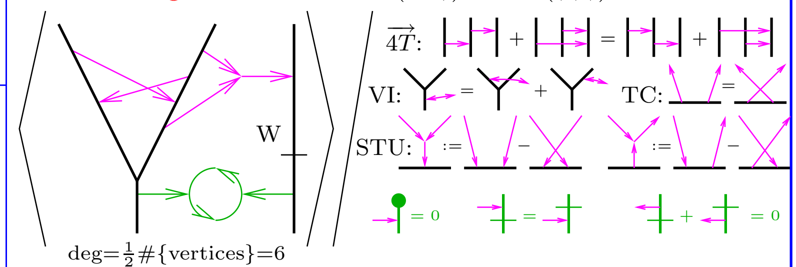
Convolutions statement (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, let $j : \mathfrak{g} \to \mathbb{R}$ be the Jacobian of the exponential map $\exp : \mathfrak{g} \to G$, and let $\Phi : \mathrm{Fun}(G) \to \mathrm{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := j^{1/2}(x)f(\exp x)$. Then if $f, g \in \mathrm{Fun}(G)$ are Ad-invariant and supported near the identity, then

$$\Phi(f) \star \Phi(g) = \Phi(f \star g).$$

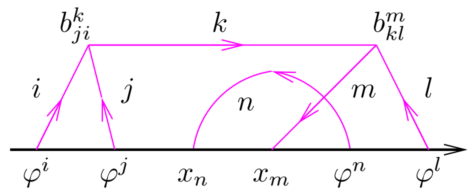
From wTT to $\mathcal{A}^w$. $\mathrm{gr}_m\,\mathrm{wTT} := \{m\text{−cubes}\}/\{(m+1)\text{−cubes}\}$:



w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is



Diagrammatic to Algebraic. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i, x_j] = \sum b^k_{ij}x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via



$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}} b^k_{ij}b^m_{kl}\varphi^i\varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g})$$

Unitary $\Longleftrightarrow$ Algebraic. The key is to interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$:

- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\mathrm{ad}\,x$: $(x\varphi)(y) := \varphi([x,y])$.
- $c : \hat{\mathcal{U}}(I\mathfrak{g}) \to \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{S}}(\mathfrak{g}^*)$ is "the constant term".

Unitary $\Longrightarrow$ Group-Algebra. $\iint \omega^2_{x+y}e^{x+y}\phi(x)\psi(y)$

$$= \langle \omega_{x+y}, \omega_{x+y}e^{x+y}\phi(x)\psi(y)\rangle = \langle V\omega_{x+y}, Ve^{x+y}\phi(x)\psi(y)\omega_{x+y}\rangle$$
$$= \langle \omega_x\omega_y, e^xe^yV\phi(x)\psi(y)\omega_{x+y}\rangle = \langle \omega_x\omega_y, e^xe^y\phi(x)\psi(y)\omega_x\omega_y\rangle$$
$$= \iint \omega^2_x\omega^2_y e^xe^y\phi(x)\psi(y).$$

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G \to A$ is multiplicative then $(\mathrm{Fun}(G), \star) \cong (A, \cdot)$ via $L : f \mapsto \sum f(a)\tau(a)$. For Lie $(G, \mathfrak{g})$,

$$\begin{tikzcd}
(\mathfrak{g},+) \ni x \arrow[r, "\tau_0=\exp_{\mathcal{S}}"] \arrow[d, "\exp_G"] \arrow[dr, "\exp_{\mathcal{U}}"] & e^x \in \hat{\mathcal{S}}(\mathfrak{g}) \arrow[d, "\chi"] \\
(G,\cdot) \ni e^x \arrow[r, "\tau_1"] & e^x \in \hat{\mathcal{U}}(\mathfrak{g})
\end{tikzcd}
\quad \text{so} \quad
\begin{tikzcd}
\mathrm{Fun}(\mathfrak{g}) \arrow[r, "L_0"] \arrow[d, "\Phi^{-1}"] & \hat{\mathcal{S}}(\mathfrak{g}) \arrow[d, "\chi"] \\
\mathrm{Fun}(G) \arrow[r, "L_1"] & \hat{\mathcal{U}}(\mathfrak{g})
\end{tikzcd}$$

with $L_0\psi = \int \psi(x)e^x dx \in \hat{\mathcal{S}}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi = \int \psi(x)e^x \in \hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i \in \mathrm{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1 \star \psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $L_{0/1}$ are "Laplace transforms")

$\star$ in G: $\iint \psi_1(x)\psi_2(y)e^xe^y$ $\star$ in $\mathfrak{g}$: $\iint \psi_1(x)\psi_2(y)e^{x+y}$

We skipped...
- The Alexander polynomial and Milnor numbers.
- u-Knots, Alekseev-Torossian, and Drinfel'd associators.
- v-Knots, quantum groups and Etingof-Kazhdan.
- BF theory and the successful religion of path integrals.
- The simplest problem hyperbolic geometry solves.