An explicit formula for quantization of Lie bialgebras

Notes of the talk at the Les Daibrelets workshop given on 27.08.2015 by **Sergei Merkulov**.
The talk is based on a joint work (in progress) with **Thomas Willwacher**

1. Introduction

We show two new explicit formulae — one for deformation quantizations of Poisson structures and one for quantization of Lie bialgebras. Both extend to associated formality maps.

The main idea in both formulae is to consider an intermediate object — a *quantizable Poisson structure* and, respectively, a *quantizable Lie bialgebra* — so that the quantization process splits in two steps as follows

$$\boxed{\begin{array}{c}\text{Poisson}\\ \text{structures in } \mathbb{R}^d\end{array}} \xrightarrow[\textit{associators}]{\textit{depends on}} \boxed{\begin{array}{c}\text{Quantizable}\\ \text{Poisson}\\ \text{structures in } \mathbb{R}^d\end{array}} \xrightarrow[\textit{for associators}]{\textit{easy: no need}} \boxed{\begin{array}{c}\text{Star products}\\ *_\hbar \text{ in } C^\infty(\mathbb{R}^d)[[\hbar]]\end{array}}$$

$$\boxed{\begin{array}{c}\text{Lie bialgebra}\\ \text{structures in } \mathbb{R}^d\end{array}} \xrightarrow[\textit{associators}]{\textit{depends on}} \boxed{\begin{array}{c}\text{Quantizable}\\ \text{Lie bialgebra}\\ \text{structures in } \mathbb{R}^d\end{array}} \xrightarrow{\textit{easy}} \boxed{\begin{array}{c}\text{Bialgebra structures}\\ (*_\hbar, \Delta_\hbar) \text{ in } \odot^\bullet(\mathbb{R}^d)[[\hbar]]\end{array}}$$

Here d is any natural number and $\hbar$ a formal parameter. In fact, we have explicit formulae for the associated (to these deformation quantization problems) formality maps.

2. A new explicit formula for quantization of Poisson structures

2.1. **Poisson structures and their deformation quantization.** Let $\mathcal{T}_{poly}(\mathbb{R}^d) = C^\infty(\mathbb{R}^d)[\psi_1, \ldots, \psi_d]$ (ψ_i being a formal variable of homological degree 1 which stands for the partial derivative $\partial/\partial x^i$) be the vector space of smooth (or formal) polyvector fields on $\mathbb{R}^d = \{x^1, \ldots, x^i, \ldots, x^d\}$, equipped with the standard Schouten Lie brackets (of degree -1)

$$[f_1, f_2]_S = \sum_{i=1} \frac{\partial f_1}{\partial \psi_i}\frac{\partial f_2}{\partial x^i} + (-1)^{|f_1||f_2|}\frac{\partial f_1}{\partial x^i}\frac{\partial f_2}{\partial \psi_i}.$$

A *Poisson structure* in $\mathbb{R}^d$ is, by definition, a Maurer-Cartan element of this Lie algebra, that is a bivector field $\pi = \sum_{i,j=1^d} \pi^{ij}(x)\psi_i\psi_j$ satisfying the equation $[\pi, \pi]_S = 0$.

A *start product* in $C^\infty(\mathbb{R}^d)$ is an associative product,

$$\begin{array}{rccc} \sqrt[n]{}*_\hbar : & C^\infty(\mathbb{R}^d) \times C^\infty(\mathbb{R}^d) & \longrightarrow & C^\infty(\mathbb{R}^d) \\ & (f(x), g(x)) & \longrightarrow & f *_\hbar g = fg + \sum_{k\geq 1}^\infty \hbar^k B_k(f,g) \end{array}$$

where all operators B_k are bi-differential. One can check that the associativity condition for $*_\hbar$ implies that $\pi(f,g) := B_1(f,g) - B_1(g,f)$ is a Poisson structure in $\mathbb{R}^d$; then $*_\hbar$ is called a *deformation quantization* of $\pi \in \mathcal{T}_{poly}(\mathbb{R}^d)$.

Deformation quantization problem: given a Poisson structure in $\mathbb{R}^d$, does there exists $*_\hbar$ which is its deformation quantization?

This problem was solved by Maxim Kontsevich [Ko] by giving an explicit direct map between the two sets

$$\boxed{\begin{array}{c}\text{Poisson}\\ \text{structures in } \mathbb{R}^d\end{array}} \xrightarrow[\textit{associators}]{\textit{depends on}} \boxed{\begin{array}{c}\text{Star products}\\ *_\hbar \text{ in } C^\infty(\mathbb{R}^d)[[\hbar]]\end{array}}$$

Later Dmitry Tamarkin proved [Ta2] existence theorem for deformation quantizations which exhibited a non-trivial role of Drinfeld's associators.

We show a new explicit formula for a formality map behind quantizations of Poisson structures by considering an intermediate structure called *quantizable Poisson structure*. To show an explicit formula defining these intermediate structures, and its relation to ordinary Poisson structures, we have to introduce a certain operad of graphs and an associated graph complex which we discuss next (the operadic structure can be ignored at the first reading, what is really important is an association "a graph Γ" $\longrightarrow$ a "polydifferential operator Φ_Γ on $\mathcal{T}_{poly}(\mathbb{R}^d)$").

2.2. **An operad of graphs and polyvector fields.** For any integers $n \geq 1$ and $l \geq 0$ we denote by $\mathsf{G}_{n,l}$ a set of graphs[1], $\{\Gamma\}$, with n vertices and l edges such that (i) the vertices of Γ are labelled by elements of $[n] := \{1, \ldots, n\}$, (ii) the set of edges, $E(\Gamma)$, is totally ordered up to an even permutation, (iii) the edges are oriented, i.e. there is a choice of arrow on any edge. For example, $\overset{1}{\bullet}\!\!\longrightarrow\!\!\overset{2}{\bullet} \in \mathsf{G}_{2,1}$. The group $\mathbb{Z}_2$ acts freely on $\mathsf{G}_{n,l}$ for $l \geq 2$ by changes of the total ordering (which are often called *orientations* of graphs); its orbit is denoted by $\{\Gamma, \Gamma_{opp}\}$. Let $\mathbb{K}\langle \mathsf{G}_{n,l}\rangle$ be the vector space over a field $\mathbb{K}$ spanned by isomorphism classes, $[\Gamma]$, of elements of $\mathsf{G}_{n,l}$ modulo the relation[2] $\Gamma_{opp} = -\Gamma$, and consider a $\mathbb{Z}$-graded $\mathbb{S}_n$-module,

$$\mathcal{G}ra(n) := \bigoplus_{l=0}^{\infty} \mathbb{K}\langle \mathsf{G}_{n,l}\rangle[l].$$

Note that graphs with two or more edges between any fixed pair of vertices do not contribute to $\mathsf{Gra}(n)$ so that we could have assumed right from the beginning that the sets $\mathsf{G}_{n,l}$ do not contain graphs with multiple edges. The $\mathbb{S}$-module, $\mathcal{G}\mathsf{ra} := \{\mathsf{Gra}(n)\}_{n\geq 1}$, is naturally an operad with the operadic compositions given by

$$\begin{array}{rccc} \circ_i: & \mathcal{G}ra(n) \otimes \mathcal{G}ra(m) & \longrightarrow & \mathcal{G}ra(m+n-1) \\ & \Gamma_1 \otimes \Gamma_2 & \longrightarrow & \sum_{\Gamma \in \mathsf{G}^i_{\Gamma_1,\Gamma_2}} (-1)^{\sigma_\Gamma}\Gamma \end{array}$$

where $\mathsf{G}^i_{\Gamma_1,\Gamma_2}$ is the subset of $\mathsf{G}_{n+m-1,\#E(\Gamma_1)+\#E(\Gamma_2)}$ consisting of graphs, Γ, satisfying the condition: the full subgraph of Γ spanned by the vertices labeled by the set $\{i, i+1, \ldots, i+m-1\}$ is isomorphic to Γ_2 and the quotient graph, Γ/Γ_2, obtained by contracting that subgraph to a single vertex, is isomorphic to Γ_1. The sign $(-1)^{\sigma_\Gamma}$ is determined by the equality

$$\bigwedge_{e\in E(\Gamma)} e = (-1)^{\sigma_\Gamma} \bigwedge_{e'\in E(\Gamma_1)} e' \wedge \bigwedge_{e''\in E(\Gamma_2)} e''.$$

where the wedge product is taken with respect The unique element in $\mathsf{G}_{1,0}$ serves as the unit element in the operad Gra. The associated Lie algebra of $\mathbb{S}$-invariants, $((\mathsf{Gra}\{-2\})^{\mathbb{S}}, [\ ,\ ])$ is denoted, following notations of [Wi1], by fGC_2. Its elements can be understood as graphs from $\mathsf{G}_{n,l}$ but with labeling of vertices forgotten, e.g.

$$\Upsilon_2 := \bullet\!\!\longrightarrow\!\!\bullet = \frac{1}{2}\left(\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet} + \overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\right) \in \mathsf{fGC}_2.$$

The cohomological degree of a graph with n vertices and l edges is $2(n-1)-l$. It is easy to check that $\bullet\!\!-\!\!\!-\!\!\bullet$ is a Maurer-Cartan element in the Lie algebra fGC_2. Hence we obtain a dg Lie algebra

$$\left(\mathsf{fGC}_2, [\ ,\ ], \delta_0 := [\bullet\!\!\longrightarrow\!\!\bullet,\ ]\right).$$

One may define a dg Lie subalgebra, GC_2, spanned by connected graphs with at least trivalent vertices and no edges beginning and ending at the same vertex. It is called the *Kontsevich graph complex*. The cohomologies of GC_2 and fGC_2 were partially computed by Thomas Willwacher [Wi1]. The complex fGC_2 can be identified with the deformation complex of morphism of operads $\mathsf{Def}(\mathcal{L}ie\{1\} \to \mathcal{G}ra)$ so that non-trivial MC elements of $(\mathsf{fGC}_2, [\ ,\ ])$ give us non-trivial morphisms of operads $\mathcal{L}ie\{1\} \to \mathcal{G}ra$.

There is a canonical representation of the operad $\mathcal{G}ra$ in the vector space of polyvector fields $\mathcal{T}_{poly}(\mathbb{R}^d) = C^\infty(\mathbb{R}^d)[\psi_1, \ldots, \psi_d]$,

$$\begin{array}{rccc} \rho: & \mathcal{G}ra(n) & \longrightarrow & \mathcal{E}nd_{\mathcal{T}_{poly}(\mathbb{R}^d)}(n) = \mathrm{Hom}(\mathcal{T}_{poly}(\mathbb{R}^d), \mathcal{T}_{poly}(\mathbb{R}^d)) \\ & \Gamma & \longrightarrow & \Phi_\Gamma \end{array} \tag{1}$$

[1] A *graph* Γ is, by definition, a 1-dimensional CW-complex whose 0-cells are called *vertices* and 1-dimensional cells are called *edges*. The set of vertices of Γ is denoted by $V(\Gamma)$ and the set of edges by $E(\Gamma)$.

[2] Abusing notations we identify from now an equivalence class $[\Gamma]$ with any of its representative Γ.

given explicitly as follows:

$$\Phi_\Gamma(f_1,\ldots,f_n) := \mu\left(\prod_{e\in E(\Gamma)} \Delta_e\left(f_1(x_{(1)},\psi_{(1)})\otimes f_2(x_{(2)},\psi_{(2)})\otimes\ldots\otimes f_n(x_{(n)},\psi_{(n)})\right)\right)$$

where, for an edge e connecting vertices labeled by integers i and j,

$$\Delta_e = \sum_{a=1}^{n} \frac{\partial}{\partial x^a_{(i)}}\frac{\partial}{\partial \psi_{a(j)}} + \frac{\partial}{\partial \psi_{a(i)}}\frac{\partial}{\partial x^a_{(j)}}$$

with the subscript (i) or (j) indicating that the derivative operator is to be applied to the i-th of j-th factor in the tensor product. The symbol μ above denotes the multiplication map,

$$\begin{array}{rccc} \pi: & \mathcal{T}_{poly}(\mathbb{R}^n)^{\otimes n} & \longrightarrow & \mathcal{T}_{poly}(\mathbb{R}^n) \\ & f_1\otimes f_2\otimes\ldots\otimes f_n & \longrightarrow & f_1 f_2\cdots f_n. \end{array}$$

The above representation induces a map of Lie algebras

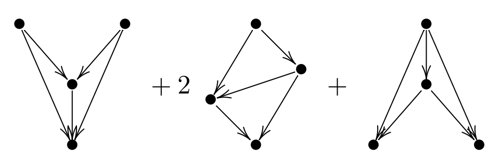
$$\mathsf{fGC}_2 \longrightarrow \mathrm{Coder}(\odot^\bullet(\mathcal{T}_{poly}(\mathbb{R}^d)))$$

so that any MC element in fGC_2 gives us a $\mathcal{L}ie_\infty$ structure in $\mathcal{T}_{poly}(\mathbb{R}^d))$. In particular, the element $\bullet\!\!\longrightarrow\!\!\bullet$ gives us the standard Schouten brackets in $\mathcal{T}_{poly}(\mathbb{R}^d))$, and its infinitesimal universal deformations are controlled by the cohomology group $H^1(\mathsf{fGC}_2,\delta_0)$. It is a folk conjecture that $H^1(\mathsf{fGC}_2,\delta)=0$ (but the implication $\delta_0\gamma=0\Rightarrow\gamma=\delta_0\alpha$ must depend on the choice of an associator).

There is another more exotic $\mathcal{L}ie_\infty$ structure in $\mathcal{T}_{poly}(\mathbb{R}^d))$ which is in a sense unique and which leads us to the notion of *quantizable* Poisson structures.

2.3. **Oriented graph complexes and an exotic $\mathcal{L}ie_\infty$ structure in $\mathcal{T}_{poly}(\mathbb{R}^d))$.** The above representation, $\Gamma\to\Phi_\gamma$, of $\mathcal{G}ra$ in $\mathcal{T}_{poly}(\mathbb{R}^d))$ makes sense only for *finite* values of the parameter d. There is a suboperad $\mathcal{G}ra^\uparrow$ spanned by graphs without wheels, that is, by graphs with no directed paths of edges making a closed path. The associated subcomplex of fGC_2 is denoted by $\mathsf{fGC}_2^\uparrow$. The MC elements of $\mathsf{fGC}_2^\uparrow$ give us those $\mathcal{L}ie_\infty$ structures in $\mathcal{T}_{poly}(\mathbb{R}^d))$ which make sense when $d\to+\infty$. One such structure is given by the graph $\bullet\!\!\longrightarrow\!\!\bullet$ as it belongs to the subcomplex $\mathsf{fGC}_2^\uparrow\subset\mathsf{fGC}_2$, and its infinitesimal deformations are controlled by the group $H^1(\mathsf{fGC}_2^\uparrow,\delta_0)$ which is, thanks to Thomas Willwacher [W2], is completely known: $H^1(\mathsf{fGC}_2^\uparrow,\delta_0)=\mathbb{K}$ and it is generated as 1-dimensional vector space by the following graph

$\Upsilon_4 :=$



Moreover $H^2(\mathsf{fGC}_2^\uparrow,\delta_0)=\mathbb{K}$ and is generated by a graph with four vertices. This means that one can construct by induction a new Maurer-Cartan element (the integer subscript stand for the number of vertices)

$$\Upsilon_{KS} = \bullet\!\!\longrightarrow\!\!\bullet + \Upsilon_4 + + \Upsilon_6 + \Upsilon_8 + \ldots$$

as all obstructions have more than 8 vertices and hence do not hit the unique cohomology class in $H^2(\mathsf{fGC}_2^\uparrow,\delta_0)=\mathbb{K}$. Up to gauge equivalence, this new MC element Υ is the *only* non-trivial deformation of the standard MC element $\bullet\!\!\longrightarrow\!\!\bullet$.

2.3.1. **Definition.** Any MC element in the Lie algebra $(\mathsf{fGC}_2^\uparrow,[\ ,\ ])$ which is gauge equivalent to Υ_{KS} above is a called a *Kontsevich-Shoikhet* MC element, and the associated $\mathcal{L}ie_\infty$ structure in $\mathcal{T}_{poly}(\mathbb{R}^d)$ is called a *Kontsevich-Shoikhet* one. It was introduced by Boris Shoikhet in [Sh2] with a reference to an important contribution by Maxim Kontsevich via an informal communication.

2.3.2. **Definition.** Let $\{\mu^{KS}_{2k} : \wedge^{2k}\mathcal{T}_{poly}(\mathbb{R}^d) \to \mathcal{T}_{poly}(\mathbb{R}^d)\}_{k\geq 1}$ be a Kontsevich-Shoikhet $\mathcal{L}ie_\infty$ structure. A bivector field $\pi \in \mathcal{T}_{poly}(\mathbb{R}^d)$ is called *quantizable* if it satisfies the equation

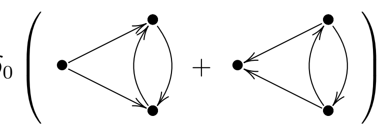
$$\mu_2(\pi,\pi) + \hbar^2\mu_4(\pi,\pi,\pi,\pi) + \hbar^4\mu_6(\pi,\pi,\pi,\pi,\pi,\pi) + \ldots = 0$$

Do these structures have anything to do with ordinary Poisson structures and their quantization? The answer is "yes" as we explain in the next subsections.

2.4. **Quantizable versus ordinary Poisson structures.** Boris Shoikhet conjectured in [Sh2] that $\mathcal{L}ie_\infty$ algebras $(\mathcal{T}_{poly}(\mathbb{R}^d), [\ ,\ ]_S)$ and $(\mathcal{T}_{poly}(\mathbb{R}^d), \mu^{KS}_\bullet)$ are quasi-isomorphic for any finite $d \in \mathbb{N}$. Stated in terms of graphs, this conjecture says that as MC elements in fGC_2 (rather than in $\mathsf{fGC}_2^{\uparrow}$) Υ_2 and Υ^{KS} are gauge-equivalent,

$$\Upsilon_2 = e^{ad_\Theta}\Upsilon_{KS} \equiv e^{ad_\Theta}\left(\sum_{k=1}^{\infty}\Upsilon_{2k}\right) \tag{2}$$

for some degree zero element Θ in fGC_2. That this relation hold true is far from obvious. Indeed, let us attempt to construct Θ by induction (as we managed to construct Υ_{KS}). The first step is easy — the term Υ_4 is δ_0 exact in fGC_2,

$\Upsilon_4 = \delta_0$ 

and we can use the sum of two degree zero graphs inside the brackets to gauge away Υ_4. However the next obstruction becomes a cyclic *wheeled* graph Υ_6' from fGC_2 so that starting with this second step all the obstruction classes land in $H^1(\mathsf{fGC}_2)$, the same cohomology group where obstructions for the universal deformation of Poisson structures lie. Therefore, the formula for Θ must be transcendental (as the Kontsevich formula). One of our main results in this subsection is such an explicit formula for Θ.

2.4.1. **Theorem.** *The relation (2) holds true for Θ defined as follows:*

$$\Theta = \sum_{n\geq 1}\sum_{\Gamma\in\mathsf{G}_{n,2n-2}} c_\Gamma \Gamma$$

with

$$c_\Gamma = \int_{\mathcal{C}_n(\mathbb{R}^2)} \Omega_\Gamma$$

where

(i) $\mathcal{C}_n(\mathbb{R}^2)$ *be the smooth $(2n-2)$-dimensional manifold of all injections*

$$\begin{array}{rccc} i: & [n] & \longrightarrow & \mathbb{R}^2 = \mathbb{C} \\ & (1,2,\ldots,n) & \longrightarrow & (z_1,\ldots,z_n) \end{array}$$

modulo the action of the 2-dimensional translation group $\mathbb{R}^2$,

$$(z_1,\ldots,z_n) \to (z_1+a,\ldots,z_1+a)\ \forall\ z_1,\ldots,z_n \in \mathbb{C}, a\in\mathbb{C},$$

(ii) $\Omega_\Gamma = \bigwedge_{e\in E(\Gamma)}\Omega_e$ *where $\Omega_e := \omega_g(z_i,z_j)$ for every directed edge $e = \overset{i}{\bullet}\!\longrightarrow\!\overset{j}{\bullet} \in E(\Gamma)$, and the smooth differential 1-form $\omega_g(z_i,z_j)$ is given by the formula*

$$\omega_g(z_1,z_2) = \frac{1}{2\pi}Arg(z_i-z_j) + \frac{|z_1-z_2|}{1+|z_1-z_2|}d\Psi_g\left(\frac{z_i-z_j}{|z_i-z_g|}\right) + \Psi_g\left(\frac{z_i-z_j}{|z_i-z_g|}\right)d\left(\frac{|z_1-z_2|}{1+|z_1-z_2|}\right)$$

where $\Psi_g\left(\frac{z_i-z_j}{|z_i-z_g|}\right)$ is a smooth function on $S^1 = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$ satisfying the condition

$$\frac{1}{2\pi}Arg(z_i-z_j) + d\Psi_g\left(\frac{z_i-z_j}{|z_i-z_g|}\right) = g\left(\frac{z_i-z_j}{|z_i-z_g|}\right)dArg(z_i-z_j)$$

for some function g on the circle S^1 with compact support in the upper half circle given by $y>0$, and normalized so that $\int_0^{2\pi} g(\theta)d\theta = 1$.

Idea of the proof: the space $\mathcal{C}_n(\mathbb{R}^2)$ admits a compactification (introduced in [Me1]) whose codimension 1 boundary strata capture the combinatorics of the 2-coloured operad of $\mathcal{L}ie_\infty$ morphisms. All the forms Ω_Γ extend to this compactification (so that weights c_Γ are well-defined) and factorize on the boundary strata as required in [Me1].

2.4.2. **Corollary.** *(i) For any finite natural number d there is a 1-1 correspondence between ordinary Poisson structures in $\mathcal{T}_{poly}(\mathbb{R}^d)[[\hbar]]$ and quantizable Poisson structures in the same vector space.*

(ii) The quantizable Poisson structure π^{qua} associated to an ordinary Poisson structure π in $\mathcal{T}_{poly}(\mathbb{R}^d)$ (or $\mathcal{T}_{poly}(\mathbb{R}^d)[[\hbar]]$) is given by the following explicit formula

$$\pi^{qua} = \pi + \sum_{n=2}^{\infty} \hbar^{n-1} \sum_{\Gamma\in \mathsf{G}_{n,2n-2}} c_\Gamma \Phi_\Gamma(\pi,\pi,\dots,\pi).$$

2.5. **From quantizable Poisson structures to star products.** This step is easy: it can be done either by induction (see [W2, B]) or using Kontsevich formulae from [Ko] with the crucial difference that instead of the hyperbolic propagator

$$\omega(z_i,z_j) = \frac{1}{2\pi} Arg\left(\frac{z_i - z_j}{\bar{z}_i - z_j}\right)$$

one uses the following smooth differential form,

$$\omega^{new}(z_i,z_j) = g(Arg(z_i - z_j))dArg(z_i - z_j)$$

3. A new explicit formula for quantization of Poisson structures

3.1. **A quantization problem suggested by Drinfeld** [D]**.** Let V be a $\mathbb{Z}$-graded real vector space, and let $\mathcal{O}_V := \odot^\bullet V = \oplus_{n\geq 0} \odot^n V$ the be the space of polynomial functions on V^* equipped with the standard graded commutative and cocommutative bialgebra structure. If $\mathcal{A}ss\mathcal{B}$ stands for the prop of bialgebras, then the standard product and coproduct in $\mathcal{O}_V$ give us a representation,

(3) $$\rho_0 : \mathcal{A}ss\mathcal{B} \longrightarrow \mathcal{E}nd_{\mathcal{O}_V}.$$

A *formal deformation* of the standard bialgebra structure in $\mathcal{O}_V$ is a continuous morphisms of props,

(4) $$\rho_\hbar : \mathcal{A}ss\mathcal{B}[[\hbar]] \longrightarrow \mathcal{E}nd_{\mathcal{O}_V}[[\hbar]],$$

$\hbar$ being a formal parameter, such that $\rho_\hbar|_{\hbar=0} = \rho_0$. It is well-known [D] that if $\rho_\hbar$ is a formal deformation of ρ_0, then $\frac{d\rho_\hbar}{d\hbar}|_{\hbar=0}$ makes the vector space V into a Lie bialgebra, that is, induces a representation,

$$\nu : \mathcal{L}ie\mathcal{B} \longrightarrow \mathcal{E}nd_V$$

of the prop of Lie bialgebras, $\mathcal{L}ie\mathcal{B}$, in the vector space V. Thus Lie bialgebra structures, ν, in V control infinitesimal formal deformations of ρ_0. Drinfeld formulated a deformation quantization problem: given ν in V, does $\rho_\hbar$ exist such that $\frac{d\rho_\hbar}{d\hbar}|_{\hbar=0}$ induces ν? This problem was solved affirmatively by Etingof and Kazhdan in [EK]; later Tamarkin gave a second proof of the Etingof-Kazhdan deformation quantization theorem in [Ta1], and recently Pavol Severa showed [**?**] a third proof; all proofs of existence of $\rho_\hbar$ are very inexplicit. Boris Shoikhet claimed in [Sh2] an explicit formula for $\rho_\hbar$.

In this subsection we show a new proof of the Etingof-Kazhdan theorem for finite-dimensional Lie bialgebras based on an explicit formula in the form

(5) $$\rho_\hbar^{explicit}(\text{a generator of } \mathcal{A}ss\mathcal{B}) = \sum_\Gamma c_\Gamma \Phi_\Gamma,$$

where the sum runs over a certain family of graphs, Φ_Γ is an element of $\mathcal{E}nd_{\mathcal{O}_V}$ uniquely determined by each graph Γ, and c_Γ is an absolutely convergent integral, $\int_{\mathfrak{C}(\Gamma)} \Omega_\Gamma$, of a smooth differential form Ω_Γ over a certain configuration space of points, $\mathfrak{C}(\Gamma)$, in a 3-dimensional subspace, $\mathcal{H}$, of the Cartesian product, $\overline{\mathbb{H}}\times\overline{\mathbb{H}}$, of two copies of the closed upper-half planes. This formula is a relatively straightforward generalization of our formula for quantization of Poisson structures discussed in the previous section..

3.2. **An explicit formula for deformation quantization of Lie bialgebras.** Let $\nu : \mathcal{L}ie\mathcal{B} \to \mathcal{E}nd_V$ be a Lie bialgebra structure in a finite-dimensional vector space V. We want to give an explicit formula for an associated bialgebra structure

$$\rho_\hbar : \mathcal{A}ss\mathcal{B} \longrightarrow \mathcal{E}nd_{\odot^\bullet V}[[\hbar]]$$

in $\odot^\bullet V[[\hbar]]$ such that $\frac{d\rho_\hbar}{d\hbar}|_{\hbar=0}$ induces precisely ν in V.

Step 1: *From Lie bialgebras to quantizable Lie bialgebras.* Let $g : [-\pi, \pi] \to \mathbb{R}$ be an even function with compact support in $(-\pi/2, \pi/2)$. Then $\omega_g := g(Arg(t+ix))dArg(t+ix) \wedge g(Arg(t+iy))dArg(t+iy)$ is a well-defined smooth 2-form on the 2-sphere

$$S^2 = \{(x,y,t) \in \mathbb{R}^3 \mid x^2 + y^2 + t^2 = 1\}$$

and has compact support in the upper semisphere $t > 0$; we can always choose g in such a way that $\int_{S^2} \omega_g = 1$. If Vol_{S^2} is the standard homogeneous volume form on S^2 (normalized so that $\int_{S^2} Vol_{S^2} = 1$) then we have

$$\omega_g = Vol_{S^2} + d\Psi_g$$

for some 1-form Ψ_g on S^2.

Let $\mathcal{C}_n(\mathbb{R}^3)$ be the smooth $(3n-3)$-dimensional manifold of all injections

$$\begin{array}{cccc} i: & [n] & \longrightarrow & \mathbb{R}^3 \\ & (1,2,\dots,n) & \longrightarrow & (p_1,\dots,p_n) \end{array}$$

modulo the action of the 3-dimensional translation group $\mathbb{R}^3$,

$$(p_1,\dots,p_n) \to (p_1+a,\dots,p_1+a) \ \forall\, p_1,\dots,p_n \in \mathbb{R}^3, a \in \mathbb{R}^3,$$

so that we have a smooth isomorphism

$$\begin{array}{ccc} \mathcal{C}_2(\mathbb{R}^3) & \xrightarrow{\simeq} & S^2 \times \mathbb{R}^+ \\ (p_1,p_2) & \longrightarrow & (\frac{p_1-p_2}{|p_1-p_2|}, |p_1-p_2|) \end{array}$$

Consider a propagator on $\mathcal{C}_2(\mathbb{R}^3)$

$$\omega(p_1,p_2) = \mathsf{Vol}_{S^2}\left(\frac{p_1-p_2}{|p_1-p_2|}\right) + h\left(|p_1-p_2|\right) d\Psi_g\left(\frac{p_1-p_2}{|p_1-p_2|}\right) - \Psi_g\left(\frac{p_1-p_2}{|p_1-p_2|}\right) \wedge dh(|p_1-p_2|)$$

where h is any smooth function on $\mathbb{R}$ with $h(0)=0$ and $h(+\infty)=1$, for example

$$h(|p_1-p_2|) = \frac{|p_1-p_2|}{1+|p_1-p_2|}$$

This is a differential 2-form on $\mathcal{C}_2(\mathbb{R}^3)$. Similarly, $\omega(p_i,p_j)$ is a well-defined differential 2-form on $\mathcal{C}_n(\mathbb{R}^3)$ for any pair of different indices $i,j \in [n]$.

Let $G_{4k+1,6k}$, $k>0$, be the set of directed graphs Γ with $4k+1$ vertices and $6k$ edges; we assume that vertices of Γ are labelled bijectively by integers from set $[4k+1]$. To every such a graph we can associated a

- a differential operator $\phi_\Gamma : \wedge^{4k+1}\mathfrak{g}_V \to \mathfrak{g}_V$ (whose construction is explained in §?.?).
- a real number c_Γ which is given by the converging integral

$$c_\Gamma = \int_{C_{4k+1}(\mathbb{R}^3)} \Omega_\Gamma.$$

where

$$\Omega_\Gamma = \bigwedge_{e \in E(\Gamma)} \Omega_e \tag{6}$$

with $\Omega_e := \omega(p_i,p_j)$ for every directed edge $e = \overset{i}{\bullet}\!\longrightarrow\!\overset{j}{\bullet} \in E(\Gamma)$.

Then the associated *quantizable Lie bialgebra* structure $\nu^{qua} \in \mathfrak{g}_V[[\hbar]]$ is defined by

$$\nu^{qua} := \nu + \sum_{k\geq 1} \sum_{\Gamma\in G_{4k+1,6k}} \frac{\hbar^{4k+1}}{(4k+1)!} c_\Gamma \phi_\Gamma(\underbrace{\nu,\ldots,\nu}_{4k+1})$$

This structure amounts to a pair consisting of "brackets", $\wedge^2 V \to V[[\hbar]]$ and "cobrackets", $V \to \wedge^2[[\hbar]]$ which satisfy a "quantizability" equation

$$\mu_2(\nu^{qua},\nu^{qua}) + \mu_9(\nu^{qua},\ldots,\nu^{qua}) + \ldots = 0$$

where $\{\mu_{4k+1} : \wedge^{2k+1}\mathfrak{g}_V \to \mathfrak{g}_V\}_{k\geq 2}$ is a $\mathcal{L}ie_\infty$ structure in $\mathfrak{g}_V$ which is a deformation of the standard Lie algebra structure $\mu_2 = \{\ ,\ \}$ in $\mathfrak{g}_V$ and which is given explicitly below in §?.?.

There is a 1-1 correspondence between ordinary Lie bialgebras and quantizable Lie bialgebras, but such a correspondence depends on the choice of an associator and hence must be highly non-trivial as in the particular explicit formula above (we do not know which associator this particular formula corresponds to).

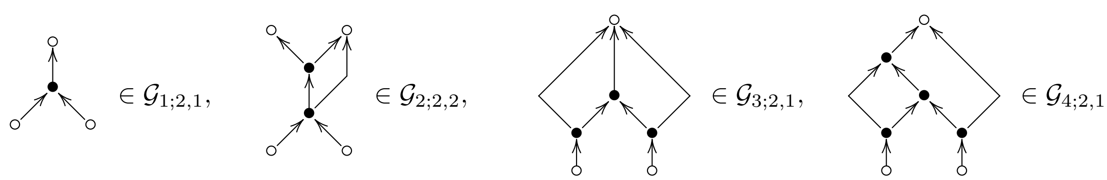
Step 2: *From quantizable Lie bialgebras to bialgebra structures in* $\odot^\bullet(V)[[\hbar]]$. Let $k \geq 0$, $m \geq 1$ and $n \geq 1$ be integers satisfying inequality $3k + m + n \geq 3$. The set $\mathcal{G}_{k;m,n}$ consists, by definition, of directed graphs Γ with k vertices called *internal*, m vertices called *in*-vertices and n vertices called *out*-vertices and satisfying the following conditions

(i) every internal vertex of Γ is at least trivalent, and has at least one incoming edge and one outgoing edge,
(ii) every in-vertex can have outgoing edges (called *out-legs*) but no incoming ones,
(iii) every out-vertex can have ingoing edges (called *in-legs*) but no outgoing ones,
(iv) Γ has no loops (edges beginning and ending at the same vertex) an no wheels (a wheel is a sequence of directed edges making a closed path in the standard pictorial representation of directed graphs).
(v) if $E_{int}(\Gamma)$ stands for the set of *internal* edges (that is, the edges connecting two internal vertices), $E_{in}(\Gamma)$ for the set of in-legs, and $E_{out}(\Gamma)$ for the set of out-legs, then the following equality,

$$2\#E_{int}(\Gamma) + \#E_{in}(\Gamma) + \#E_{out}(\Gamma) = 3k + m + n - 3 \tag{7}$$

holds,
(vi) there are no edges connecting in-vertices to out-vertices,
(vii) bijections $V_{internal}(\Gamma) \to [k]$, $V_{in}(\Gamma) \to [n]$, $V_{out}(\Gamma) \to [m]$ are fixed,
(viii) the sets $E_{in}(\Gamma)$ and $E_{out}(\Gamma)$ are totally ordered up to an even permutation,

For example (we omit labellings of vertices by integers),



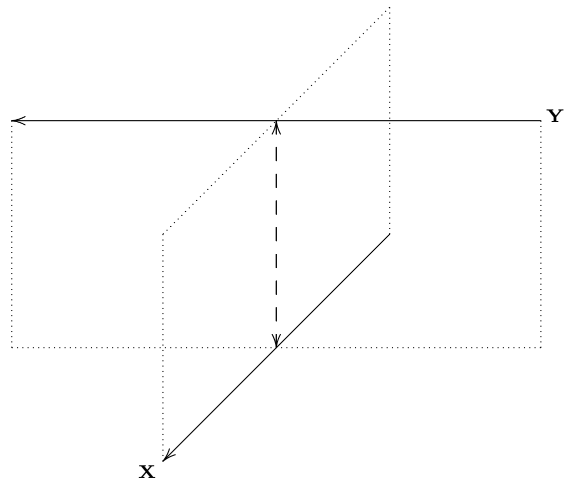
Thus graphs from $\mathcal{G}_{k;m,n}$ admit a flow which we always assume in our pictures to be directed from the bottom to the top (so that there is no need to show directions of the edges anymore). To any graph $\Gamma \in \mathcal{G}_{k;m,n}$ one can associate a linear map

$$\begin{array}{rccc} \Phi_\Gamma : & \otimes^k \mathfrak{g}_V \otimes \otimes^n(\odot^\bullet V) & \longrightarrow & \otimes^m(\odot^\bullet V) \\ & (\gamma_1,\ldots,\gamma_k,f_1,\ldots,f_n) & \longrightarrow & \Phi_\Gamma(\gamma_1,\ldots,\gamma_k,f_1,\ldots,f_n) \end{array}$$

In fact one can construct a 2-coloured properad $\mathcal{B}ra$ out of these graphs such that the association $\Gamma \to \Phi_\Gamma$ gives us a representation.

Let $\mathbb{H}' = \{(x,t) \in \mathbb{R}\times\mathbb{R}^{>0}\}$ and $\mathbb{H}'' = \{(y,\hat{t}) \in \mathbb{R}\times\mathbb{R}^{>0}\}$ be two copies of the upper-half plane, and let $\overline{\mathbb{H}}' = \{(x,t) \in \mathbb{R}\times\mathbb{R}^{\geq 0}\}$ and $\overline{\mathbb{H}}'' = \{(y,\hat{t}) \in \mathbb{R}\times\mathbb{R}^{\geq 0}\}$ be their closures. Consider a subspace $\mathcal{H} \subset \mathbb{H}'\times\mathbb{H}''$ given by the equation $t\hat{t} = 1$, and denote by $\overline{\mathcal{H}}$ its closure under the embedding into $\overline{\mathbb{H}}'\times\overline{\mathbb{H}}''$. The space

$\mathcal{H}$ has two distinguished lines, $\mathbf{X} := \{(x \in \mathbb{R}, y = 0, t = 0\}$ and $\mathbf{Y} := \{(x = 0, y \in \mathbb{R}, \widehat{t} = 0\}$; it also has a natural structure of a smooth manifold with boundary.



The group $G_3 := R^+ \rtimes \mathbb{R}^2$ acts on $\overline{\mathcal{H}}$,

$$\begin{array}{cccccc} R^+ \rtimes \mathbb{R}^2 & \times & \widehat{\mathcal{H}} & \longrightarrow & \widehat{\mathcal{H}} \\ (\lambda, a, b) & \times & (x, y, t) & \longrightarrow & (\lambda x + a, \lambda^{-1} y + b, \lambda t). \end{array}$$

Consider a configuration space of injections

$$Conf_{k;m,n} := \{ i_{int} : [k] \rightarrow \mathcal{H}, \quad [n] : V_{in}(\Gamma) \hookrightarrow \mathbf{X}, \quad [m] : V_{out}(\Gamma) \longrightarrow \mathbf{Y} \}\,, \tag{8}$$

where injections i_{int} are required to satisfy an extra condition that its composition with each of the natural projections $\mathcal{H} \rightarrow \mathbb{H}'$ and $\mathcal{H} \rightarrow \mathbb{H}''$ is also an injection, and that injection i_{in} (respectively, i_{out}) respects the total orders[3] in the sets $[n]$ and $\mathbf{X}$ (respectively, in $[m]$ and $\mathbf{Y}$). The group G_3 acts on $Conf_{k;m,n}(\Gamma)$ freely so that the quotient

$$C_{k;m,n} := Conf_{k;m,n}/G_3$$

is an open smooth manifold of dimension $3k + m + n - 3$.

For any graph $\Gamma \in \mathcal{G}_{k;m,n}$ we define a smooth top degree differential form Ω_Γ on $C_{k;m,n}$,

$$\Omega_\Gamma := \bigwedge_{e \in E_{in}(\Gamma)} \omega'_e \ \wedge \bigwedge_{e \in E_{int}(\Gamma)} \Omega_e \ \wedge \bigwedge_{e \in E_{out}(\Gamma)} \omega''_e$$

where ω'_e and ω''_e are 1-forms and Ω_e is a 2-form defined as follows.
There are two natural projections,

$$\begin{array}{ccccc} \overline{\mathbb{H}}' & \overset{\pi''}{\longleftarrow} & \overline{\mathcal{H}} & \overset{\pi''}{\longrightarrow} & \overline{\mathbb{H}}'' \\ z'(p) = x + it & \longleftarrow & p = (x, y, t) & \longrightarrow & z''(p) = y + \frac{i}{t}. \end{array}$$

If we identify vertices of Γ with their images in $\overline{\mathcal{H}}$ under injections in (8), then

(i) for any edge $e = \overset{v_1}{\circ}\!\!\longrightarrow\!\!\overset{v_2}{\bullet} \in E_{in}(\Gamma)$ we set $\omega'_e := g(Arg(z'(v_1) - z'(v_2)))dArg(z'(v_1) - z'(v_2))$,

(ii) for any edge $e = \overset{v_1}{\bullet}\!\!\longrightarrow\!\!\overset{v_2}{\circ} \in E_{out}(\Gamma)$ we set $\omega''_e := g(Arg(z''(v_1) - z''(v_2)))dArg(z''(v_1) - z''(v_2))$,

(iii) for any edge $e = \overset{v_1}{\bullet}\!\!\longrightarrow\!\!\overset{v_2}{\bullet} \in E_{int}(\Gamma)$ we set $\Omega_e := \omega'_e \wedge \omega''_e$.

The wedge product of 1-forms in the formula for Ω_Γ is taken with respect to the chosen orderings (up to an even permutations) in the sets $E_{in}(\Gamma)$ and $E_{out}(\Gamma)$. The integral

$$C_\Gamma := \int_{C_{k;m,n}} \Omega_\Gamma$$

[3] The real lines $\mathbf{X}$ and $\mathbf{Y}$ are equipped with their standard total orders.

converges for any graph $\Gamma \in \mathcal{G}_{k;m,n}$.
Then the required bialgebra structures in $\odot^\bullet V[[\hbar]]$ which quantizes the given Lie bialgebra structure ν consists of compatible multiplication map,

$$\begin{array}{rccc} \star_\hbar : & \odot^\bullet V \otimes \odot^\bullet V & \longrightarrow & \odot^\bullet V[[\hbar]] \\ & (f_1, f_2) & \longrightarrow & f_1 \star_\hbar f_2 \end{array}$$

and comultiplication map,

$$\begin{array}{rccc} \Delta_\hbar : & \odot^\bullet V & \longrightarrow & \odot^\bullet V \otimes \odot^\bullet V[[\hbar]] \\ & f & \longrightarrow & \Delta_\hbar(f) \end{array}$$

and is given explicitly by

$$f_1 \star_\hbar f_2 = f_1 f_2 + \sum_{k\geq 1} \frac{\hbar^k}{k!} \sum_{\Gamma\in\mathcal{G}_k,1,2} C_\Gamma \Phi_\Gamma(f_1, f_2, \nu^{qua}, \ldots, \nu^{qua})$$

and

$$\Delta_\hbar(f) = \Delta_0(f) + \sum_{k\geq 1} \frac{\hbar^k}{k!} \sum_{\Gamma\in\mathcal{G}_k,2,1} C_\Gamma \Phi_\Gamma(f_1, f_2, \nu^{qua}, \ldots, \nu^{qua})$$

where Δ_0 is the standard cocommutative coalgebra structure in $\odot^\bullet V$.

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