The LMO functor: from associators to cobordisms & tangles

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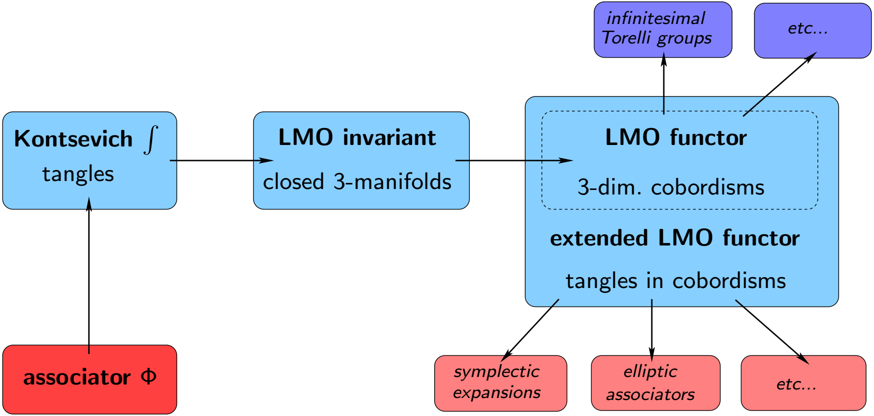
(IRMA, Strasbourg)

GRT, MZV's and associators
Les Diablerets, August 2015

Overview



Kontsevich $\int$
tangles

LMO invariant
closed 3-manifolds

LMO functor
3-dim. cobordisms

extended LMO functor
tangles in cobordisms

infinitesimal Torelli groups

etc...

associator Φ

symplectic expansions

elliptic associators

etc...

Plan

1. Review of the Kontsevich integral
2. Review of the LMO invariant
3. Construction of the LMO functor
4. The LMO homomorphism

The monoidal category $\mathcal{T}_q$ of quasi-tangles

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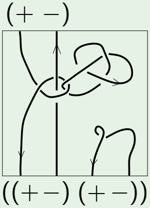

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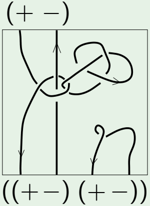

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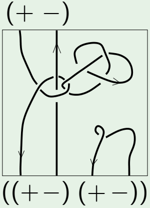

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Jacobi diagrams on 1-manifolds

X : an oriented 1-manifold

A Jacobi diagram on X is a finite graph whose vertices are either

- trivalent and oriented,
- or, univalent and embedded into X.

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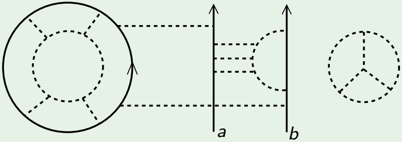

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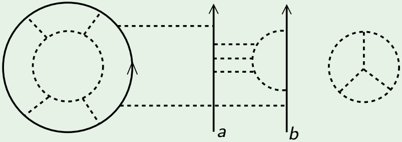
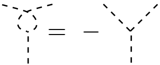
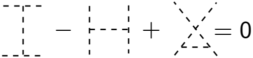
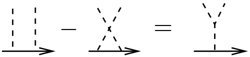


$$\mathcal{A}(X) := \frac{\mathbb{Q} \cdot \{\text{Jacobi diagrams on } X\}}{\text{AS, IHX, STU}}$$


AS


IHX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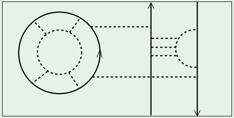
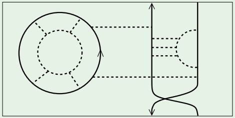
$D :=$  $\in \mathrm{Mor}_{\mathcal{A}}(-+, -+)$

$D' :=$  $\in \mathrm{Mor}_{\mathcal{A}}(-+, +-)$

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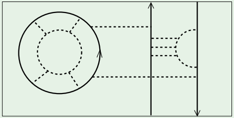
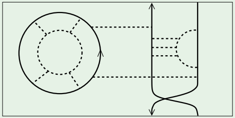
$D :=$  $\in \mathrm{Mor}_{\mathcal{A}}(-+, -+)$

$D' :=$  $\in \mathrm{Mor}_{\mathcal{A}}(-+, +-)$

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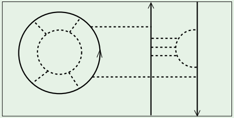
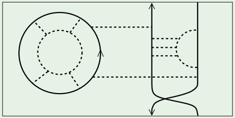
$D :=$  $\in \mathrm{Mor}_{\mathcal{A}}(-+,-+)$

$D' :=$  $\in \mathrm{Mor}_{\mathcal{A}}(-+,+-)$

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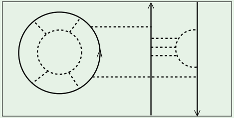
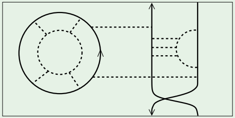
$D :=$  $\in \mathrm{Mor}_{\mathcal{A}}(-+, -+)$

$D' :=$  $\in \mathrm{Mor}_{\mathcal{A}}(-+, +-)$

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Let $\mathcal{A}^{\mathsf{l}} \subset \mathcal{A}$ be the subcategory spanned by Jacobi diagrams without free component.

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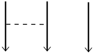

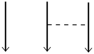

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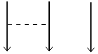
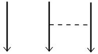
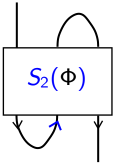
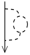




and define

$$\nu := \left(\quad \right)^{-1} = \downarrow + \frac{1}{48} \quad + \cdots \in \mathcal{A}(\downarrow).$$





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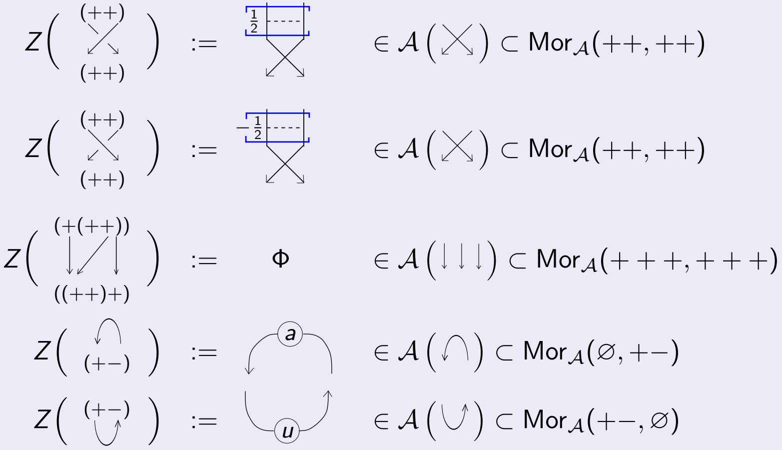

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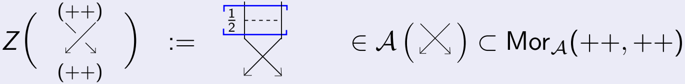

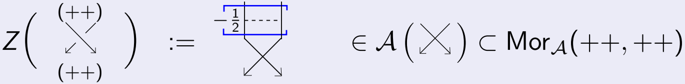

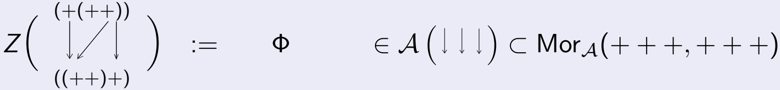

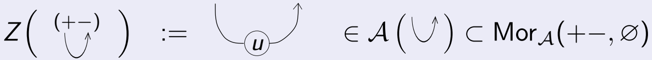

where $a, u \in \mathcal{A}(\downarrow)$ *satisfy* $a \cdot u = \nu = Z(\mathit{unknot})$. *(E.g.* $a := \downarrow$, $u := \nu$.*)*

Plan

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2. Review of the LMO invariant
3. Construction of the LMO functor
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The monoidal category $\mathcal{T}_q\mathcal{C}ub$ of quasi-tangles in homology cubes

A homology cube is a compact oriented 3-manifold C such that $\partial C \cong \partial[0,1]^3$ and $H_*(C;\mathbb{Q}) \simeq H_*([0,1]^3;\mathbb{Q})$.

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There is a "short exact sequence" $\mathcal{T}_q \rightarrowtail \mathcal{T}_q\mathcal{C}ub \twoheadrightarrow \mathcal{C}ub$.

The LMO invariant

Theorem (Le–Murakami–Ohtsuki'98)

There is a tensor-preserving functor $Z : \mathcal{T}_q\mathcal{C}ub \longrightarrow \mathcal{A}$ *which extends the Kontsevich integral:*

$$\begin{tikzcd}
\mathcal{T}_q \arrow[r, tail] \arrow[d, "Z"] & \mathcal{T}_q\mathcal{C}ub \arrow[r, two heads] \arrow[d, dotted, "Z"] & \mathcal{C}ub \arrow[d, dotted, "Z"] \\
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$Z : \mathcal{T}_q\mathcal{C}ub \longrightarrow \mathcal{A}$ is universal among "finite-type invariants": using surgery, one can define a filtration

$$\mathbb{Q}\mathcal{T}_q\mathcal{C}ub = F_0(\mathbb{Q}\mathcal{T}_q\mathcal{C}ub) \supset F_1(\mathbb{Q}\mathcal{T}_q\mathcal{C}ub) \supset F_2(\mathbb{Q}\mathcal{T}_q\mathcal{C}ub) \supset \cdots$$

s.t. Z is filtration-preserving and $\operatorname{Gr} Z$ is an isomorphism (Le'97 for $\mathcal{C}ub$).

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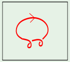
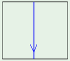
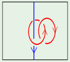
	(2)	(punctured $\mathbb{R}P^3$, $\varnothing$)
	$()$	($[0,1]^3$, trivial string)
	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	($[0,1]^3$, trivial string)

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Theorem (Kirby'78)

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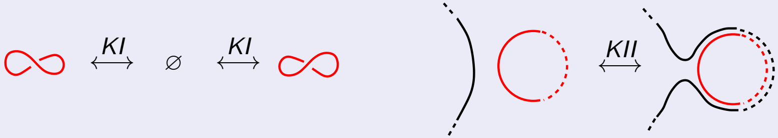

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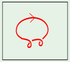
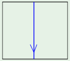
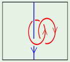
	(2)	(punctured $\mathbb{R}P^3$, $\varnothing$)
	$()$	($[0,1]^3$, trivial string)
	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	($[0,1]^3$, trivial string)

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where $(\sigma_+(L), \sigma_-(L))$ is the signature of B_L.

Plan

1. Review of the Kontsevich integral
2. Review of the LMO invariant
3. Construction of the LMO functor
4. The LMO homomorphism

3-dimensional cobordisms

For all $g \in \mathbb{N}$, fix a model surface of genus g:

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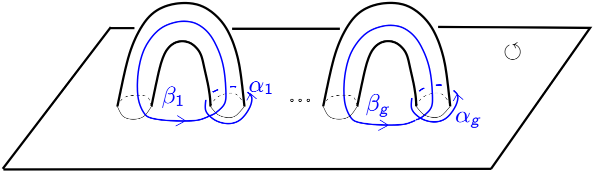

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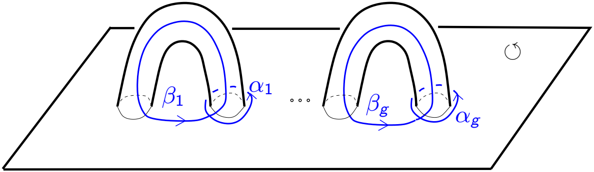

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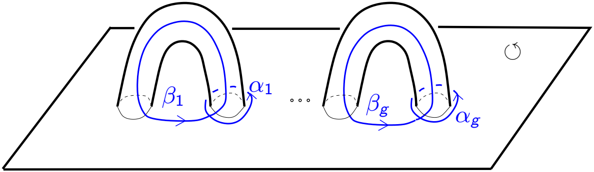

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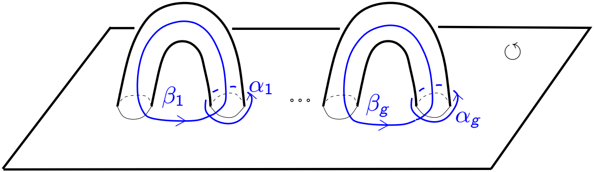

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It is Lagrangian if it satisfies certain homological conditions which involve A_g and A_h.

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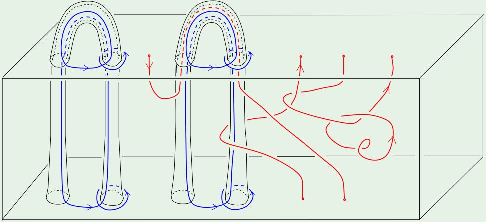

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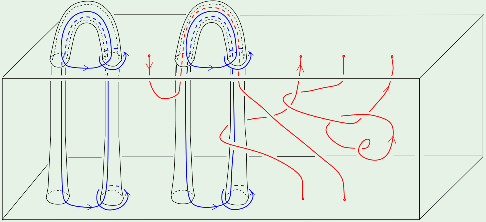

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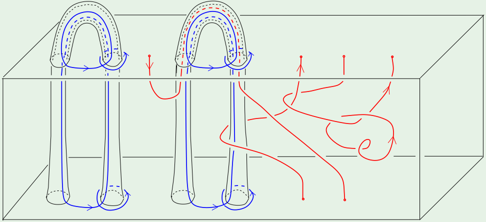

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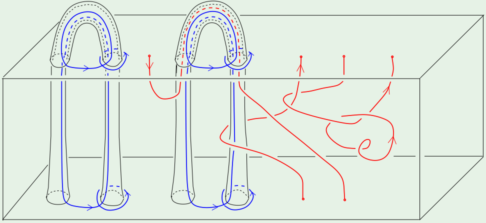

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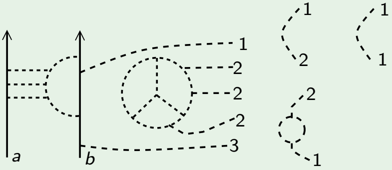
X : an oriented 1-manifold, C : a finite set

A *C-colored Jacobi diagram on X* is a finite graph whose vertices are either

- trivalent and oriented,
- or, univalent and embedded into X,
- or, univalent and colored by C.

$X := \uparrow_a \uparrow_b\,,\ C := \{1, 2, 3, 4\}$



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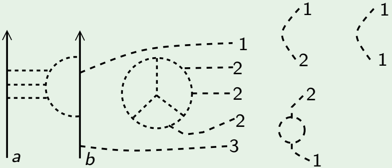
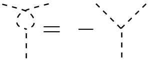


$$\mathcal{A}(C, X) := \frac{\mathbb{Q} \cdot \{C\text{-colored Jacobi diagrams on } X\}}{\text{AS, IHX, STU}}$$



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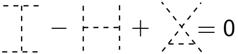

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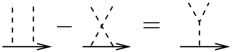

STU

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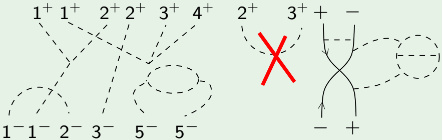

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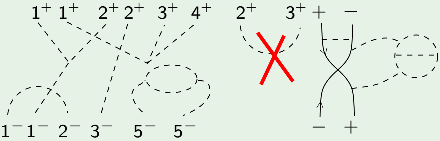

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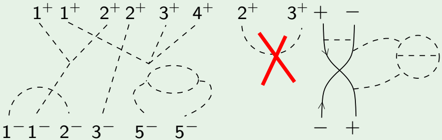

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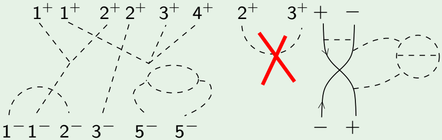

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The LMO functor

Theorem (Cheptea–Habiro–M.'08 for $\mathcal{LC}ob$)

There is a tensor-preserving functor $\widetilde{Z} : \mathcal{T}_q\mathcal{LC}ob \longrightarrow {}^{ts}\!\mathcal{A}$ *which extends the LMO invariant:*

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\mathcal{T}_q\mathcal{C}ub \arrow[r, tail] \arrow[d, "Z"] & \mathcal{T}_q\mathcal{LC}ob \arrow[d, dotted, "\widetilde{Z}"] \\
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Furthermore, (a reduction of) $\widetilde{Z}$ *is universal among finite-type invariants.*

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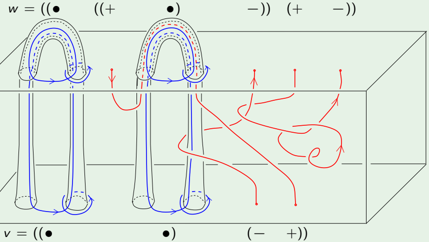

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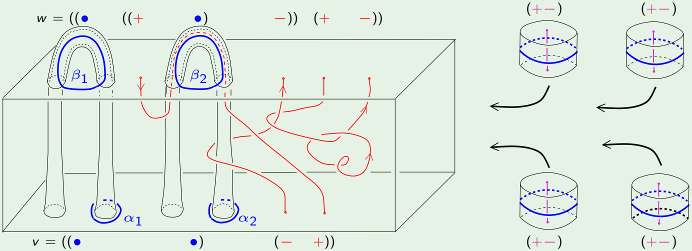

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Obtain a quasi-tangle $\gamma \cup \tau$ in a homology cube C, where γ consists of the co-cores of the 2-handles and τ is the initial tangle.

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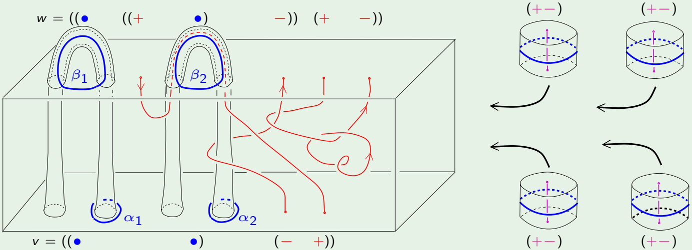

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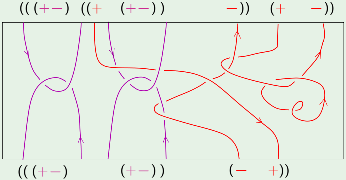

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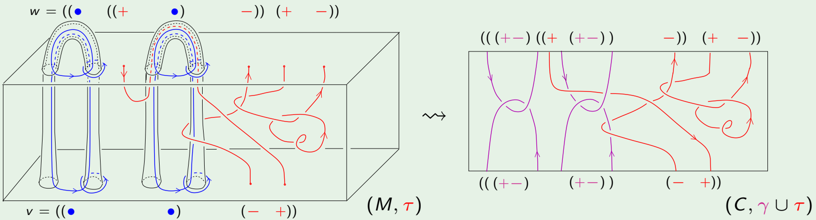

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Construction of the LMO functor (2/3)

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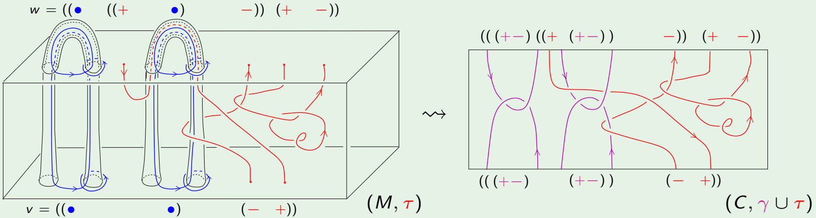
$$\mathcal{A}\big(\{1^+,\dots,g^+\}\cup\{1^-,\dots,f^-\},\tau\big)\ \underset{\simeq}{\xrightarrow{\chi}}\ \mathcal{A}\big(\gamma\cup\tau\big)$$



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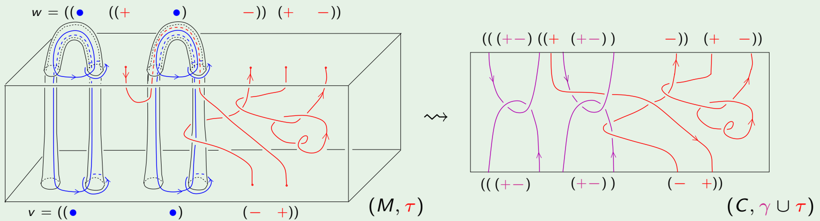

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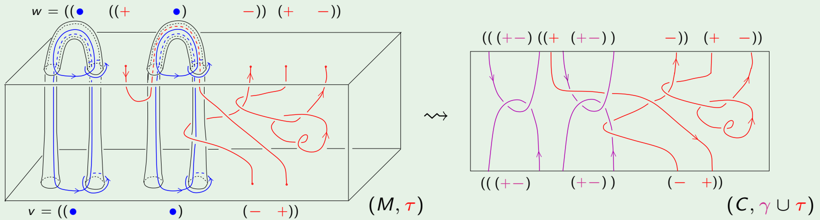

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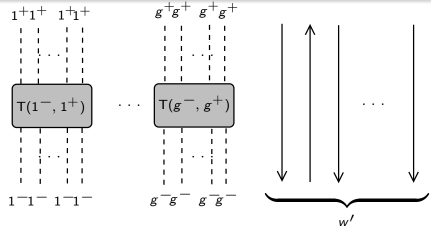
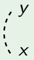
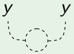
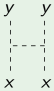
$\mathrm{T}_{g,w'} :=$ 

where $\mathrm{T}(x,y) \in \mathcal{A}(\{x,y\})$ is defined in terms of $Z($  $)$ and BCH:

$\mathrm{T}(x,y) \overset{(\Phi:even)}{=} \exp_\sqcup\Big($ 

$\Big) \sqcup \Big(\varnothing - \frac{1}{8}\cdot$ 

$- \frac{1}{48}\cdot$ 

$+ \frac{1}{8}\cdot$ 

$+ \cdots \Big)$

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- **objects:** non-associative words in the letters $+, -, \bullet$;
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- **composition:** vertical gluing $(M_2, \tau_2) \circ (M_1, \tau_1) := \begin{array}{|c|} \hline (M_1, \tau_1) \\ \hline (M_2, \tau_2) \\ \hline \end{array}$.

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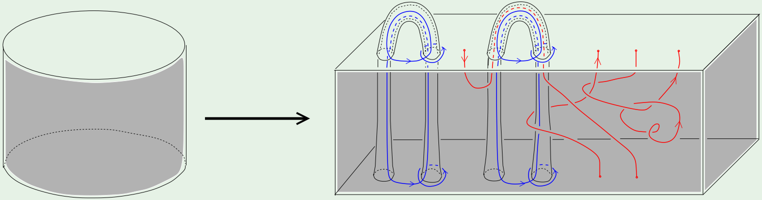
The attachment of a 2-handle defines a functor $\mathcal{T}_q\mathcal{LC}ob \twoheadrightarrow \widehat{\mathcal{T}_q\mathcal{LC}ob}$.



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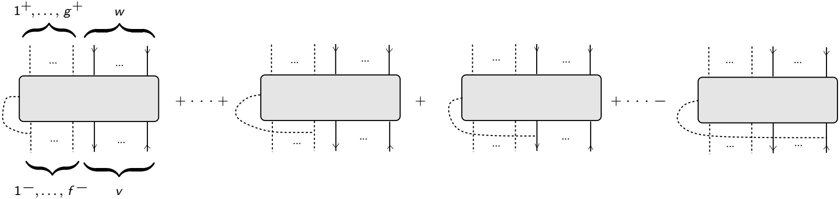
For all $f, g \in \mathbb{N}$ and for all associative words v, w in $+, -$, the subspaces of $\mathrm{Mor}_{{}^{ts}\mathcal{A}}\big((w, g), (v, f)\big)$ spanned by diagrams of the form



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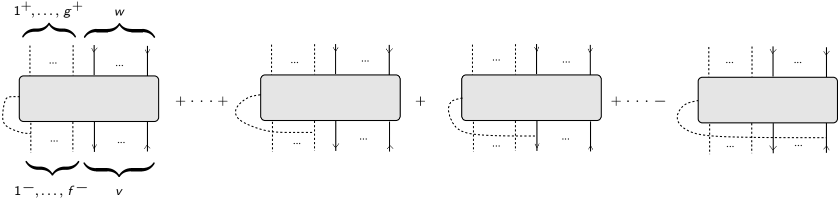

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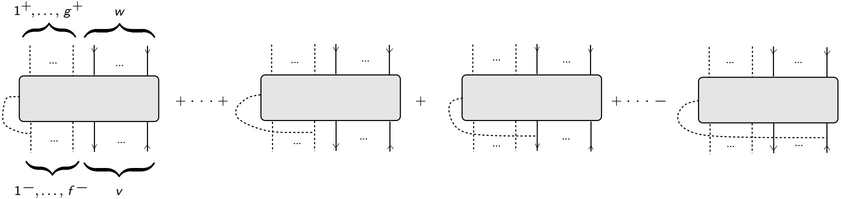

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Theorem (CHM'08 for $\mathcal{LC}ob$)

There exists a unique functor $\widetilde{Z} : \widehat{\mathcal{T}_q\mathcal{LC}ob} \longrightarrow \widehat{{}^{ts}\mathcal{A}}$ *such that*

$$\begin{tikzcd}
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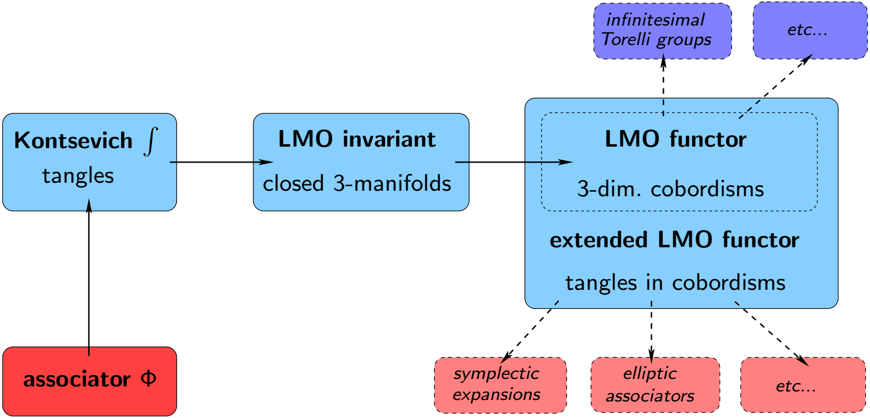
So far...



Kontsevich $\int$

tangles

LMO invariant

closed 3-manifolds

LMO functor

3-dim. cobordisms

extended LMO functor

tangles in cobordisms

associator Φ

infinitesimal Torelli groups

etc...

symplectic expansions

elliptic associators

etc...

Plan

1. Review of the Kontsevich integral
2. Review of the LMO invariant
3. Construction of the LMO functor
4. The LMO homomorphism

The monoid of string-links in homology cylinders

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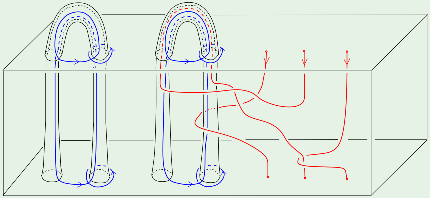

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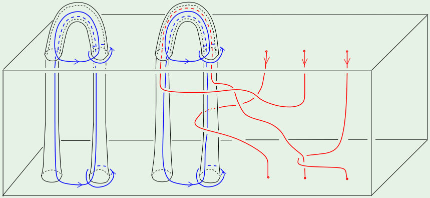

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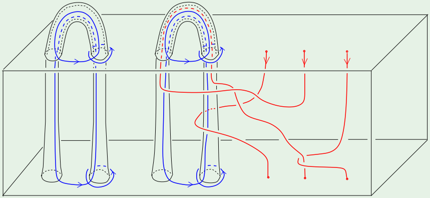

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There is a similar monoid $\widehat{\mathcal{SC}yl}_{g,n}$ if the surface F_g is replaced by $\widehat{F_g}$.

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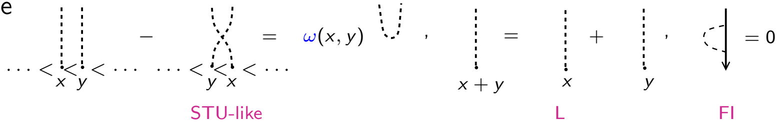

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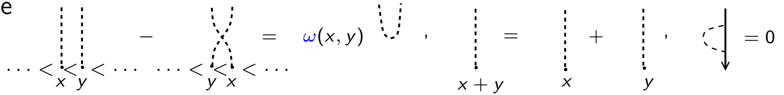

STU-like

L

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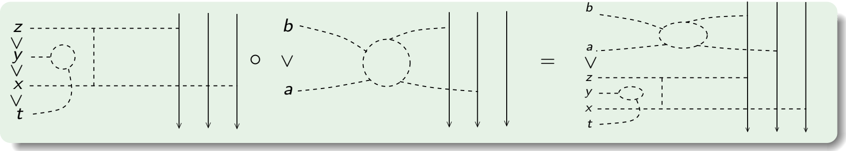
There is an associative multiplication $\circ$ on $\mathcal{A}^{<}_{g,n}$:



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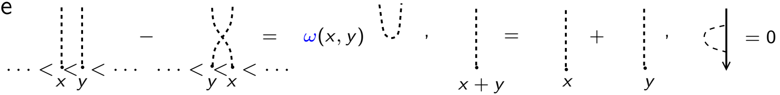

STU-like L FI

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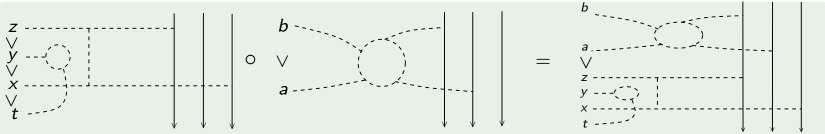
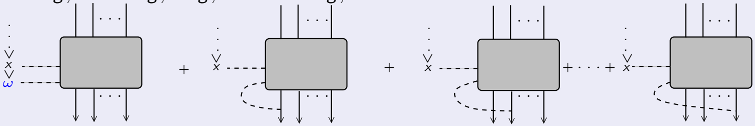


Set $\widehat{\mathcal{A}}^{<}_{g,n} := \mathcal{A}^{<}_{g,n}/I^{<}_{g,n}$ where $I^{<}_{g,n}$ is spanned by



The LMO homomorphism

Theorem (CHM'08 & HM'09 for $n = 0$)

The LMO functor $\widetilde{Z}$ induces monoid homomorphisms

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\mathcal{S}\mathcal{C}yl_{g,n} \arrow[r, dotted, "Z^<"] \arrow[d, two heads] & \mathcal{A}^<_{g,n} \arrow[d, two heads] \\
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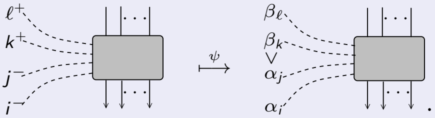
$$\forall (M,\tau) \in \mathcal{S}\mathcal{C}yl_{g,n}, \quad \widetilde{Z}(M,\tau) = \exp_\sqcup \left(\sum_{i=1}^{g} \begin{smallmatrix} i^+ \\ \vdots \\ i^- \end{smallmatrix} \right) \sqcup \underbrace{\widetilde{Z}^Y(M,\tau)}_{\in \mathcal{A}^Y_{g,n}}$$

Set $Z^< := \psi \circ \widetilde{Z}^Y$ where $\psi : \mathcal{A}^Y_{g,n} \xrightarrow{\simeq} \mathcal{A}^<_{g,n}$ is defined by



□

Application of the LMO homomorphism to some groups (1/2)

Each of the following groups G embeds into a monoid M of string-links in homology cylinders, and it is thus mapped to a diagrammatic algebra A:

$$\begin{tikzcd} G \arrow[r, hook] \arrow[rr, dotted, bend right, "Z^{<}"'] & M \arrow[r, "Z^{<}"] & A \end{tikzcd}$$

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⚠ This map $Z^< : G \longrightarrow A$ depends on the associator Φ and the system of meridians & parallels $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ on F_g.

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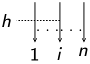
$PB_n(\widehat{F_g})$	$\widehat{\mathcal{A}}^<_{g,n}$	$\frac{T\left(H^{\oplus n}\right)}{\langle \text{quad. \& cubic rel.} \rangle}, g \geq 1$ (Bezrukavnikov'94, Nakamura–Takao–Ueno'95)	$h^{(i)} \mapsto$  (Humbert'12)	probably YES OK if $g = 1$ and $n \in \{2, 3\}$ (Katz'15)

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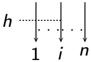
$PB_n(\widehat{F_g})$	$\widehat{\mathcal{A}}^{<}_{g,n}$	$\frac{T\left(H^{\oplus n}\right)}{\langle \text{quad. \& cubic rel.} \rangle}, g \geq 1$ (Bezrukavnikov'94, Nakamura–Takao–Ueno'95)	$h^{(i)} \mapsto$  (Humbert'12)	probably YES OK if $g = 1$ and $n \in \{2,3\}$ (Katz'15)
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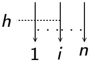
$PB_n(\widehat{F_g})$	$\widehat{\mathcal{A}}^<_{g,n}$	$\frac{T\left(H^{\oplus n}\right)}{\langle \text{quad. \& cubic rel.} \rangle}$, $g \geq 1$ (Bezrukavnikov'94, Nakamura–Takao–Ueno'95)	$h^{(i)} \mapsto$  (Humbert'12)	probably YES OK if $g = 1$ and $n \in \{2,3\}$ (Katz'15)
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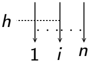
$PB_n(\widehat{F_g})$	$\widehat{\mathcal{A}}^<_{g,n}$	$\frac{T\left(H^{\oplus n}\right)}{\langle \text{quad. \& cubic rel.}\rangle}, g \geq 1$ (Bezrukavnikov'94, Nakamura–Takao–Ueno'95)	$h^{(i)} \mapsto$  (Humbert'12)	probably YES OK if $g = 1$ and $n \in \{2,3\}$ (Katz'15)
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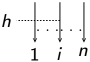
$PB_n(\widehat{F_g})$	$\widehat{\mathcal{A}}^<_{g,n}$	$\frac{T\left(H^{\oplus n}\right)}{\langle\text{quad. \& cubic rel.}\rangle}$, $g \geq 1$ (Bezrukavnikov'94, Nakamura–Takao–Ueno'95)	$h^{(i)} \mapsto$  (Humbert'12)	probably YES OK if $g = 1$ and $n \in \{2,3\}$ (Katz'15)
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After "homotopic" reduction, $Z^< : PB_2(\widehat{F_1}) \longrightarrow \widehat{\mathcal{A}}^<_{1,2}$ recovers Enriquez' formulas building an elliptic associator $(\Phi, X(\Phi), Y(\Phi))$ from Φ (Katz'15).