DIABLERETS TALK (1)

(0) Rough sketch

KV conjecture 1978, A-M 2006, A-T 2008 reformulation

Find $F \in TAut(\widehat{lie}(x,y))$, "a" power series s.t. (not say)

- $F(x+y) = \cancel{BCH(x,y)}\ \log(e^x e^y) = x + y + \frac{1}{2}[x,y] + \dots$
- $jF = a(x) + a(y) - a(\log(e^x e^y))$ (not say)

j cocycle: $j(gh) = j(g) + g \cdot j(h)$ $\quad j: TAut_n \to tr_n$

$\frac{d}{ds} j(\exp(sD))\big|_{s=0} = div\, D$

$div: tder_n \to tr_n \quad div(a_1, \dots, a_n) = \sum_{k=1}^{n} tr(\underbrace{(\partial_k a_k)\, x_k}_{\text{delete all words in } a_k \text{ which do not end in } x_k})$

A-T 2008

Soln of KV $\rightsquigarrow$ associator (w/ values in "KRV_3^0")

F

proven but not explicit

explicit formula by A-E-T 2009 (from Drinfel'd assoc.)
(also Bazarchenko)

Relnship to topology

Topological approach to "eqns in graded spaces"

$K \circlearrowleft$ ops — some space of "knotted objects" w/ operations
e.g. braids w/ multiplication ~~+ strand double~~
Awenall's q-tangles w/ comp. + strand double

forms an "algebraic structure" (kinds of objects = skeleta e.g. for braids $B_n \to S_n$)
assume K finitely presented

$K = \langle gens \mid rels \rangle$

$\mathbb{Q}K$ (same kinds) I = augm. ideal $= \langle \sum c_n k_n \mid \sum c_n = 0 \rangle$

$\mathbb{Q}K \supseteq I \supseteq I^2 \supseteq \dots$ filtration

$\mathcal{A} := gr\, K = \prod_{n=0}^{\infty} I^n / I^{n+1}$ ($\mathcal{A}$ is always degree completed)

expansion $Z: \mathbb{Q}K \to \mathcal{A}$ lin. filtered map
s.t. $gr\, Z: \mathcal{A} \xrightarrow{id} \mathcal{A}$

DIABLERETS TALK (2)

Ops on $\mathcal{K}$ ⇝ induced ops on $\mathcal{A}$

Z homomorphic if

$$\begin{tikzcd} \mathcal{K} \arrow[r, "op"] \arrow[d, "Z"'] & \mathcal{K} \arrow[d, "Z"] \\ \mathcal{A} \arrow[r, "\text{gr op}"'] & \mathcal{A} \end{tikzcd}$$

for all ops.

How to find a homomorphic expansion?

Need Z(gens) in $\mathcal{A}$

relns ⇝ eqns in $\mathcal{A}$ }

homomorphic ⇝ eqns in $\mathcal{A}$ } solve these eqns.

$Z \xleftrightarrow{1-1}$ solns of eqns in some graded space ~~Associators~~ ~~KV~~

"3D knot theory" $\xleftrightarrow{\text{M-O, C-L, BN, D}}$ Drinfel'd Associators

"4D knot theory" $\xleftrightarrow{\text{BN - D}}$ KV

$$\begin{tikzcd} \Phi \arrow[d, "\text{Ass} \leftrightarrow Z^u"'] & \text{3D knotted object} \arrow[rr, "a"] \arrow[rr, "\text{"tubing"/"spinning"}"'] \arrow[d] & & \text{4D knotted object} \arrow[d, "Z^w \leftrightarrow \text{Sol KV}"] \\ \Phi & \mathcal{A}^u \arrow[rr, "\alpha = \text{gr } a"'] & & \mathcal{A}^w \end{tikzcd}$$

explains $F ⇝ \Phi$ direction of AT

If $\exists Z^w$ s.t. this diag. commutes then get Φ in KRV^0_s

Drinf. ass.

$\mathcal{A}^u$ ∪ "$U(t_n)$" as in Gwenael, Benoit & others

$\mathcal{A}^w$ ∪ $TAut_n$, tr_n, KRV^0_n

But AET comes from where?

No "dimension decreasing map..."

The alg. sh. are very different & 3D is awkward

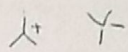
<u>Plan</u>: Show that gens of 4D can be expressed in terms of a(3D) & 4D ops, so $Z^u ⇝ Z^w$. Non-triv!

DIABLERETS TALK ③

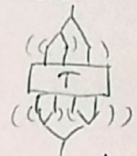
① 3D Knot Theory

KTGs = framed oriented trivalent 1- or 2-tangles

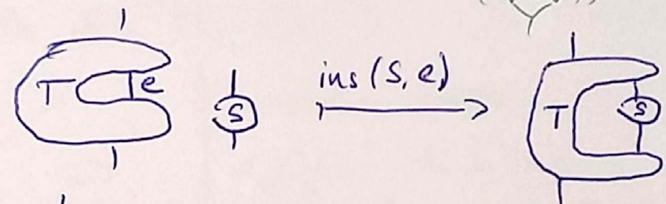
ignoring: vertex signs,
cyclic ori @ vertices ...



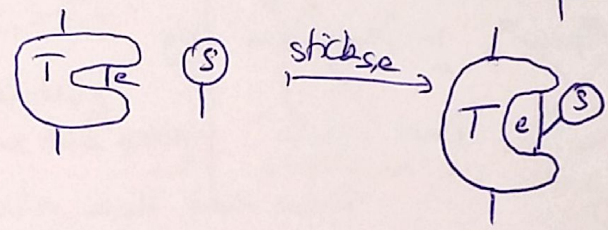
Very close to Gwenael's q-tangles



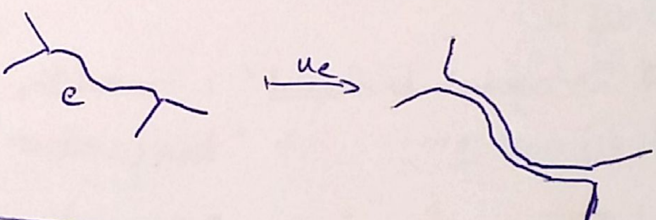
Operations:
- ori switch
- "insertion"



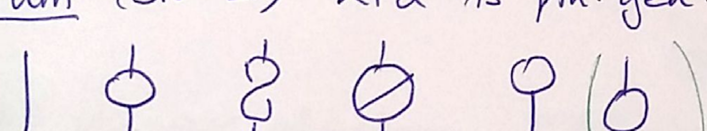
- "stick-on"



- ~~"ori switch"~~ "unzip"



Thm (BN-D) KTG is fin. gen. by



strand, bubble, twist, assoc, balloon, (noose)

also lt twist lt assoc noose

$\mathcal{A}^{u}$: Chord diags (Jacobi diags) on "trivalent skeleta" /relns (*)
(not necessarily horizontal chords but ignore)

Relns pentagon & hexagon for $Z(\text{twist}) =: R$, $Z(\text{assoc}) := \Phi$
(R, Φ) Drinfeld assoc.
the rest is trivial ("n·b = 0")

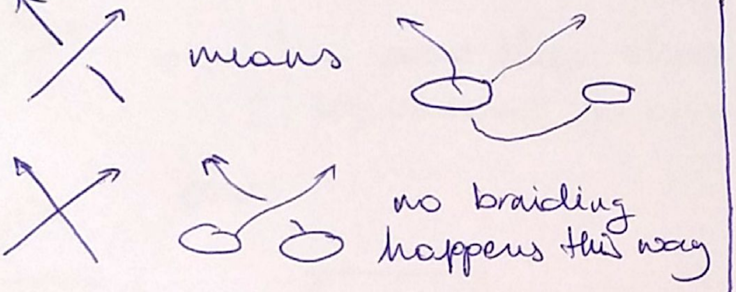
DIABLERETS TALK ④

② 4D Knot theory

The Braid group is "the group of crawling ants"
= π_1 (config. space of n disj. pts in the plane)
also has a "Reidemeister" ~~description~~ presentation in terms of gens & rels

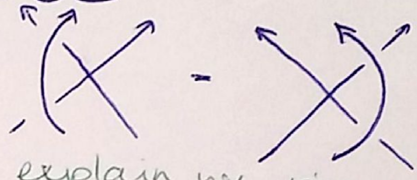
wB_n = "The group of flying rings"
= π_1 (config. space of n horizontal geom. circles in $\mathbb{R}^3$)

= ⟨  | R3, VR3, LS, M, OC ⟩



~~explain R2~~

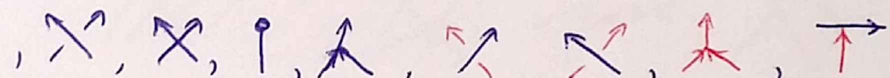
∘ OC



explain w/ rings

= braided tubes in $\mathbb{R}^4$

We need something a bit richer: "knotted tubes in $\mathbb{R}^4$ w/ trivalent vertices + strings attached"

wTF := PA ⟨  | rels / ops ⟩

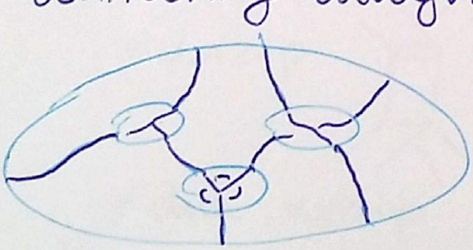
several kinds

(actually a bit more but ignoring now)

PA = planar algebra:
- elts are pictures
- PA product given by

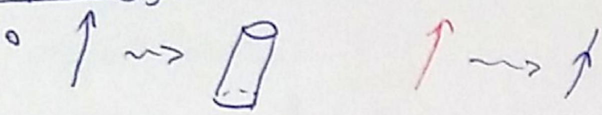
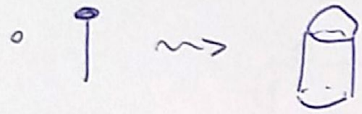
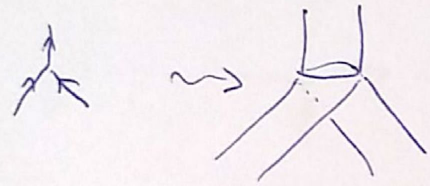
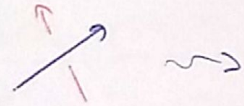
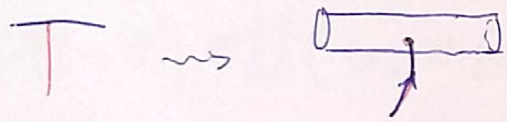
"connecting diagrams": planar connections

(there are colours & oris which have to match)

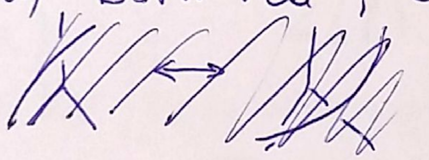
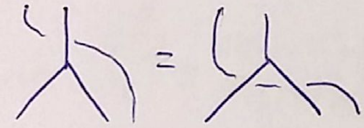
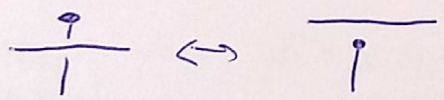


DIABLERETS TALK 5

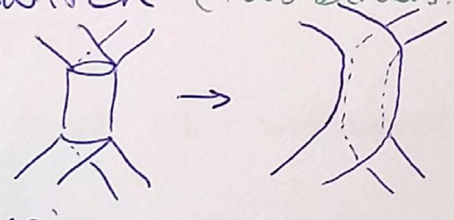
Topology

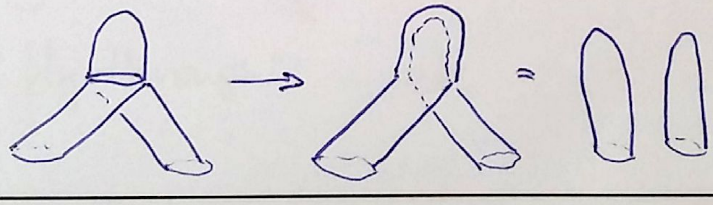
- 
- crossings as before
-  cap (ring shrinks & disappears)
-  "foam vertex" not pair of pants
  show w/ rings
-  "a point flies through a ring"
  (red cannot go over)
-  obvi
- 

Relns

- R-moves w/ both red & black as makes sense
  incl: R4  note 
  etc...
- OC
- CP 

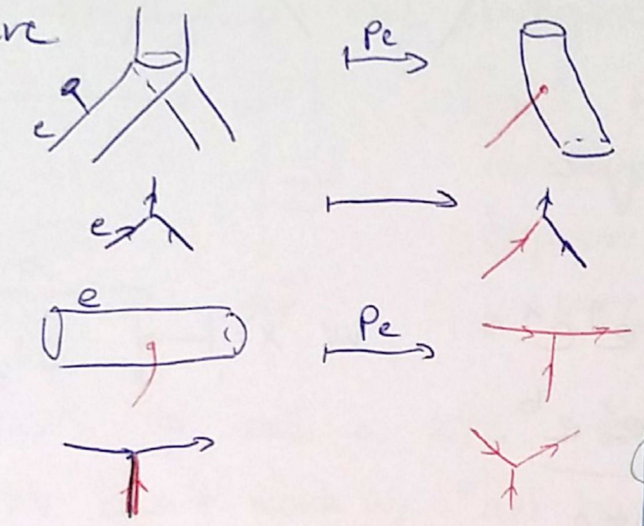
OPs

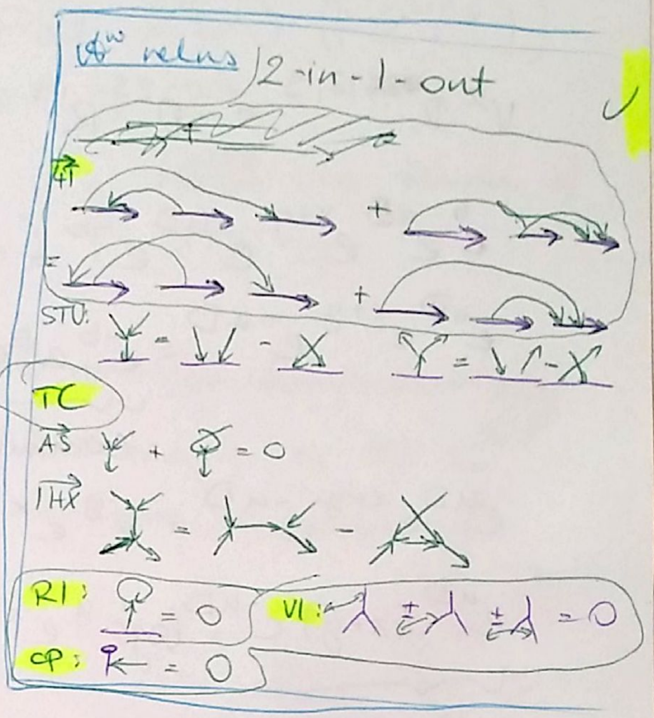
- PA products
- ori switch (two kinds...)
- unzip  "double along framing"
- disk unzip



DIABLERETS TALK ⑥

• puncture





Assoc graded

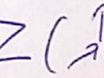
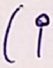
$\mathcal{A}^w$ = "arrow diagrams on wTF skeleta" / relns

Turns out

$\mathcal{A}^w(\uparrow\uparrow\cdots\uparrow) \cong U(\mathrm{tr}_n \rtimes (\mathrm{tder}_n \oplus \underset{\text{easy}}{\overset{a_n}{\text{sg}}}))$

(tder_n acts on tr_n : commute wheel across a tree.) NOT TO SAY

Homom expansion

determined by Z() = V Z() = C

V is the soln F to KV [V is an arrow diag on 2 strands  $F(x) = V^{-1}xV$, $F(y) = V^{-1}yV$]

R4  $\rightsquigarrow$ $F(x+y) = \log(e^x e^y)$ ⊛

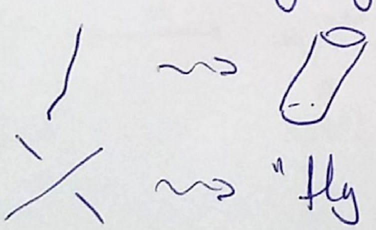
~~homom~~ homom. of unzip & disk unzip } $\rightsquigarrow$ $jF = a(x) + a(y) - a(\log(e^x e^y))$ w/ $a = -2\log C$

check the $jF \in \operatorname{im}\tilde{\delta}$ formulation
$\tilde{\delta}: \mathrm{tr}_1 \to \mathrm{tr}_2$
$\tilde{\delta}(a)(x,y) = a(x) + a(y) - a(BCH(x,y))$

③ The 3D / 4D relationship

a: KTG $\longrightarrow$ wTF

combinatorially: just re-interpret the diagrams



⊛ topol. descr. of tubing

DIABLERETS TALK ⑦

We are studying the diagram

$$\begin{array}{ccc} KTG & \xrightarrow{a} & wTF \\ Z^u \downarrow & & \vdots\, Z^w \\ \mathcal{A}^u & \xrightarrow{\alpha} & \mathcal{A}^w \end{array}$$

<u>Goal</u>: Find $Z^w(\curlywedge)$ ($;\ Z^w(\uparrow)$) in terms of Z^u.
Express $\curlywedge$ in terms of $a(KTG)$; wTF ops.

<u>Problem</u> $\curlywedge$ is <u>not</u> a KTG. ~~KTG is not a planar alg~~.
We can't work w/ "all trivalent tangles in $\mathbb{R}^3$" bc/ those don't have a homom. expansion.

<u>Thm</u> (Bar-Natan - D)

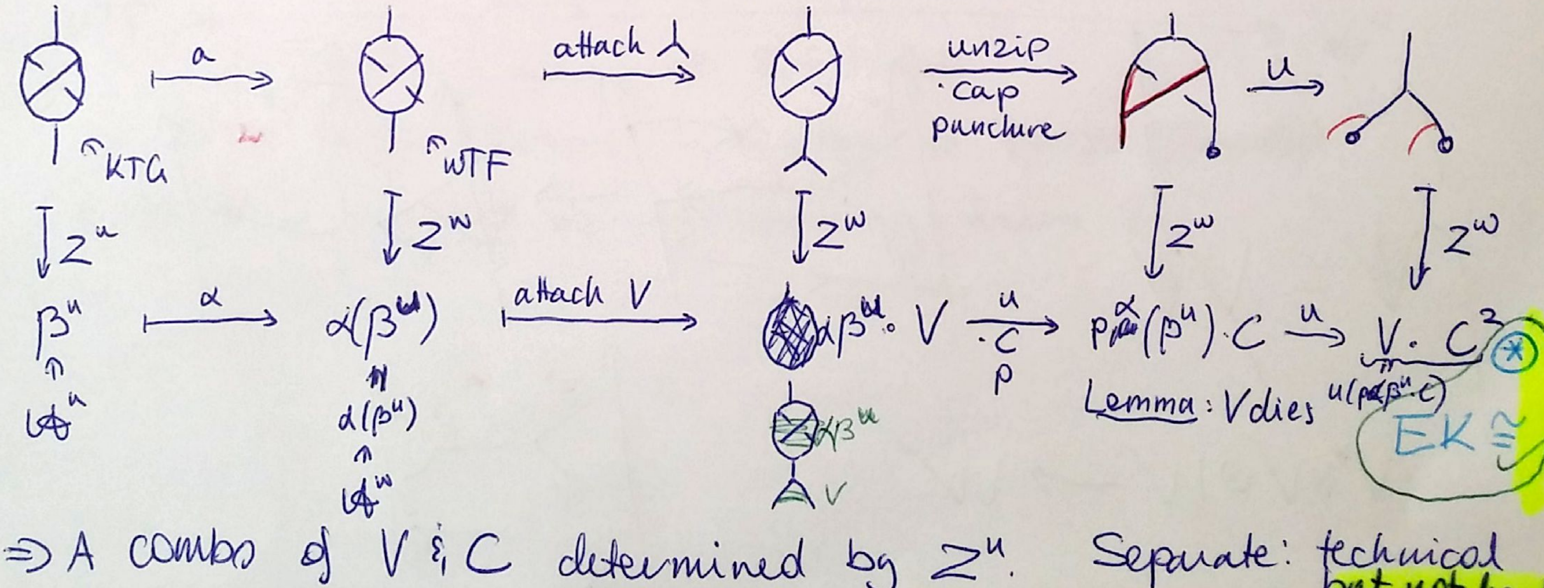
① <u>If</u> a compatible Z^w exists, it is determined by Z^u.

② The formula for V in terms of Z^u is exactly the AET formula for F from Φ.

③ For any Z^u (given by a horiz. chord Φ) a compatible Z^w exists. ($\Rightarrow$ KV has solns).

<u>Proof</u>

① $Z^u \rightsquigarrow V, C$, <u>assuming</u> Z^w exists.

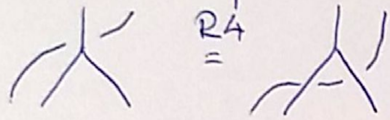


$\Rightarrow$ A combo of V ; C determined by Z^u. Separate: technical but not hard

DIABLERETS TALK (8)

(2) An exercise in translation

(3) "Only" need to check that the V ; C obtained from (1) satisfies the equations. Hard :(.

If only  $\subset$ uTT "usual trivalent tangles"

3D

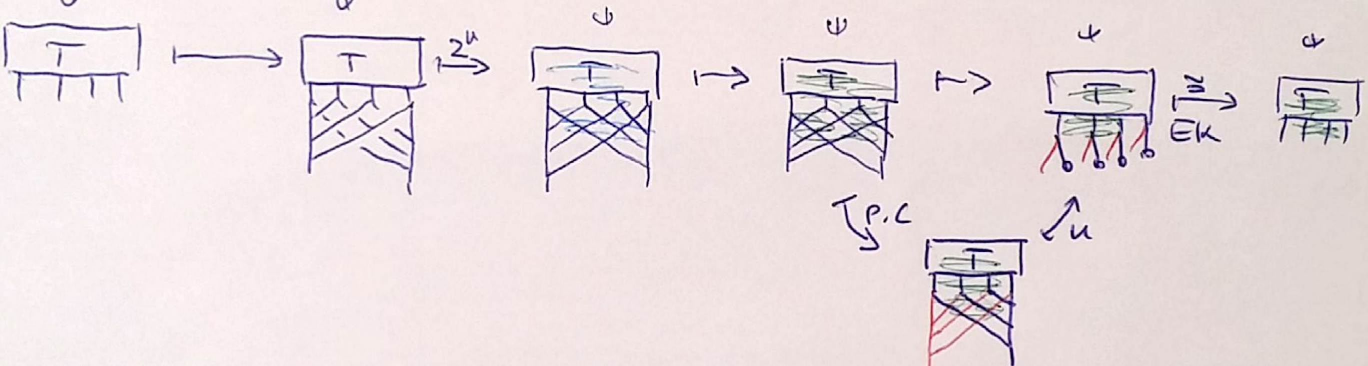
$$\begin{tikzcd}
R4 \arrow[r, "\alpha\ \cancel{\text{PA map}}"] & R4 \\
uTT \arrow[u, phantom, "\cap"] \arrow[d, "\cancel{2^u}\ \text{"PA map"}"] & wTF \arrow[u, phantom, "\cap"] \arrow[d, "2^w"] \\
\text{"}\mathcal{A}^u\text{"} \arrow[r, "\alpha", "\cancel{\text{PA}}\ \text{map}"'] & \mathcal{A}^w
\end{tikzcd} \qquad V := \alpha(2^u(\lambda))$$

But we can imitate this

wTF (no X, ?, |)

$\cup$

$$uTT \xrightarrow{\varphi} KTG \xrightarrow{2^u} \mathcal{A}^u(\varphi(T)) \xrightarrow{\alpha} \mathcal{A}^w(\varphi(T)) \xrightarrow{p,c,u} \mathcal{A}^w(\check{T}) \cong \mathcal{A}^w(T)$$



uTT $\xrightarrow{\text{composition}}$ $\mathcal{A}^w$

- PA - map! $\Rightarrow$ R4 holds
- can extract V, C from it ~~[illegible]~~

+ The V, C we get is the same as from (1)

∎