Expansions: A Loosely Tied Traverse from Feynman Diagrams to Quantum Algebra

Geometric, Algebraic, and Topological Methods for Quantum Field Theory,
Villa de Leyva, Colombia

Dror Bar-Natan, July 2011,
http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107/

Abstract. Assuming lots of luck, in six classes we'll talk about

1. Perturbed Gaussian integration in $\mathbb{R}^n$ and Feynman diagrams.
2. Chern-Simons theory, knots, holonomies and configuration space integrals.
3. Finite type invariants, chord and Jacobi diagrams and "expansions".
4. Drinfel'd associators and knotted trivalent graphs.
5. w-Knotted objects and co-commutative Lie bi-algebras.
6. My dreams on virtual knots and and quantization of Lie bi-algebras.

Each class will be closely connected to the next, yet the first and last will only be very loosely related.

The u→v→w & p Stories

explained | sketched | could explain | could explain, gaps remain | more gaps then explains | mystery

	Topology	Combinatorics	Low Algebra	High Algebra	Counting Coincidences Conf. Space Integrals	Quantum Field Theory	Graph Homology
u-Knots	The usual Knotted Objects (KOs) in 3D — braids, knots, links, tangles, knotted graphs, etc.	Chord diagrams and Jacobi diagrams, modulo $4T$, STU, IHX, etc.	Finite dimensional metrized Lie algebras, representations, and associated spaces.	The Drinfel'd theory of associators.	Today's work. Not beautifully written, and some detour-forcing cracks remain.	Perturbative Chern-Simons-Witten theory.	The "original" graph homology.
↓ v-Knots	Virtual KOs — "algebraic", "not embedded"; KOs drawn on a surface, mod stabilization.	Arrow diagrams and v-Jacobi diagrams, modulo $6T$ and various "directed" STUs and IHXs, etc.	Finite dimensional Lie bi-algebras, representations, and associated spaces.	Likely, quantum groups and the Etingof-Kazhdan theory of quantization of Lie bi-algebras.	No clue.	No clue.	No clue.
↓ w-Knots	Ribbon 2D KOs in 4D; "flying rings". Like v, but also with "overcrossings commute".	Like v, but also with "tails commute". Only "two in one out" internal vertices.	Finite dimensional co-commutative Lie bi-algebras ($\mathfrak{g} \ltimes \mathfrak{g}^*$), representations, and associated spaces.	The Kashiwara-Vergne-Alekseev-Torossian theory of convolutions on Lie groups / algebras.	No clue.	Probably related to 4D BF theory.	Studied.
p-Objects	No clue.	"Acrobat towers" with 2-in many-out vertices.	Poisson structures.	Deformation quantization of poisson manifolds.	Configuration space integrals are key, but they don't reduce to counting.	Work of Cattaneo. 	Studied.

Video and more at http://www.math.toronto.edu/~drorbn/Talks//Tennessee-1103/

Stonehenge

From Stonehenge to Witten Skipping all the Details

Oporto Meeting on Geometry, Topology and Physics, July 2004

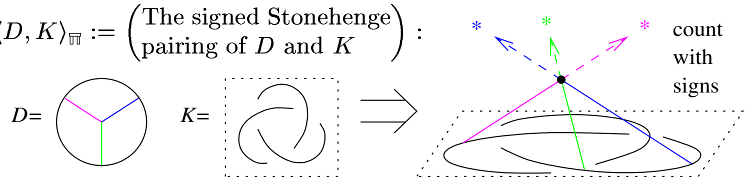
Dror Bar–Natan, University of Toronto

Witten

It is well known that when the Sun rises on midsummer's morning over the "Heel Stone" at Stonehenge, its first rays shine right through the open arms of the horseshoe arrangement. Thus astrological lineups, one of the pillars of modern thought, are much older than the famed Gaussian linking number of two knots.

Recall that the latter is itself an astrological construct: one of the standard ways to compute the Gaussian linking number is to place the two knots in space and then count (with signs) the number of shade points cast on one of the knots by the other knot, with the only lighting coming from some fixed distant star.

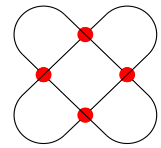
$\langle D, K\rangle_{\pi} :=$ (The signed Stonehenge pairing of D and K) :



The Gaussian linking number

$lk(\text{link}) = \frac{1}{2}\sum_{\text{vertical chopsticks}} (\text{signs})$

Carl Friedrich Gauss



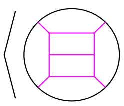
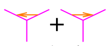
lk=2

Thus we consider the generating function of all stellar coincidences:

Dylan Thurston

$$Z(K) := \lim_{N\to\infty} \sum_{\text{3-valent } D} \frac{1}{2^c c! \binom{N}{e}} \langle D, K\rangle_{\pi} D \cdot \begin{pmatrix}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{pmatrix} \in \mathcal{A}(\circlearrowleft)$$

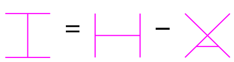
N := # of stars
c := # of chopsticks
e := # of edges of D

$\mathcal{A}(\circlearrowleft)$:=Span ⟨  ⟩ / oriented vertices, AS:  + = 0 & more relations

Theorem. Modulo Relations, $Z(K)$ is a knot invariant!

When deforming, catastrophes occur when:

A plane moves over an intersection point – Solution: Impose IHX,



(see below)

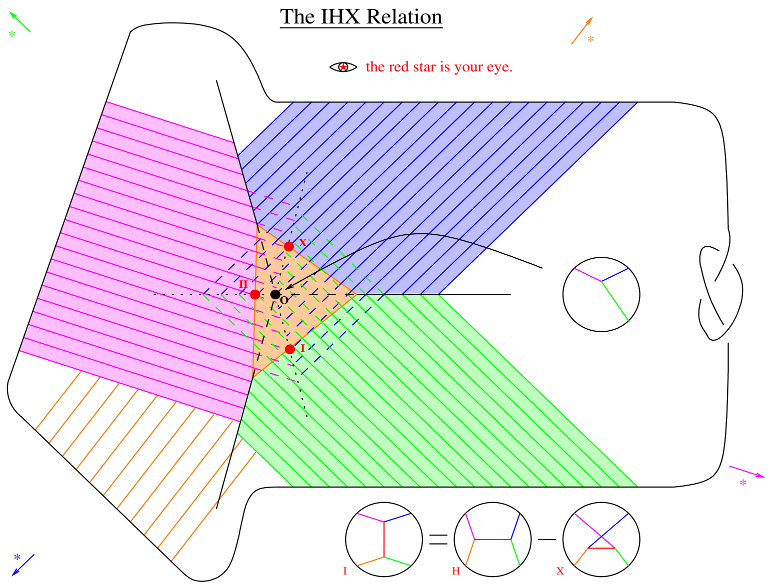
An intersection line cuts through the knot – Solution: Impose STU,

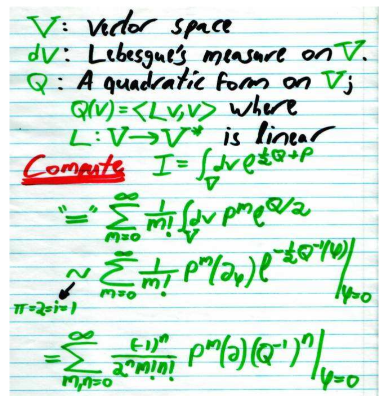
(similar argument)

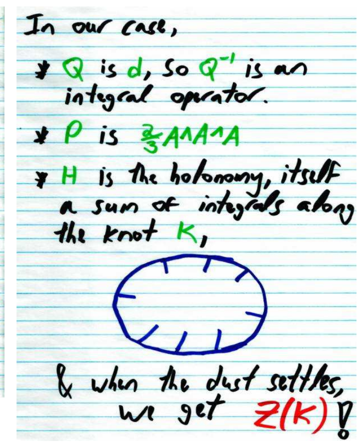
The Gauss curve slides over a star – Solution: Multiply by a framing–dependent counter–term.

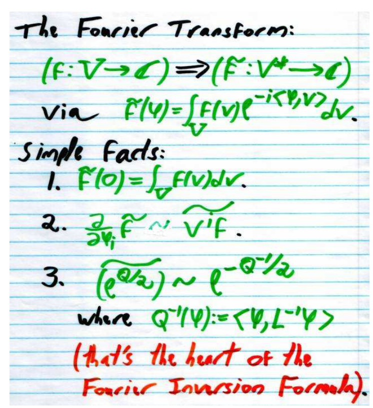
(not shown here)

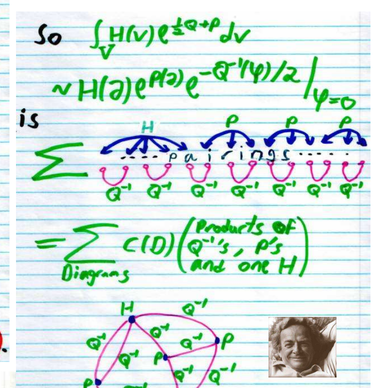
The IHX Relation



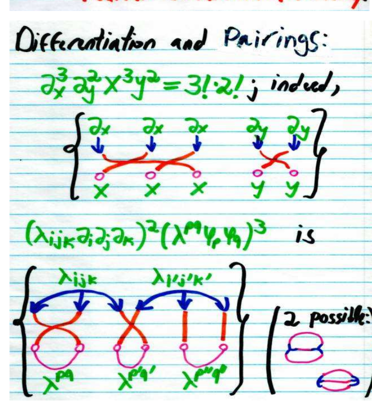




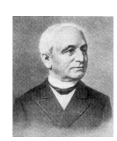




Richard Feynman



"God created the knots, all else in topology is the work of man."



Leopold Kronecker

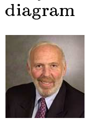
(modified)

It all is perturbative Chern–Simons–Witten theory:

$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A\, hol_K(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right]$$

$$\to \sum_{D:\text{ Feynman diagram}} W_{\mathfrak{g}}(D) \sum\!\!\!\!\!\!/\ \mathcal{E}(D) \to \sum_{D:\text{ Feynman diagram}} D \sum\!\!\!\!\!\!/\ \mathcal{E}(D)$$

Shiing–shen Chern



James H Simons

This handout is at http://www.math.toronto.edu/~drorbn/Talks/Oporto–0407

More at `http://www.math.toronto.edu/~drorbn/Talks/Oporto-0407/`

From Stonehenge to Witten – Some Further Details

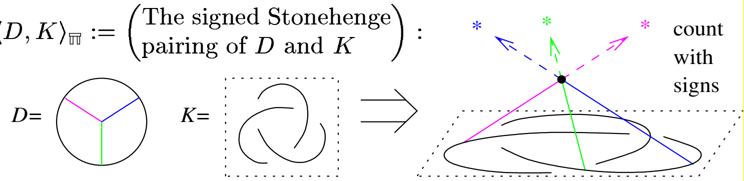
Oporto Meeting on Geometry, Topology and Physics, July 2004

Dror Bar–Natan, University of Toronto

We the generating function of all stellar coincidences:

$$Z(K) := \lim_{N\to\infty} \sum_{\text{3-valent } D} \frac{1}{2^c c! \binom{N}{e}} \langle D, K\rangle_{\Pi} D \cdot \begin{pmatrix}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{pmatrix} \in \mathcal{A}(\circlearrowleft)$$

$$\langle D, K\rangle_{\Pi} := \begin{pmatrix}\text{The signed Stonehenge}\\ \text{pairing of } D \text{ and } K\end{pmatrix}:$$



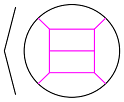
Theorem. Modulo Relations, $Z(K)$ is a knot invariant!

Dylan Thurston

N := # of stars

c := # of chopsticks

e := # of edges of D

$\mathcal{A}(\circlearrowleft)$:=Span $\left\langle\right.$  $\left.\right\rangle$ / oriented vertices, AS: + =0 & more relations

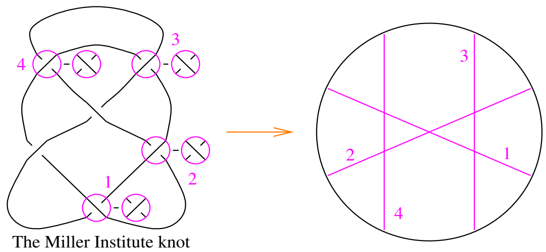
When deforming, catastrophes occur when:

A plane moves over an intersection point – Solution: Impose IHX,

An intersection line cuts through the knot – Solution: Impose STU,

The Gauss curve slides over a star – Solution: Multiply by a framing–dependent counter–term.

$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A\, hol_K(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right] \longrightarrow \sum_{D:\text{ Feynman diagram}} W_{\mathfrak{g}}(D) \sum\!\!\!\!\!\!\int \mathcal{E}(D) \longrightarrow \sum_{D:\text{ Feynman diagram}} D \sum\!\!\!\!\!\!\int \mathcal{E}(D)$$



The Miller Institute knot

Definition. V is finite type (Vassiliev, Goussarov) if it vanishes on sufficiently large alternations as on the right

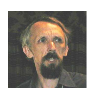
Theorem. All knot polynomials (Conway, Jones, etc.) are of finite type.

Goussarov

Conjecture. (Taylor's theorem) Finite type invariants separate knots.

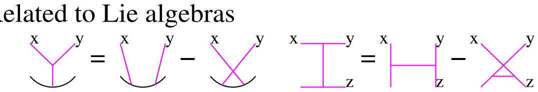
Theorem. $Z(K)$ is a universal finite type invariant!

(sketch: to dance in many parties, you need many feet).

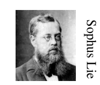


Vassiliev

Related to Lie algebras



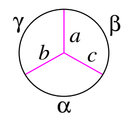
[x,y] = xy − yx   [[x,y],z]=[x,[y,z]]−[y,[x,z]]



More precisely, let $\mathfrak{g} = \langle X_a\rangle$ be a Lie algebra with an orthonormal basis, and let $R = \langle v_\alpha\rangle$ be a representation. Set

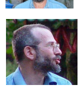
$$f_{abc} := \langle [a,b], c\rangle \qquad X_a v_\beta = \sum_\beta r^\beta_{a\gamma} v_\gamma$$

and then

$W_{\mathfrak{g},R}$:  $\longrightarrow \displaystyle\sum_{abc\alpha\beta\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$

$W_{\mathfrak{g},R} \circ Z$ is often interesting:

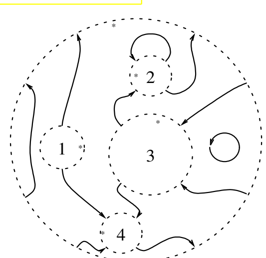
$\mathfrak{g} = sl(2)$ →  The Jones polynomial

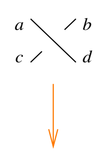
$\mathfrak{g} = sl(N)$ →  The HOMFLYPT polynomial

Przytycki

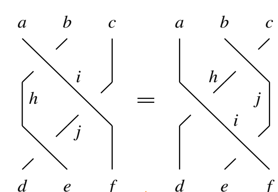
$\mathfrak{g} = so(N)$ →  The Kauffman polynomial

Planar algebra and the Yang–Baxter equation



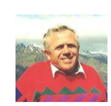


R^{ab}_{cd}

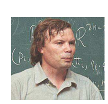


$$R^{ab}_{hi} R^{ic}_{jf} R^{hj}_{de} = R^{ah}_{di} R^{bc}_{hj} R^{ij}_{ef}$$

Yang

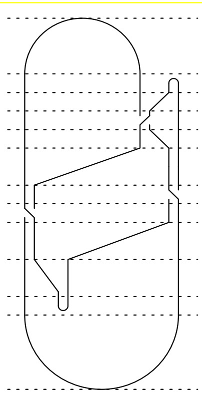


Baxter

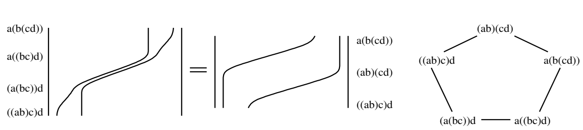


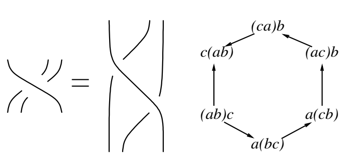
Reshetikhin

Turaev



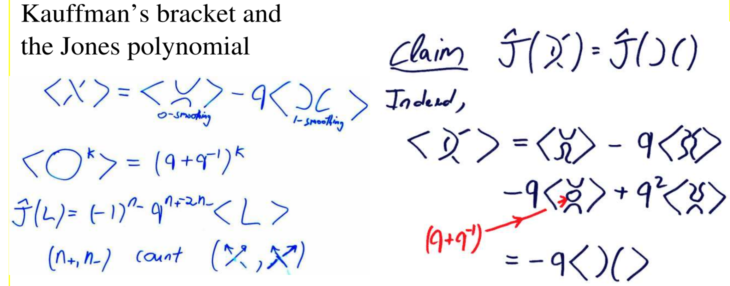
Parenthesized tangles, the pentagon and hexagon





Drinfel'd

Kauffman's bracket and the Jones polynomial



"God created the knots, all else in topology is the work of man."

This handout is at http://www.math.toronto.edu/~drorbn/Talks/Oporto–0407

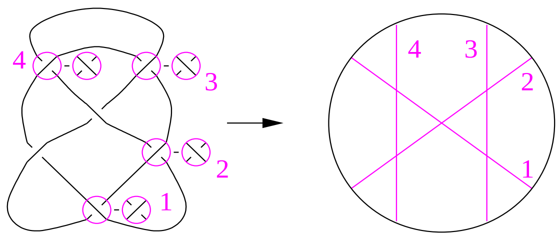
More at http://www.math.toronto.edu/~drorbn/Talks/Oporto-0407/

Knotted Trivalent Graphs, Tetrahedra and Associators

HUJI Topology and Geometry Seminar, November 16, 2000

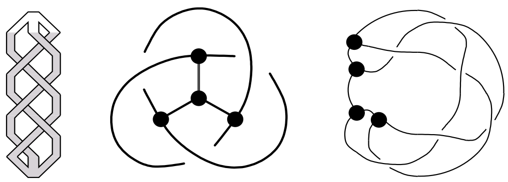
Dror Bar−Natan

Goal: Z:{knots}−>{chord diagrams}/4T so that

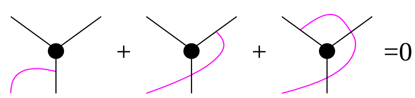


The Miller Institute knot

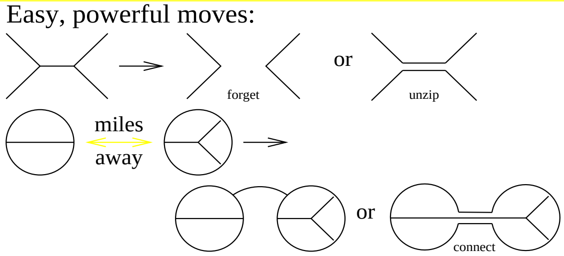
Extend to Knotted Trivalent Graphs (KTG's):

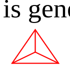


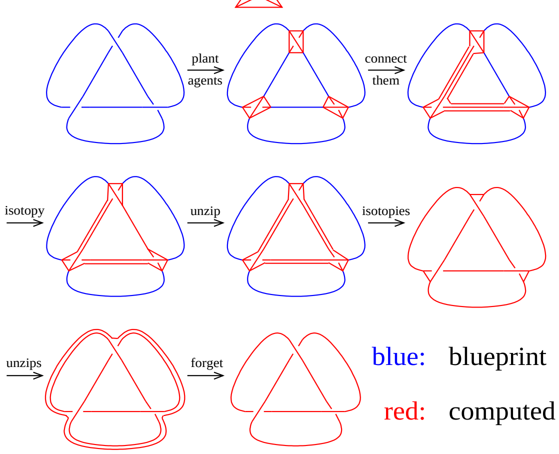
Need a new relation:

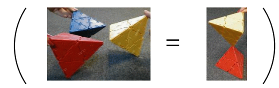


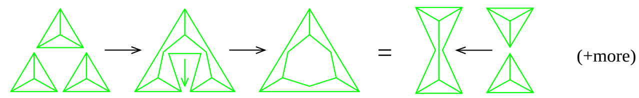
Easy, powerful moves:



Using moves, KTG is generated by ribbon twists and the tetrahedron  :

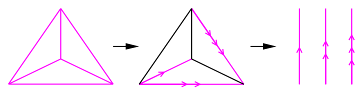


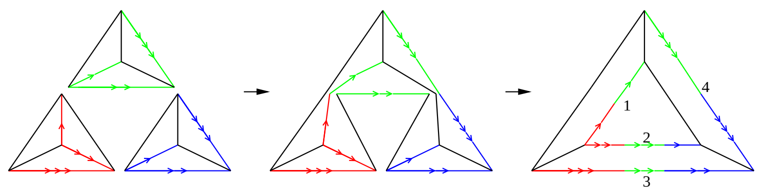
Modulo the relation(s): 

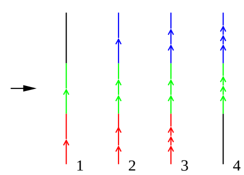


Claim. With $\Phi := Z(\triangle)$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi Hopf algebras.

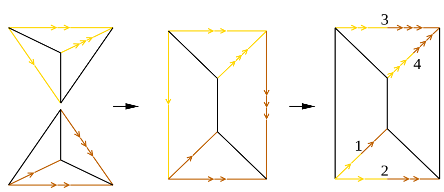
Proof.

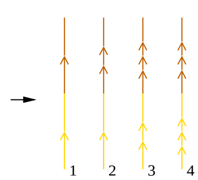
 $:= \Phi \in \mathcal{A}(\uparrow_3)$





$= (\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1)(\Phi) \cdot (1 \otimes \Phi) \in \mathcal{A}(\uparrow_4)$





$= (\Delta \otimes 1 \otimes 1)(\Phi) \cdot (1 \otimes 1 \otimes \Delta)(\Phi) \in \mathcal{A}(\uparrow_4)$

Further directions:

1. Relations with perturbative Chern−Simons theory.
2. Relations with the theory of 6j symbols
3. Relations with the Turaev−Viro invariants.
4. Can this be used to prove the Witten asymptotics conjecture?
5. Does this extend/improve Drinfel'd's theory of associators?

This handout is at http://www.ma.huji.ac.il/~drorbn/Talks/HUJI−001116

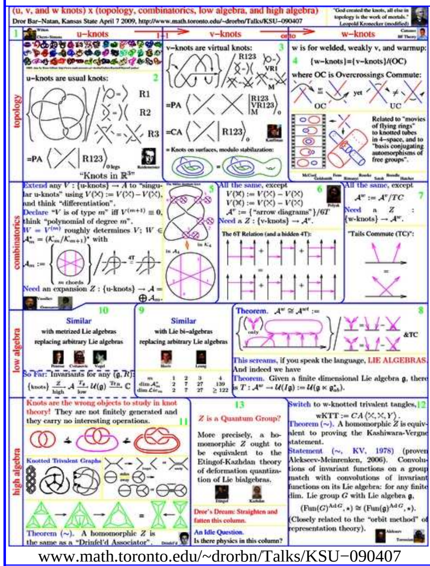
More at `http://www.math.toronto.edu/~drorbn/Talks/HUJI-001116/`

Convolutions on Lie Groups and Lie Algebras and Ribbon 2−Knots

Dror Bar−Natan, Bonn August 2009, http://www.math.toronto.edu/~drorbn/Talks/Bonn−0908

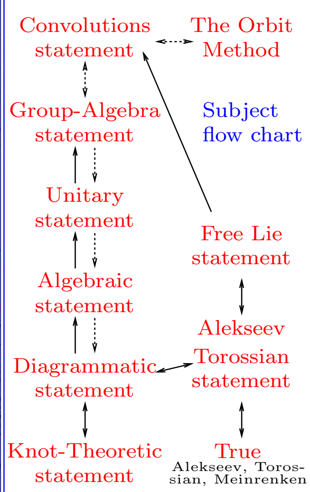
Disclaimer: Rough edges remain!

"God created the knots, all else in topology is the work of mortals." Leopold Kronecker (modified)



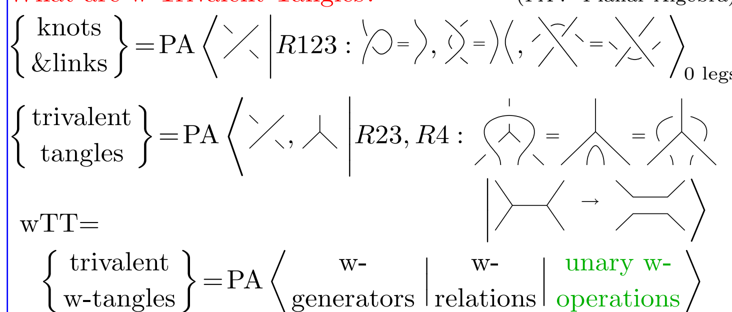
www.math.toronto.edu/~drorbn/Talks/KSU−090407

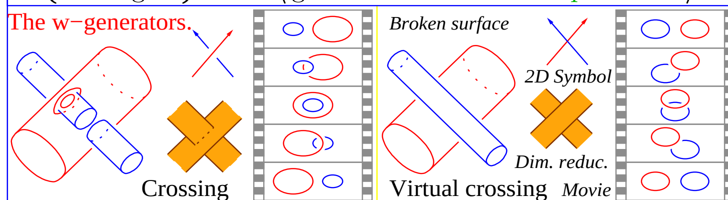
The Bigger Picture...

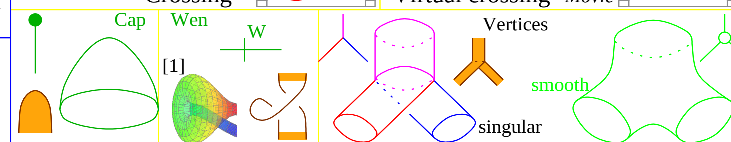


What are w-Trivalent Tangles?

(PA :=Planar Algebra)







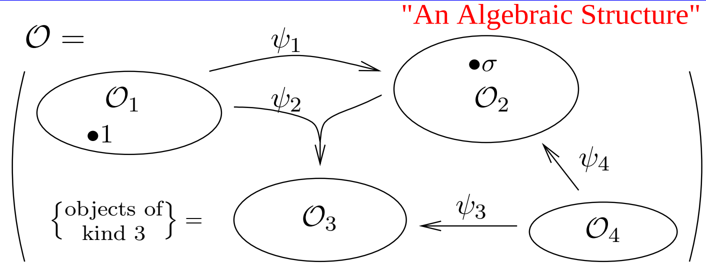
Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

$$\begin{array}{ccccccccc} \text{ops} \circlearrowright \mathcal{K} & = & \mathcal{K}_0 & \supset & \mathcal{K}_1 & \supset & \mathcal{K}_2 & \supset & \mathcal{K}_3 & \supset \dots \\ \Downarrow & & & & & & \downarrow Z & & \\ \text{ops} \circlearrowright \operatorname{gr}\mathcal{K} & := & \mathcal{K}_0/\mathcal{K}_1 & \oplus & \mathcal{K}_1/\mathcal{K}_2 & \oplus & \mathcal{K}_2/\mathcal{K}_3 & \oplus & \mathcal{K}_3/\mathcal{K}_4 & \oplus \dots \end{array}$$

An **expansion** is a filtration respecting $Z : \mathcal{K} \to \operatorname{gr}\mathcal{K}$ that "covers" the identity on $\operatorname{gr}\mathcal{K}$. A **homomorphic expansion** is an expansion that respects all relevant "extra" operations.

Filtered algebraic structures are cheap and plenty. In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle (\mathcal{K}_1)^m \rangle$ (using all available "products").

"An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

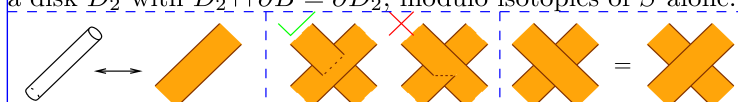
Example: Pure Braids. PB_n is generated by x_{ij}, "strand i goes around strand j once", modulo "Reidemeister moves". $A_n := \operatorname{gr} PB_n$ is generated by $t_{ij} := x_{ij} - 1$, modulo the $4T$ relations $[t_{ij}, t_{ik} + t_{jk}] = 0$ (and some lesser ones too). Much happens in A_n, including the Drinfel'd theory of associators.

Our case(s).

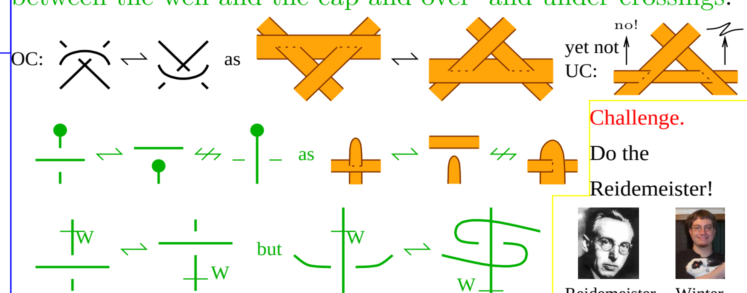
$$\mathcal{K} \xrightarrow[\text{solving finitely many equations in finitely many unknowns}]{Z\text{: high algebra}} \mathcal{A} := \operatorname{gr}\mathcal{K} \xrightarrow[\text{low algebra: pictures represent formulas}]{\text{given a "Lie" algebra } \mathfrak{g}} \text{"}\mathcal{U}(\mathfrak{g})\text{"}$$

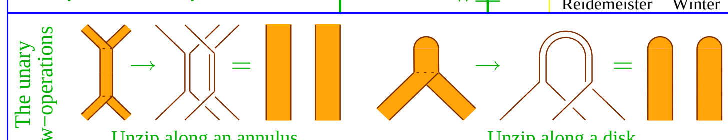
$\mathcal{K}$ is knot theory or **topology**; $\operatorname{gr}\mathcal{K}$ is finite **combinatorics**: bounded-complexity diagrams modulo simple relations.

A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.

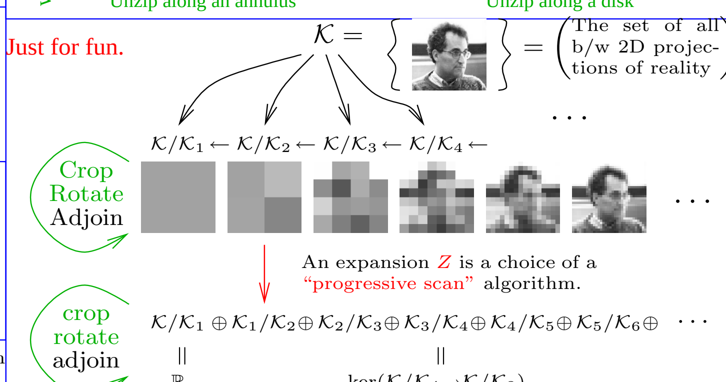


The w-relations include R234, VR1234, M, Overcrossings Commute (OC) but not UC, $W^2 = 1$, and funny interactions between the wen and the cap and over- and under-crossings:





Just for fun.

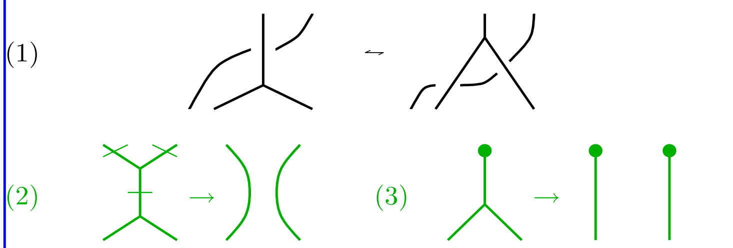


[1] http://qlink.queensu.ca/~4lb11/interesting.html 29/5/10, 8:42am

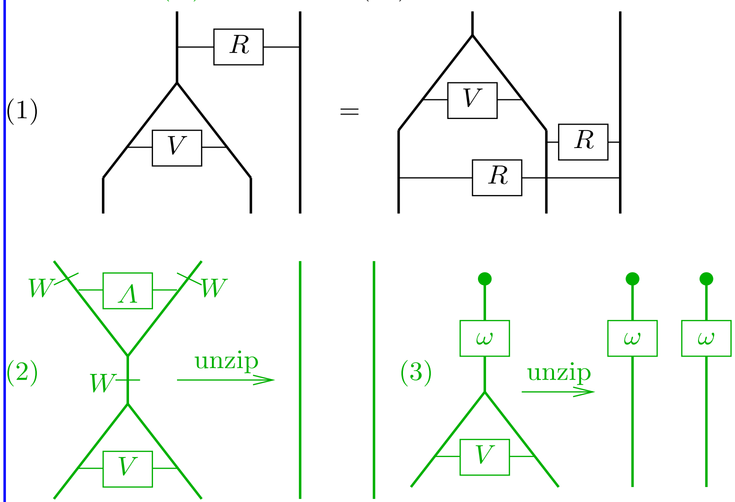
Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/Bonn-0908/`

Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:



Diagrammatic statement. Let $R = \exp \overrightarrow{H} \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that



Algebraic statement. With $I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g}$, with $c : \hat{\mathcal{U}}(I\mathfrak{g}) \to \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{S}}(\mathfrak{g}^*)$ the obvious projection, with S the antipode of $\hat{\mathcal{U}}(I\mathfrak{g})$, with W the automorphism of $\hat{\mathcal{U}}(I\mathfrak{g})$ induced by flipping the sign of $\mathfrak{g}^*$, with $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{\mathcal{U}}(I\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$ there exist $\omega \in \hat{\mathcal{S}}(\mathfrak{g}^*)$ and $V \in \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that
(1) $V(\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2} \otimes \hat{\mathcal{U}}(\mathfrak{g})$
(2) $V \cdot SWV = 1$ (3) $(c \otimes c)(V\Delta(\omega)) = \omega \otimes \omega$

Unitary statement. There exists $\omega \in \mathrm{Fun}(\mathfrak{g})^G$ and an (infinite order) tangential differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that
(1) $V\widehat{e^{x+y}} = \widehat{e^x}\widehat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)
(2) $VV^* = I$ (3) $V\omega_{x+y} = \omega_x\omega_y$

Group-Algebra statement. There exists $\omega^2 \in \mathrm{Fun}(\mathfrak{g})^G$ so that for every $\phi, \psi \in \mathrm{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $\omega^2 = j^{1/2}$)

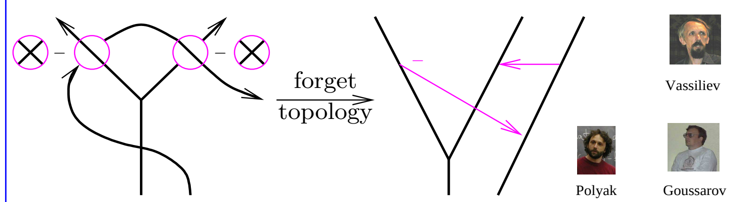
$$\iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2_{x+y}e^{x+y} = \iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2_x\omega^2_y e^x e^y.$$

(shhh, this is Duflo)

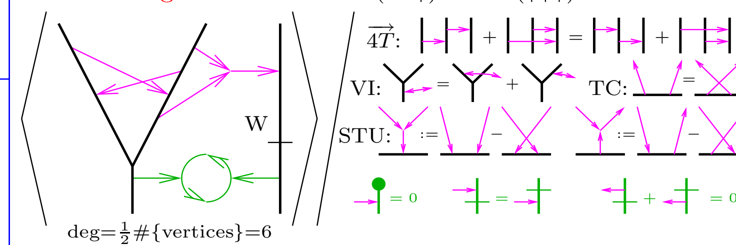
Convolutions statement (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, let $j : \mathfrak{g} \to \mathbb{R}$ be the Jacobian of the exponential map $\exp : \mathfrak{g} \to G$, and let $\Phi : \mathrm{Fun}(G) \to \mathrm{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := j^{1/2}(x)f(\exp x)$. Then if $f, g \in \mathrm{Fun}(G)$ are Ad-invariant and supported near the identity, then

$$\Phi(f) \star \Phi(g) = \Phi(f \star g).$$

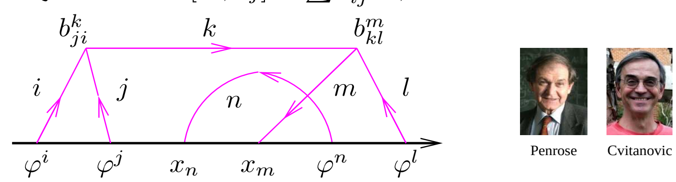
From wTT to $\mathcal{A}^w$. $\mathrm{gr}_m\,\mathrm{wTT} := \{m\text{−cubes}\}/\{(m+1)\text{−cubes}\}$:



w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is



Diagrammatic to Algebraic. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i, x_j] = \sum b^k_{ij}x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via



$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}} b^k_{ij}b^m_{kl}\varphi^i\varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g})$$

Unitary $\Longleftrightarrow$ Algebraic. The key is to interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$:
- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\mathrm{ad}\,x$: $(x\varphi)(y) := \varphi([x, y])$.
- $c : \hat{\mathcal{U}}(I\mathfrak{g}) \to \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{S}}(\mathfrak{g}^*)$ is "the constant term".

Unitary $\Longrightarrow$ Group-Algebra. $\iint \omega^2_{x+y}e^{x+y}\phi(x)\psi(y)$

$$= \langle \omega_{x+y}, \omega_{x+y}e^{x+y}\phi(x)\psi(y)\rangle = \langle V\omega_{x+y}, Ve^{x+y}\phi(x)\psi(y)\omega_{x+y}\rangle$$
$$= \langle \omega_x\omega_y, e^xe^yV\phi(x)\psi(y)\omega_{x+y}\rangle = \langle \omega_x\omega_y, e^xe^y\phi(x)\psi(y)\omega_x\omega_y\rangle$$
$$= \iint \omega^2_x\omega^2_y e^xe^y\phi(x)\psi(y).$$

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G \to A$ is multiplicative then $(\mathrm{Fun}(G), \star) \cong (A, \cdot)$ via $L : f \mapsto \sum f(a)\tau(a)$. For Lie $(G, \mathfrak{g})$,

$$\begin{array}{ccc} (\mathfrak{g}, +) \ni x & \xrightarrow{\tau_0 = \exp_{\mathcal{S}}} & e^x \in \hat{\mathcal{S}}(\mathfrak{g}) \\ \downarrow \exp_G & \searrow \exp_{\mathcal{U}} & \downarrow \chi \\ (G, \cdot) \ni e^x & \xrightarrow{\tau_1} & e^x \in \hat{\mathcal{U}}(\mathfrak{g}) \end{array} \quad \text{so} \quad \begin{array}{ccc} \mathrm{Fun}(\mathfrak{g}) & \xrightarrow{L_0} & \hat{\mathcal{S}}(\mathfrak{g}) \\ \downarrow \Phi^{-1} & & \downarrow \chi \\ \mathrm{Fun}(G) & \xrightarrow{L_1} & \hat{\mathcal{U}}(\mathfrak{g}) \end{array}$$

with $L_0\psi = \int \psi(x)e^x dx \in \hat{\mathcal{S}}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi = \int \psi(x)e^x \in \hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i \in \mathrm{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1 \star \psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $L_{0/1}$ are "Laplace transforms")

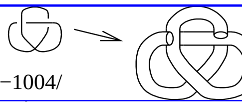
$\star$ in G: $\iint \psi_1(x)\psi_2(y)e^xe^y$ $\star$ in $\mathfrak{g}$: $\iint \psi_1(x)\psi_2(y)e^{x+y}$

We skipped...
- The Alexander polynomial and Milnor numbers.
- u-Knots, Alekseev-Torossian, and Drinfel'd associators.
- v-Knots, quantum groups and Etingof-Kazhdan.
- BF theory and the successful religion of path integrals.
- The simplest problem hyperbolic geometry solves.

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/Bonn-0908/`

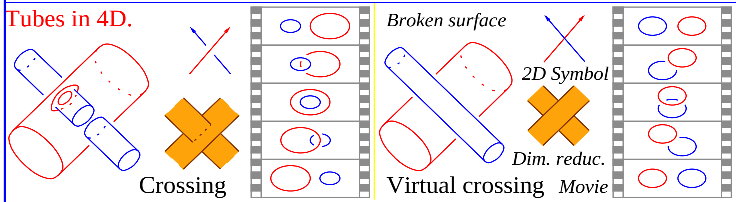
w−Knots from Z to A

Dror Bar−Natan, Luminy, April 2010
http://www.math.toronto.edu/~drorbn/Talks/Luminy−1004/

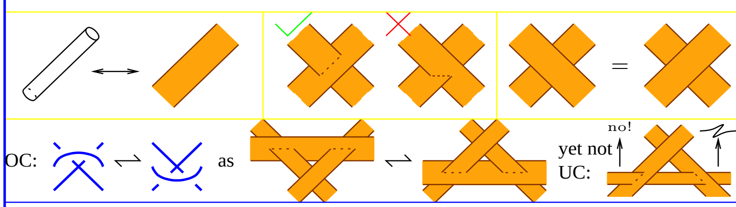


Abstract I will define w-knots, a class of knots wider than ordinary knots but weaker than virtual knots, and show that it is quite easy to construct a universal finite invariant Z of w-knots. In order to study Z we will introduce the "Euler Operator" and the "Infinitesimal Alexander Module", at the end finding a simple determinant formula for Z. With no doubt that formula computes the Alexander polynomial A, except I don't have a proof yet.

Tubes in 4D.



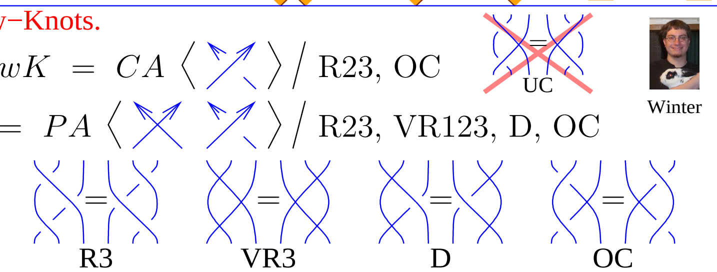
A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.



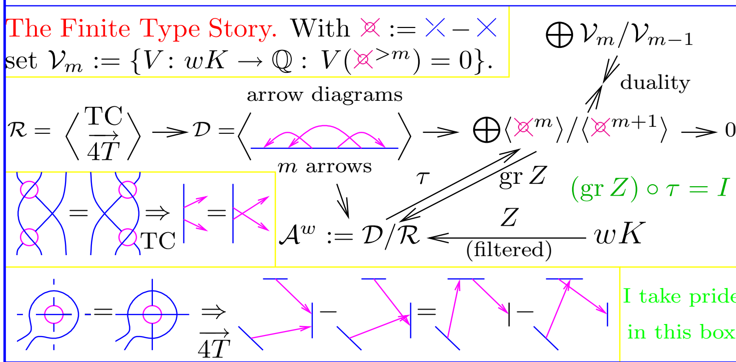
w−Knots.

$$wK = CA \langle \ \rangle / \text{R23, OC}$$

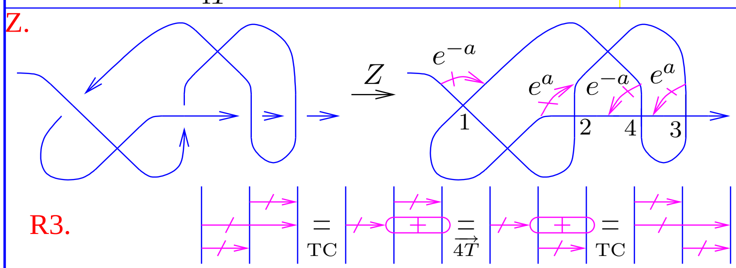
$$= PA \langle \ \rangle / \text{R23, VR123, D, OC}$$

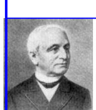


The Finite Type Story. With $\times := \times - \times$ set $\mathcal{V}_m := \{V : wK \to \mathbb{Q} : V(\times^{>m}) = 0\}$.



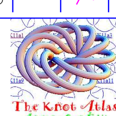
Z.



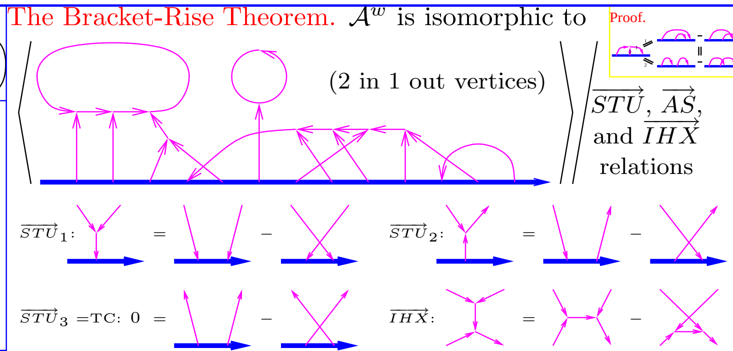


"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org



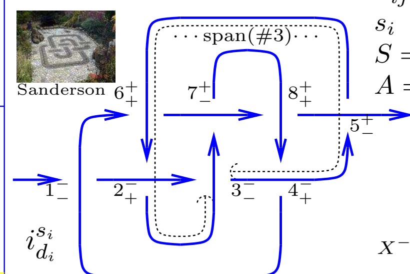
The Bracket-Rise Theorem. $\mathcal{A}^w$ is isomorphic to



Corollaries. (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.

Habiro - can you do better?

The Alexander Theorem. $T_{ij} = |\text{low}(\#j) \in \text{span}(\#i)|$, $s_i = \text{sign}(\#i)$, $d_i = \text{dir}(\#i)$, $S = \text{diag}(s_i d_i)$, $A = \det\left(I + T(I - X^{-S})\right)$.



$$T = \begin{pmatrix} 0&1&1&1&1&0&1&0 \\ 0&0&1&0&1&0&0&0 \\ 0&1&0&0&1&0&0&0 \\ 0&1&0&0&1&0&1&0 \\ 0&1&0&1&0&1&1&1 \\ 0&1&0&1&0&0&1&0 \\ 0&0&0&1&0&1&0&0 \\ 0&0&0&1&0&1&0&0 \end{pmatrix},$$

$$X^{-S} = \text{diag}\left(\tfrac{1}{X}, X, \tfrac{1}{X}, X, X, \tfrac{1}{X}, X, \tfrac{1}{X}\right).$$

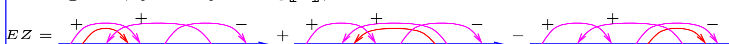
Conjecture. For u-knots, A is the Alexander polynomial.

Theorem. With $w : x^k \mapsto w_k =$ (the k-wheel),

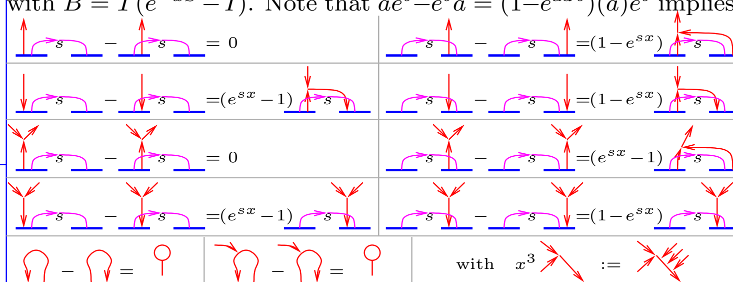
$$Z = N \exp_{\mathcal{A}^w}\left(-w\left(\log_{\mathbb{Q}[\![x]\!]} A(e^x)\right)\right)$$

mod $w_k w_l = w_{k+l}$, $Z = N \cdot A^{-1}(e^x)$

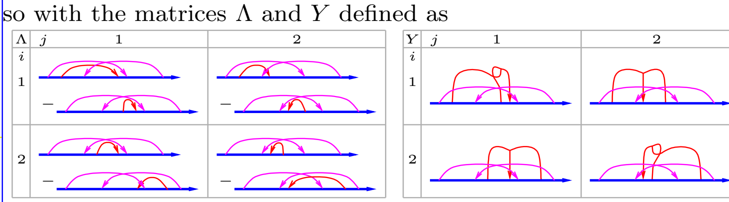
Proof Sketch. Let E be the Euler operator, "multiply anything by its degree", $f \mapsto xf'$ in $\mathbb{Q}[\![x]\!]$, so $Ee^x = xe^x$ and



We need to show that $Z^{-1}EZ = N' - \text{tr}\left((I-B)^{-1}TSe^{-xS}\right)w_1$, with $B = T(e^{-xS} - I)$. Note that $ae^b - e^b a = (1 - e^{\text{ad}\, b})(a)e^b$ implies



so with the matrices Λ and Y defined as



we have $EZ - N'' = \text{tr}(S\Lambda)$, $\Lambda = -BY - Te^{-xS}w_1$, and $Y = BY + Te^{-xS}w_1$. The theorem follows.

So What? • Habiro-Shima did this already, but not quite. (HS: *Finite Type Invariants of Ribbon 2-Knots, II*, Top. and its Appl. **111** (2001).)
• New (?) formula for Alexander, new (?) "Infinitesimal Alexander Module". Related to Lescop's arXiv:1001.4474?
• An "ultimate Alexander invariant": local, composes well, behaves under cabling. Ought to also generalize the multi-variable Alexander polynomial and the theory of Milnor linking numbers.
• Tip of the Alekseev-Torossian-Kashiwara-Vergne iceberg (AT: *The Kashiwara-Vergne conjecture and Drinfeld's associators*, arXiv:0802.4300).
• Tip of the v-knots iceberg. May lead to other polynomial-time polynomial invariants. "A polynomial's worth a thousand exponentials".
Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Luminy-1004/

18 Conjectures

Dror Bar-Natan, Chicago, September 2010
http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/

Abstract. I will state $18 = 3 \times 3 \times 2$ "fundamental" conjectures on finite type invariants of various classes of virtual knots. This done, I will state a few further conjectures about these conjectures and ask a few questions about how these 18 conjectures may or may not interact.

Following *"Some Dimensions of Spaces of Finite Type Invariants of Virtual Knots"*, by B-N, Halacheva, Leung, and Roukema, http://www.math.toronto.edu/~drorbn/papers/v-Dims/.

LRHB by Chu

Theorem. For u-knots, $\dim \mathcal{V}_n/\mathcal{V}_{n-1} = \dim \mathcal{W}_n$ for all n.

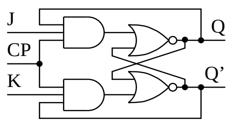
Proof. This is the Kontsevich integral, or the "Fundamental Theorem of Finite Type Invariants". The known proofs use QFT-inspired differential geometry or associators and some homological computations.

Two tables. The following tables show $\dim \mathcal{V}_n/\mathcal{V}_{n-1}$ and $\dim \mathcal{W}_n$ for $n = 1, \ldots, 5$ for 18 classes of v-knots:

relations\skeleton		round (◯)	long (⟶)	flat (╳ = ╳)
standard	mod R1	0, 0, 1, 4, 17 •	0, 2, 7, 42, 246 •	0, 0, 1, 6, 34 •
R2b R2c R3b	no R1	1, 1, 2, 7, 29	2, 5, 15, 67, 365	1, 1, 2, 8, 42
braid-like	mod R1	0, 0, 1, 4, 17 •	0, 2, 7, 42, 246 •	0, 0, 1, 6, 34 •
R2b R3b	no R1	1, 2, 5, 19, 77	2, 7, 27, 139, 813	1, 2, 6, 24, 120
R2 only	mod R1	0, 0, 4, 44, 648	0, 2, 28, 420, 7808	0, 0, 2, 18, 174
R2b R2c	no R1	1, 3, 16, 160, 2248	2, 10, 96, 1332, 23880	1, 2, 9, 63, 570

18 Conjectures. These 18 coincidences persist.

Circuit Algebras

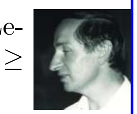


A J-K Flip Flop

Infineon HYS64T64020HDL-3.7-A 512MB RAM

Comments. 0, 0, 1, 4, 17 and 0, 2, 7, 42, 246. These are the "standard" virtual knots.

2, 7, 27, 139, 813. These best match Lie bi-algebra. Leung computed the bi-algebra dimensions to be $\geq$ 2, 7, 27, 128.



Vogel

•••. We only half-understand these equalities.

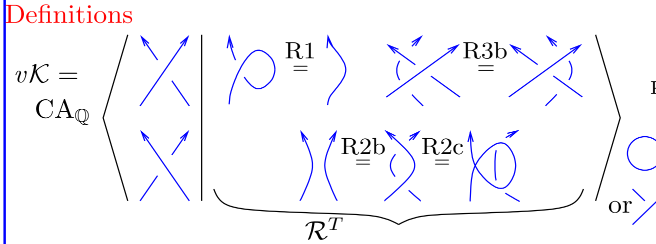
1, 2, 6, 24, 120. Yes, we noticed. Karene Chu is proving all about this, including the classification of flat knots.

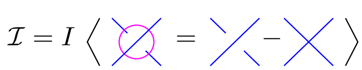
1, 1, 2, 8, 42, 258, 1824, 14664, …, which is probably http://www.research.att.com/~njas/sequences/A013999.

What about w? See other side. What about flat and round? What about v-braids? I don't know. Likely fails!

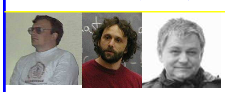
Definitions

Kauffman

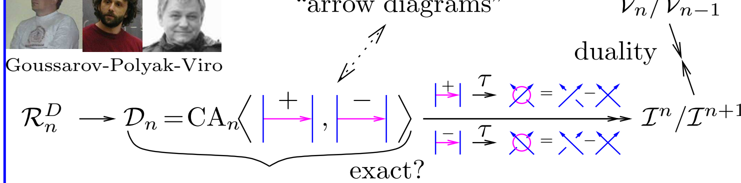


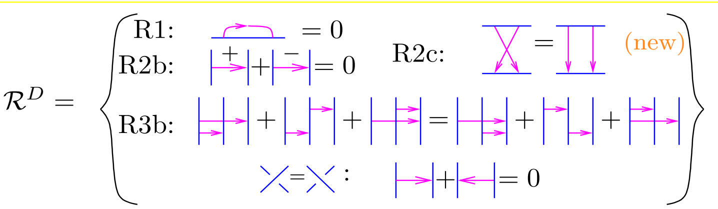


$\mathcal{V}_n = (v\mathcal{K}/\mathcal{I}^{n+1})^*$ is one thing we measure…



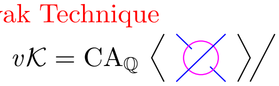
Goussarov-Polyak-Viro



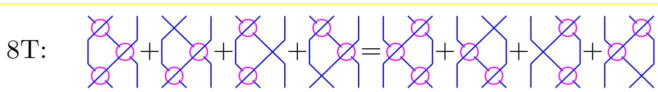


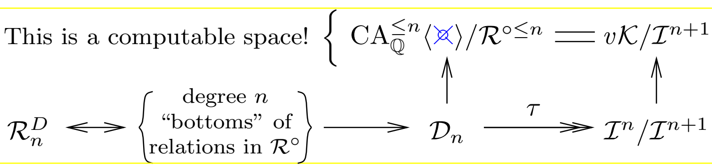
$\mathcal{W}_n = (\mathcal{D}_n/\mathcal{R}_n^D)^* = (\mathcal{A}_n)^*$ is the other thing we measure…

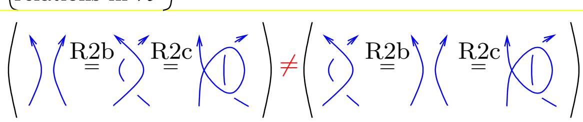
The Polyak Technique



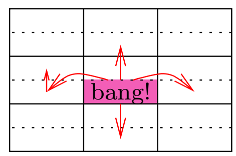
fails in the u case







The True Count



One bang! and five compatible transfer principles.

Bang. Recall the surjection $\bar{\tau} : \mathcal{A}_n = \mathcal{D}_n/\mathcal{R}_n^D \to \mathcal{I}^n/\mathcal{I}^{n+1}$. A filtered map $Z : v\mathcal{K} \to \mathcal{A} = \bigoplus \mathcal{A}_n$ such that $(\operatorname{gr} Z) \circ \bar{\tau} = I$ is called a universal finite type invariant, or an "expansion".

Theorem. Such Z exist iff $\bar{\tau} : \mathcal{D}_n/\mathcal{R}_n^D \to \mathcal{I}^n/\mathcal{I}^{n+1}$ is an isomorphism for every class and every n, and iff the 18 conjectures hold true.

The Big Bang. Can you find a "homomorphic expansion" Z — an expansion that is also a morphism of circuit algebras? Perhaps one that would also intertwine other operations, such as strand doubling? Or one that would extend to v-knotted trivalent graphs?

- Using generators/relations, finding Z is an exercise in solving equations in graded spaces.
- In the u case, these are the Drinfel'd pentagon and hexagon equations.
- In the w case, these are the Kashiwara-Vergne-Alekseev-Torossian equations. Composed with $\mathcal{T}_\mathfrak{g} : \mathcal{A} \to \mathcal{U}$, you get that the convolution algebra of invariant functions on a Lie group is isomorphic to the convolution algebra of invariant functions on its Lie algebra.
- In the v case there are strong indications that you'd get the equations defining a quantized universal enveloping algebra and the Etingof-Kazhdan theory of quantization of Lie bi-algebras. That's why I'm here!

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/

More on Chern-Simons Theory and Feynman Diagrams

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

After $A \mapsto A/\sqrt{k}$, and setting $\hbar = \frac{1}{\sqrt{k}}$:

$$Z(\gamma) = \int_{A\in\Omega^1(\mathbb{R}^3,\mathfrak{g})} \mathcal{D}A\, tr_R\, hol_\gamma(A)\, e^{\frac{i}{4\pi}\underbrace{\int_{\mathbb{R}^3} tr(A\wedge dA + \frac{2\hbar}{3}A\wedge A\wedge A)}_{CS(A)}}$$

where $tr_R hol_\gamma(A) = tr_R\left(1 + \hbar\int ds\, A(\dot\gamma(s)) + \hbar^2 \int_{s_1<s_2} A(\dot\gamma(s_1))A(\dot\gamma(s_2)) + \dots\right)$

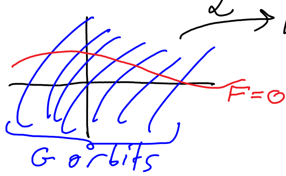
Trouble! "d" is not invertible!

Gauge Invariance: $CS(A)$ is invariant under $A \mapsto A + \delta A$, $\delta A = -(dc + \hbar[A,c])$, $c \in \Omega^0(\mathbb{R},\mathfrak{g})$

Back to the drawing board....

Suppose $\mathcal{L}(x)$ on $\mathbb{R}^n$ is invariant under a k-dimensional group G w/ Lie algebra $\mathfrak{g} = \langle g_a \rangle$, and suppose $F: \mathbb{R}^n \to \mathbb{R}^k$ is such that $F=0$ is a section of the G-action:

$$G \to \mathbb{R}^n \xrightarrow{F} \mathbb{R}^k$$



Then

$$\int_{\mathbb{R}^n} dx\, e^{i\mathcal{L}} \sim \int_{\mathbb{R}^n} dx\, e^{i\mathcal{L}}\delta(F(x))\cdot \det\left(\frac{\partial F^a}{\partial g_b}\right)(x)$$

$$\sim \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^k} d\phi\, e^{i(\mathcal{L}+F(x)\cdot\phi)} \det\left(\frac{\partial F^a}{\partial g_b}\right)(x)$$

Perturbation theory for determinants?

$$\det(J_0 + \hbar J_1(x)) = \det(J_0)\sum_m \hbar^m\, Tr(\Lambda^m J_0^{-1})\cdot(\Lambda^m J_1(x))$$

Berezin / Fermionic / Anti-commuting Variables: $\int d^k\bar{c}\, d^k c\, e^{i\bar{c}_a J^a_b c^b} \sim \det(J)$

So

$$Z \sim \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^k} d\phi \int d^k\bar{c} \int d^k c\, e^{i\mathcal{L}_{tot}} \text{ where}$$

$$\mathcal{L}_{tot} = \underbrace{\mathcal{L}(x)}_{\text{the original}} + \underbrace{F(x)\cdot\phi}_{\text{gauge-fixing}} + \underbrace{\bar{c}_a\left(\frac{\partial F^a}{\partial g_b}\right)c^b}_{\text{"ghosts"}}$$

In Chern-Simons, w/ $F(A) := d^*A = \partial_i A^i$, get

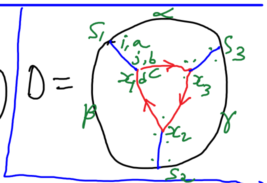
$$\mathcal{L}_{tot} = \frac{k}{4\pi}\int_{\mathbb{R}^3} tr(A\wedge dA + \tfrac{2}{3}A\wedge A\wedge A + \phi\,\partial_i A^i + \bar{c}\,\partial_i(\partial^i + ad\, A^i)c$$

So we have

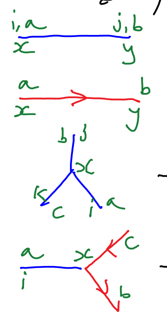
* A bosonic quadratic term involving $\binom{A}{\phi}$.
* A fermionic quadratic term involving $\bar{c}, c$.
* A cubic interaction of 3 A's.
* A cubic $A\bar{c}c$ vertex.
* Funny A and γ "holonomy" vertices along γ.

After much crunching:

$$Z(\gamma) = \sum_{m=0}^{\infty} \hbar^m \sum_{\substack{\text{Feynman}\\ \text{diags } D}} \oint \mathcal{E}(D) \qquad D =$$



where $\mathcal{E}(D)$ is constructed as follows:

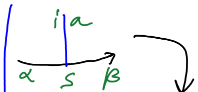


$$\longrightarrow \epsilon_{ijk}\, t^{ab}\, \frac{i(x-y)^k}{2|x-y|^3}$$

$$\longrightarrow \frac{t^{ab}}{4\pi|x-y|}$$

$$\longrightarrow \frac{i}{2\pi}\int_{\mathbb{R}^3} t_{abc}\,\epsilon^{ijk}$$

$$\longrightarrow \frac{1}{2\pi}\int_{\mathbb{R}^3} dz\, t_{abc}\,\partial^i_x \quad \text{(acting only in } b\text{-direction)}$$



$$\int ds\, R^\alpha_{a\beta}\,\dot\gamma^i(s)$$

$(-)$ sign for each red loop.

By a bit of a miracle, this boils down to a configuration space integral, which in itself, can be reduced to a pre-image count.

... But I run out of steam for tonight...

Banks like knots.

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

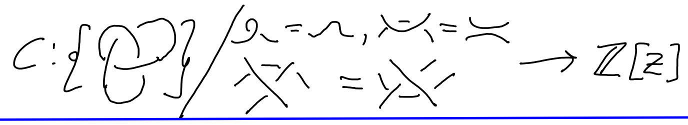
www.katlas.org

Video at http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107/

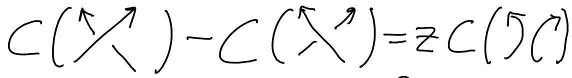
The Basics of Finite-Type Invariants of Knots

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

Definition. A knot invariant is any function whose domain is {knots}. Really, we mean a computable function whose target space is understandable; e.g.



Example. The Conway polynomial is given by

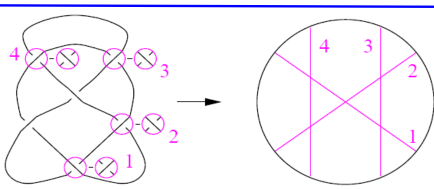


and

$$C(\bigcirc\bigcirc\cdots\bigcirc_k) = \begin{cases} 1 & k=1 \\ 0 & k>1 \end{cases}$$

Exercise. Pick your favourite bank and compute the Conway polynomial of its logo.

Definition. Any $V: \{\text{knots}\} \to$ Abelian Group A can be extended to "knots w/ double points"



using $V(\times) = V(\nearrow) - V(\nwarrow)$. (Think "differentiation")

Definition. V is of type m if always $V(\underbrace{\times\times\cdots\times}_{m+1}) = 0$ (think "polynomial")

Conjecture. Finite type invariants separate knots.

Theorem. If $C(K) = \sum_{m=0}^{\infty} V_m(K) z^m$ then V_m is of type m.

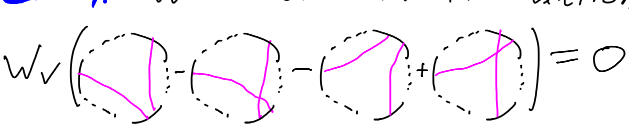
Proof. $C(\times) = C(\nearrow) - C(\nwarrow) = zC(\,)\,(\,) \quad \square$

Let V be of type m; then $V^{(m)}$ is constant:

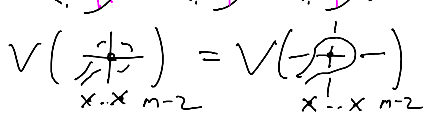
$$V(\underbrace{\times\cdots\times}_{m}\nearrow) = V(\underbrace{\times\cdots\times}_{m}\nwarrow)$$

So $W_V := V^{(m)} = V|_{m\text{-singular knots}}$ is really a function on m-chord diagrams: $W_V: \{\text{chord diagrams}\} \to A$

Claim. W_V satisfies the 4T relation:



Proof.



Exercise for Lecture 2. Use $\int_{\mathbb{R}} e^{-x^2/2} = \sqrt{2\pi}$, Fubini's theorem, and polar coordinates to compute $\int_{\mathbb{R}^n} e^{-\|x\|^2/2} d^n x$ in two different ways and hence to deduce the volume of S^{n-1}, the $(n-1)$-dimensional sphere.

Exercise. 1. Determine the "weight system" W_{V_m} of the m-th coefficient of the Conway polynomial and verify that is satisfies 4T.

2. Learn somewhere about the Jones polynomial, and do the same for its coefficients.

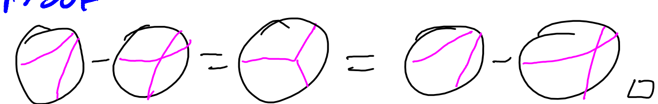
Theorem. (The Fundamental Theorem) Every "weight system", i.e. every linear functional W on $\mathcal{A} := \{\text{chord diagrams}\}/4T$ is the m'th derivative of a type m invariant: $\forall W\ \exists V$ s.t. $W = W_V$

M. Kontsevich

m	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{A}^r_m$	1	0	1	1	3	4	9	14	27	44	80	132	232
$\dim \mathcal{A}_m$	1	1	2	3	6	10	19	33	60	104	184	316	548
$\dim \mathcal{P}_m$	0	1	1	1	2	3	5	8	12	18	27	39	55

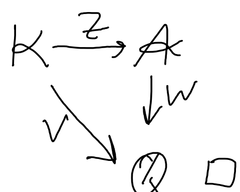
Theorem. $\mathcal{A}^{\text{today}} \cong \mathcal{A}^{\text{Monday}}$

Proof



Proposition. The fundamental thm holds iff there exists an expansion $Z: \mathcal{K} \to \hat{\mathcal{A}}$ s.t. if K is m-singular, then $Z(K) = D_K +$ higher degrees.

Proof.



Also see my old paper, "On the Vassiliev knot invariants" (google will find...)

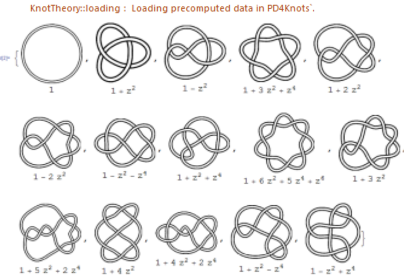
```
In[1]:= << KnotTheory`

Loading KnotTheory` version of August 22, 2010, 13:36:57.55.
Read more at http://katlas.org/wiki/KnotTheory.

In[2]:= Column[{
     Import[
      "C:\\drorbn\\AcademicPensieve\\2011-07\\RolfsenKnots\\"
       <> ToString@#[[1]] <>
       "." <> ToString@#[[2]] <> "_240.gif"],
     Conway[#][z]
    }, Center
   ] & /@ AllKnots[{0, 7}]

KnotTheory::loading : Loading precomputed data in PD4Knots`.
```

J.H. Conway

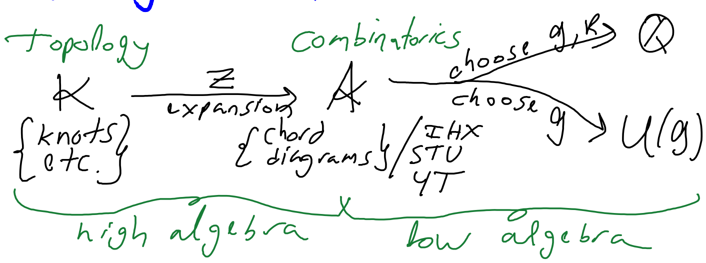


Video at http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107/

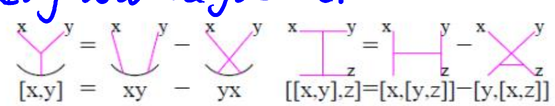
Low and High Algebra in the "u" Case

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

The big picture, "u" Case.



Very low algebra.



More precisely, let $\mathfrak{g} = \langle X_a \rangle$ be a Lie algebra with an orthonormal basis, and let $R = \langle v_\alpha \rangle$ be a representation. Set

$$f_{abc} := \langle [a,b], c \rangle \qquad X_a v_\beta = \sum_\gamma r^\beta_{a\gamma} v_\gamma$$

and then

Exercise. Find a fast method to find $W_{\mathfrak{g},R}(D)$ when $\mathfrak{g} = \mathfrak{gl}_n$, $R = \mathbb{R}^n$? Is it related to the Conway polynomial?

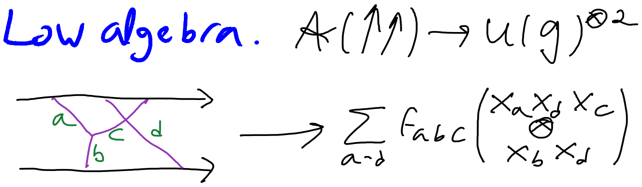
Universal Representation Theory.

Inspired by $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$, set $U(\mathfrak{g}) = \langle \text{words in } \mathfrak{g} \rangle / [x,y] = xy - yx$

* Every rep of $\mathfrak{g}$ extends to $U(\mathfrak{g})$.
* $\exists\, \Delta : U(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes 2}$ by "word splitting", as must be for $R_1 \otimes R_2$.

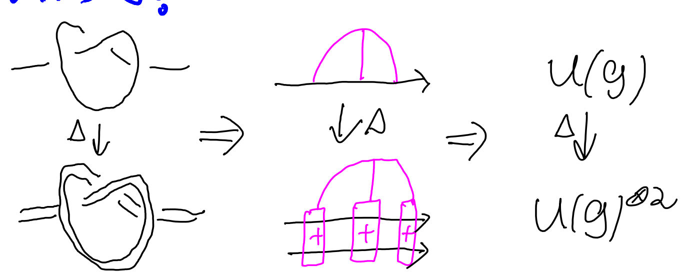
Exercise. With $\mathfrak{g} = \langle x, y \rangle / [x,y] = x$, determine $U(\mathfrak{g})$. Guess a generalization.

Low algebra. $\mathcal{A}(\uparrow\uparrow) \to U(\mathfrak{g})^{\otimes 2}$ via



& likewise, $\mathcal{A}(\uparrow_n) \to U(\mathfrak{g})^{\otimes n}$ ⇒ $\mathcal{A}(\uparrow_n)$ is "universal universal rep. theory"!

What's Δ?

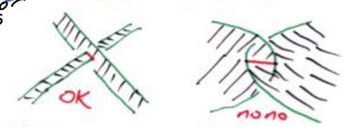


A "Homomorphic Expansion" $Z : \mathcal{K} \to \mathcal{A}$

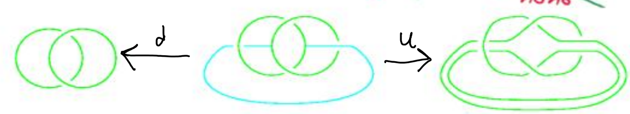
is an expansion that intertwines all relevant algebraic ops. If $\mathcal{K}$ is finitely presented, finding Z is **High Algebra.**

{Ribbon knots} = {uγ : γ ∈ K(O–O), dγ = OO} (↑ more or less)

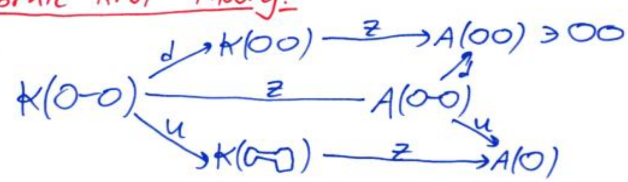
Where Ribbon means



and



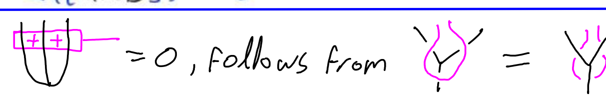
Algebraic knot theory:



So

$Z(\{\text{Ribbon knots}\}) \subset \{u\alpha : d\alpha = Z(OO)\} \subset \mathcal{A}(O\text{–}O)$

VI

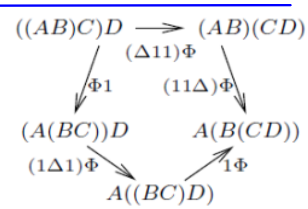


An Associator:

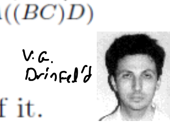
Quantum Algebra's "root object"

$$(AB)C \xrightarrow{\Phi \in \mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

satisfying the "pentagon",



$\Phi 1 \cdot (1\Delta 1)\Phi \cdot 1\Phi = (\Delta 11)\Phi \cdot (11\Delta)\Phi$



The hexagon? Never heard of it.

See Also. B-N & Dancso, arXiv:1103.1896

Video at http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107/

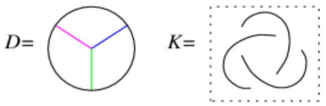
Lecture 5 Extras

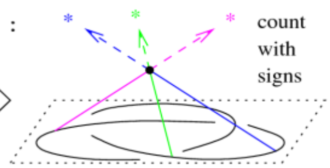
Review Material (mostly)

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

$\langle D, K\rangle_{\overline{\pi}} := \begin{pmatrix}\text{The signed Stonehenge}\\ \text{pairing of } D \text{ and } K\end{pmatrix}$: count with signs

D= K= ⇒





Thus we consider the generating function of all stellar coincidences:

$$Z(K) := \lim_{N\to\infty} \sum_{3\text{-valent } D} \frac{1}{2^c c! \binom{N}{e}} \langle D, K\rangle_{\overline{\pi}} D \cdot \begin{pmatrix}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{pmatrix} \in \mathcal{A}(\circlearrowleft)$$

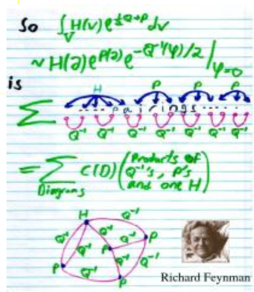
Theorem. Modulo Relations, $Z(K)$ is a knot invariant!

When deforming, catastrophes occur when:

A plane moves over an intersection point – Solution: Impose IHX, (see below)

An intersection line cuts through the knot – Solution: Impose STU, (similar argument)

The Gauss curve slides over a star – Solution: Multiply by

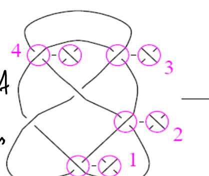


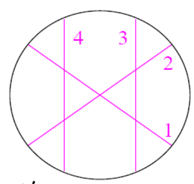
It all is perturbative Chern–Simons–Witten theory:

$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A\, hol_K(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(A\wedge dA + \frac{2}{3} A\wedge A\wedge A\right)\right]$$

$$\to \sum_{D:\text{ Feynman diagram}} W_{\mathfrak{g}}(D) \sum \mathcal{E}(D) \to \sum_{D:\text{ Feynman diagram}} D \sum \mathcal{E}(D)$$

Definition. Any $V: \{\text{knots}\} \to$ Abelian Group A can be extended to "knots w/ double points" using $V(\times) = V(\nearrow) - V(\nwarrow)$. (Think "differentiation")





Definition. V is of type m if always $V(\underbrace{\times \times \cdots \times}_{m+1}) = 0$ (think "polynomial")

Conjecture. Finite type invariants separate knots.

Theorem. If $C(K) = \sum_{m=0}^{\infty} V_m(K) z^m$ then V_m is of type m.

Proof. $C(\times) = C(\nearrow) - C(\nwarrow) = zC(\uparrow\uparrow)$

Proposition. The Fundamental thm holds iff there exists an expansion $Z: \mathcal{K} \to \hat{\mathcal{A}}$ s.t. if K is m-singular, then

$$Z(K) = D_K + \text{higher degrees}$$

The big picture, "u" case.

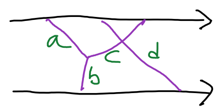
Topology — Combinatorics

$\mathcal{K} \xrightarrow[\text{expansion}]{Z} \mathcal{A} \xrightarrow{\text{choose } \mathfrak{g}, R} \mathbb{Q}$; $\mathcal{A} \xrightarrow{\text{choose } \mathfrak{g}} \mathcal{U}(\mathfrak{g})$

{knots etc.} — {chord diagrams} / IHX, STU, 4T

high algebra — low algebra

Low algebra. $\mathcal{A}(\uparrow\uparrow) \to \mathcal{U}(\mathfrak{g})^{\otimes 2}$ via

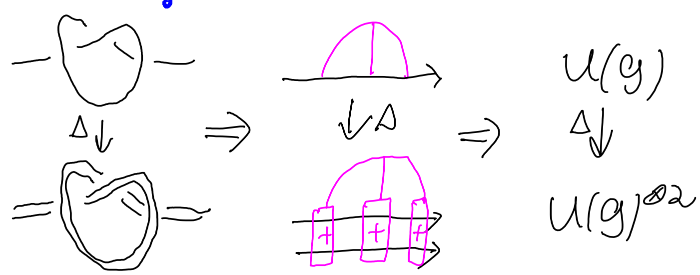


$\longrightarrow \sum_{a-d} f_{abc} \begin{pmatrix} x_a x_d x_c \\ \otimes \\ x_b x_d \end{pmatrix}$

& likewise, $\mathcal{A}(\uparrow_n) \to \mathcal{U}(\mathfrak{g})^{\otimes n} \Rightarrow$

$\mathcal{A}(\uparrow_n)$ is "universal universal rep. theory"!

What's Δ?



A "Homomorphic Expansion" $Z: \mathcal{K} \to \mathcal{A}$ is an expansion that intertwines all relevant algebraic ops. If $\mathcal{K}$ is finitely presented, finding Z is **High Algebra.**

An Associator:
Quantum Algebra's "root object"

$$(AB)C \xrightarrow{\Phi \in \mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

satisfying the "pentagon",

$((AB)C)D \longrightarrow (AB)(CD)$; $((AB)C)D \xrightarrow{\Phi 1} (A(BC))D \xrightarrow{(1\Delta 1)\Phi} A((BC)D) \xrightarrow{1\Phi} A(B(CD))$; $((AB)C)D \xrightarrow{(\Delta 11)\Phi} (AB)(CD) \xrightarrow{(11\Delta)\Phi} A(B(CD))$

$$\Phi 1 \cdot (1\Delta 1)\Phi \cdot 1\Phi = (\Delta 11)\Phi \cdot (11\Delta)\Phi$$

The hexagon? Never heard of it.

See Also. B-N & Dancso, arXiv: 1103.1896

Video at http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107/