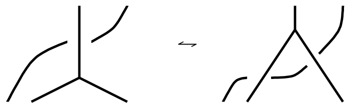
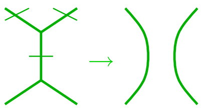
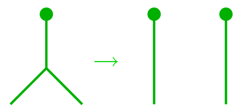
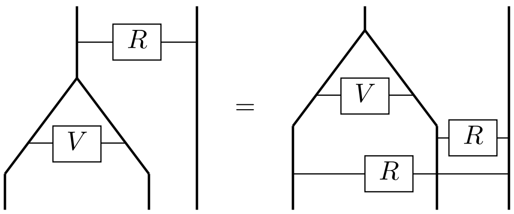
Some Details for the Caen Workshop, 1

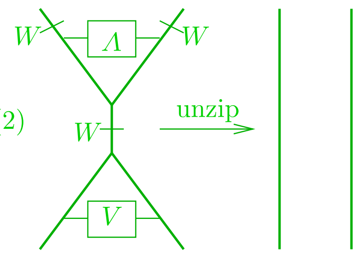
Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:

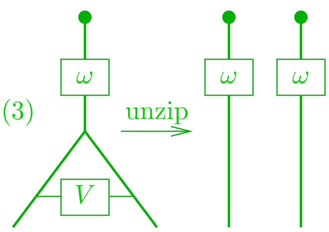
(1) 

(2)  (3) 

Diagrammatic statement. Let $R = \exp \text{↦} \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that

(1) 

(2) 

(3) 

Algebraic statement. With $I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g}$, with $c : \hat{\mathcal{U}}(I\mathfrak{g}) \to \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{S}}(\mathfrak{g}^*)$ the obvious projection, with S the antipode of $\hat{\mathcal{U}}(I\mathfrak{g})$, with W the automorphism of $\hat{\mathcal{U}}(I\mathfrak{g})$ induced by flipping the sign of $\mathfrak{g}^*$, with $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{\mathcal{U}}(I\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$ there exist $\omega \in \hat{\mathcal{S}}(\mathfrak{g}^*)$ and $V \in \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that
(1) $V(\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2} \otimes \hat{\mathcal{U}}(\mathfrak{g})$
(2) $V \cdot SWV = 1$ (3) $(c \otimes c)(V\Delta(\omega)) = \omega \otimes \omega$

Unitary statement. There exists $\omega \in \mathrm{Fun}(\mathfrak{g})^G$ and an (infinite order) tangential differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that
(1) $V\widehat{e^{x+y}} = \widehat{e^x}\widehat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)
(2) $VV^* = I$ (3) $V\omega_{x+y} = \omega_x\omega_y$

Group-Algebra statement. There exists $\omega^2 \in \mathrm{Fun}(\mathfrak{g})^G$ so that for every $\phi, \psi \in \mathrm{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $\omega^2 = j^{1/2}$)

$$\iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2_{x+y}e^{x+y} = \iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2_x\omega^2_y e^x e^y.$$

(shhh, this is Duflo)

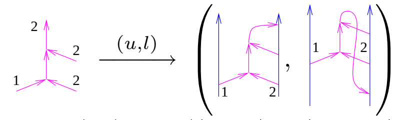
Convolutions statement (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, let $j : \mathfrak{g} \to \mathbb{R}$ be the Jacobian of the exponential map $\exp : \mathfrak{g} \to G$, and let $\Phi : \mathrm{Fun}(G) \to \mathrm{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := j^{1/2}(x)f(\exp x)$. Then if $f, g \in \mathrm{Fun}(G)$ are Ad-invariant and supported near the identity, then

$$\Phi(f) \star \Phi(g) = \Phi(f \star g).$$

Dror Bar-Natan, June 2012, Web/Caen-1206 with Web := `http://www.math.toronto.edu/~drorbn/Talks`.
Sources: Web/Bonn-0908, and Web/Montpellier-1006.

Wheels and Trees. With $\mathcal{P}$ for $\mathcal{P}$rimitives,

$$0 \longrightarrow \langle\text{wheels}\rangle \overset{\iota}{\longrightarrow} \mathcal{P}\mathcal{A}^w(\uparrow_n) \underset{l}{\overset{u}{\rightleftarrows}} \xrightarrow{\pi} \langle\text{trees}\rangle \longrightarrow 0,$$

with 

trees atop a wheel, and a little prince.

So $\mathrm{proj}\,\mathcal{K}^w(\uparrow_n) \cong \mathcal{U}(\langle\text{trees}\rangle \ltimes \langle\text{wheels}\rangle)$, or

$$\mathcal{A}^w(\uparrow_n) \cong \mathcal{U}((\mathfrak{a}_n \oplus \mathfrak{tder}_n) \ltimes \mathfrak{tr}_n).$$

Some A-T Notions. $\mathfrak{a}_n$ is the vector space with basis $x_1, \dots, x_n$, $\mathfrak{lie}_n = \mathfrak{lie}(\mathfrak{a}_n)$ is the free Lie algebra, $\mathrm{Ass}_n = \mathcal{U}(\mathfrak{lie}_n)$ is the free associative algera "of words", $\mathrm{tr} : \mathrm{Ass}^+_n \to \mathfrak{tr}_n = \mathrm{Ass}^+_n/(x_{i_1}x_{i_2}\cdots x_{i_m} = x_{i_2}\cdots x_{i_m}x_{i_1})$ is the "trace" into "cyclic words", $\mathfrak{der}_n = \mathfrak{der}(\mathfrak{lie}_n)$ are all the derivations, and

$$\mathfrak{tder}_n = \{D \in \mathfrak{der}_n : \forall i\ \exists a_i \text{ s.t. } D(x_i) = [x_i, a_i]\}$$

are "tangential derivations", so $D \leftrightarrow (a_1, \dots, a_n)$ is a vector space isomorphism $\mathfrak{a}_n \oplus \mathfrak{tder}_n \cong \bigoplus_n \mathfrak{lie}_n$. Finally, $\mathrm{div} : \mathfrak{tder}_n \to \mathfrak{tr}_n$ is $(a_1, \dots, a_n) \mapsto \sum_k \mathrm{tr}(x_k(\partial_k a_k))$, where for $a \in \mathrm{Ass}^+_n$, $\partial_k a \in \mathrm{Ass}_n$ is determined by $a = \sum_k (\partial_k a)x_k$, and $j : \mathrm{TAut}_n = \exp(\mathfrak{tder}_n) \to \mathfrak{tr}_n$ is $j(e^D) = \frac{e^D - 1}{D} \cdot \mathrm{div}\, D$.

Theorem. Everything matches. $\langle\text{trees}\rangle$ is $\mathfrak{a}_n \oplus \mathfrak{tder}_n$ as Lie algebras, $\langle\text{wheels}\rangle$ is $\mathfrak{tr}_n$ as $\langle\text{trees}\rangle$ / $\mathfrak{tder}_n$-modules, $\mathrm{div}\, D = \iota^{-1}(u - l)(D)$, and $e^{uD}e^{-lD} = e^{jD}$.

Differential Operators. Interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$:
- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\mathrm{ad}\, x$: $(x\varphi)(y) := \varphi([x, y])$.
- $c : \hat{\mathcal{U}}(I\mathfrak{g}) \to \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{S}}(\mathfrak{g}^*)$ is "the constant term".

Trees become vector fields and $uD \mapsto lD$ is $D \mapsto D^*$. So $\mathrm{div}\, D$ is $D - D^*$ and $jD = \log(e^D(e^D)^*) = \int_0^1 dt e^{tD}\, \mathrm{div}\, D$.

Alekseev-Torossian statement. There are elements $F \in \mathrm{TAut}_2$ and $a \in \mathfrak{tr}_1$ such that

$$F(x + y) = \log e^x e^y \quad\text{and}\quad jF = a(x) + a(y) - a(\log e^x e^y).$$

Theorem. The Alekseev-Torossian statement is equivalent to the knot-theoretic statement.

Proof. Write $V = e^c e^{uD}$ with $c \in \mathfrak{tr}_2$, $D \in \mathfrak{tder}_2$, and $\omega = e^b$ with $b \in \mathfrak{tr}_1$. Then (1) $\Leftrightarrow e^{uD}(x + y)e^{-uD} = \log e^x e^y$,
(2) $\Leftrightarrow I = e^c e^{uD}(e^{uD})^* e^c = e^{2c}e^{jD}$, and
(3) $\Leftrightarrow e^c e^{uD} e^{b(x+y)} = e^{b(x)+b(y)} \Leftrightarrow e^c e^{b(\log e^x e^y)} = e^{b(x)+b(y)} \Leftrightarrow c = b(x) + b(y) - b(\log e^x e^y)$.

Unitary $\Longrightarrow$ Group-Algebra. $\iint \omega^2_{x+y}e^{x+y}\phi(x)\psi(y)$

$$= \langle \omega_{x+y}, \omega_{x+y}e^{x+y}\phi(x)\psi(y)\rangle = \langle V\omega_{x+y}, Ve^{x+y}\phi(x)\psi(y)\omega_{x+y}\rangle$$
$$= \langle \omega_x\omega_y, e^x e^y V\phi(x)\psi(y)\omega_{x+y}\rangle = \langle \omega_x\omega_y, e^x e^y \phi(x)\psi(y)\omega_x\omega_y\rangle$$
$$= \iint \omega^2_x\omega^2_y e^x e^y \phi(x)\psi(y).$$

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G \to A$ is multiplicative then $(\mathrm{Fun}(G), \star) \cong (A, \cdot)$ via $L : f \mapsto \sum f(a)\tau(a)$. For Lie $(G, \mathfrak{g})$,

$$\begin{array}{ccc} (\mathfrak{g}, +) \ni x & \xrightarrow{\tau_0 = \exp_{\mathcal{S}}} & e^x \in \hat{\mathcal{S}}(\mathfrak{g}) \\ \downarrow \exp_G & \searrow \exp_{\mathcal{U}} & \downarrow \chi \\ (G, \cdot) \ni e^x & \xrightarrow{\tau_1} & e^x \in \hat{\mathcal{U}}(\mathfrak{g}) \end{array} \quad \text{so} \quad \begin{array}{ccc} \mathrm{Fun}(\mathfrak{g}) & \xrightarrow{L_0} & \hat{\mathcal{S}}(\mathfrak{g}) \\ \downarrow \Phi^{-1} & & \downarrow \chi \\ \mathrm{Fun}(G) & \xrightarrow{L_1} & \hat{\mathcal{U}}(\mathfrak{g}) \end{array}$$

with $L_0\psi = \int \psi(x)e^x dx \in \hat{\mathcal{S}}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi = \int \psi(x)e^x \in \hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i \in \mathrm{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1 \star \psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $L_{0/1}$ are "Laplace transforms")

$\star$ in G: $\iint \psi_1(x)\psi_2(y)e^x e^y$ $\quad\star$ in $\mathfrak{g}$: $\iint \psi_1(x)\psi_2(y)e^{x+y}$

Some Details for the Caen Workshop, 2