Trace Groups, Skein Modules and Khovanov Homology

Inspired by a talk by Sikora in GWU and by arXiv:math.QA/0409414 by Asaeda, Przytycki and Sikora

Conjecture: (I. Frenkel, though he may disown this version)

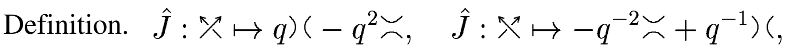
1. Every object in mathematics is the Euler characteristic of a complex.
2. Every operation in mathematics lifts to an operation between complexes.
3. Every identity in mathematics is true up to homotopy at complex–level.

Traces, dimensions, Lefschetz and Euler:

$$\tau(FG)=\tau(GF) \qquad \dim_\tau \mathcal{O} := \tau(I_{\mathcal{O}})$$

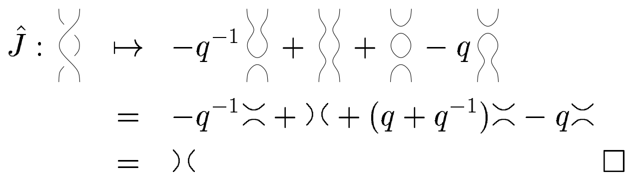
$$\tau(F) := \sum_r (-1)^r \tau(F^r) \qquad \chi_\tau(\Omega) := \tau(I_\Omega)$$

Realized for the Jones polynomial:



valued in xing–free tangles mod $\bigcirc = q+q^{-1}$

Invariance under R2:



Homotopy invariance:

$$F-G=hd+dh \quad\Longrightarrow$$

L. Euler

$$\tau(F)-\tau(G) = \sum_r (-1)^r \tau(F^r-G^r)$$
$$= \sum_r (-1)^r \tau(h^{r+1}d^r + d^{r-1}h^r)$$
$$= \sum_r (-1)^r \tau(h^{r+1}d^r - d^r h^{r+1}) = 0,$$

S. Lefschetz

$$GF\sim I_{\Omega_a},\ FG\sim I_{\Omega_b} \quad\Longrightarrow$$

$$\chi_\tau(\Omega_a)=\tau(I_{\Omega_a})=\tau(GF)=\tau(FG)=\tau(I_{\Omega_b})=\chi_\tau(\Omega_b)$$

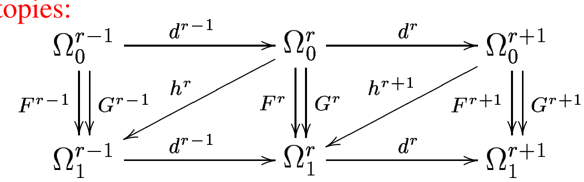
Complexes:

$$\Omega = \left(\Omega^{-n_-} \longrightarrow \Omega^{-n_-+1} \longrightarrow \cdots \qquad \longrightarrow \Omega^{n_+}\right)$$

Morphisms:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \Omega_0^{r-1} & \xrightarrow{d^{r-1}} & \Omega_0^r & \xrightarrow{d^r} & \Omega_0^{r+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle F^{r-1}} & & \downarrow{\scriptstyle F^r} & & \downarrow{\scriptstyle F^{r+1}} & & \\
\cdots & \longrightarrow & \Omega_1^{r-1} & \xrightarrow{d^{r-1}} & \Omega_1^r & \xrightarrow{d^r} & \Omega_1^{r+1} & \longrightarrow & \cdots
\end{array}$$

Homotopies:



$$F^r - G^r = h^{r+1}d^r + d^{r-1}h^r$$

All arrows in an arbitrary additive category!

The trace group:

$$\Xi(\mathcal{C}) := \bigoplus_{\mathcal{O}\in \mathrm{Obj}(\mathcal{C})} \mathrm{Mor}(\mathcal{O},\mathcal{O}) \Big/ \text{the trace relation: } FG=GF \text{ whenever } F:\mathcal{O}_1\to\mathcal{O}_2 \text{ and } G:\mathcal{O}_2\to\mathcal{O}_1.$$

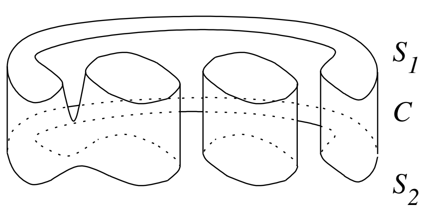
For matrices,

$$\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

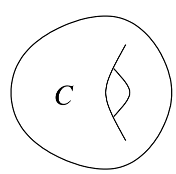
$$\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1&0\end{pmatrix}=\begin{pmatrix}1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\end{pmatrix}$$

(Thanks, Dylan)

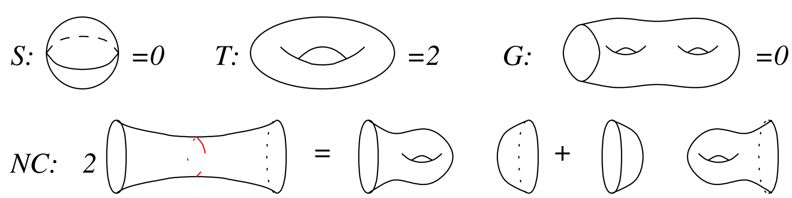
Cobordisms:

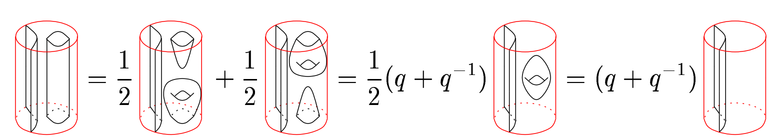


Ξ: Surfaces in a solid torus



Graded cobordisms mod S, T, G, NC:



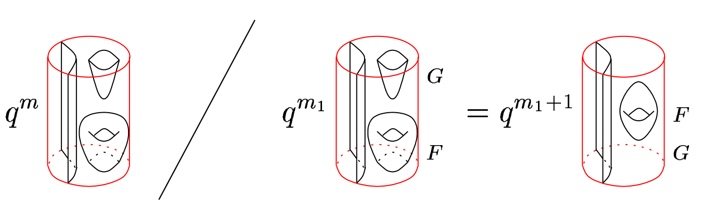


That's the Jones Skein relation!

Graded cobordisms: $\mathrm{Obj}=\{S\{m\} : m\in\mathbb{Z}\}$

$$\mathrm{Mor}\,(S_1\{m_1\}, S_2\{m_2\}) = \{C : S_1\to S_2 :\ \deg C = m_1-m_2\}$$

Ξ:



Questions:

The Big question: Do the same for Khovanov–Rozansky's arXiv:math.QA/0401268

The BIG question: Do the same for the rest of arXiv:math.QA

Dror Bar–Natan, Buffalo, October 2004

http://www.math.toronto.edu/~drorbn/Talks/Buffalo–0410 and
http://www.math.toronto.edu/~drorbn/papers/Cobordism