Meta−Groups, Meta−Bicrossed−Products, and the Alexander Polynomial, 1

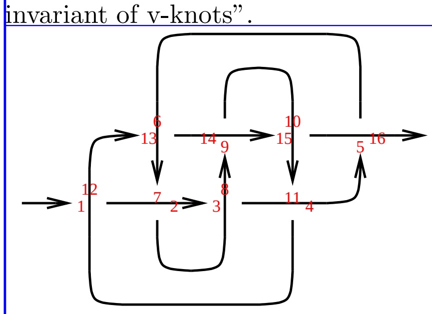
Dror Bar−Natan in Binghamton, March 2012

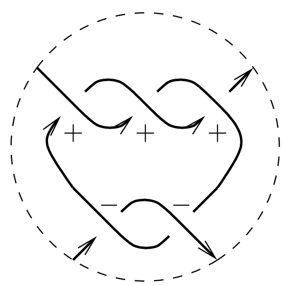
http://www.math.toronto.edu/~drorbn/Talks/Binghamton-1203/

Abstract. The a priori expectation of first year elementary school students who were just introduced to the natural numbers, if they would be ready to verbalize it, must be that soon, perhaps by second grade, they'd master the theory and know all there is to know about those numbers. But they would be wrong, for number theory remains a thriving subject, well-connected to practically anything there is out there in mathematics.

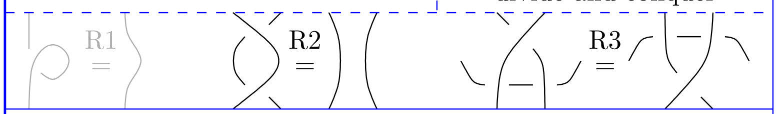
I was a bit more sophisticated when I first heard of knot theory. My first thought was that it was either trivial or intractable, and most definitely, I wasn't going to learn it is interesting. But it is, and I was wrong, for the reader of knot theory is often lead to the most interesting and beautiful structures in topology, geometry, quantum field theory, and algebra.

Today I will talk about just one minor example: A straightforward proposal for a group-theoretic invariant of knots fails if one really means groups, but works once generalized to meta-groups (to be defined). We will construct one complicated but elementary meta-group as a meta-bicrossed-product (to be defined), and explain how the resulting invariant is a not-yet-understood generalization of the Alexander polynomial, while at the same time being a specialization of a somewhat-understood "universal finite type invariant of w-knots" and of an elusive "universal finite type invariant of v-knots".

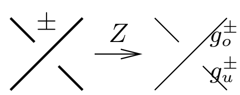


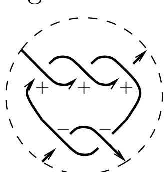


"divide and conquer"



Idea. Given a group G and two "YB" pairs $R^\pm = (g_o^\pm, g_u^\pm) \in G^2$, map them to xings and "multiply along", so that

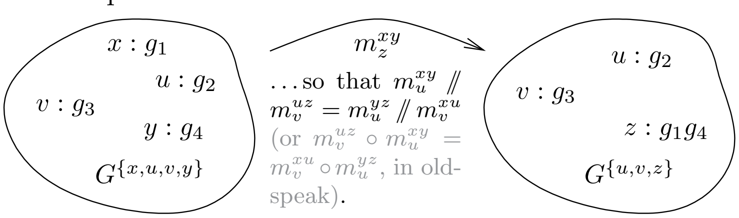




$$\overset{Z}{\longmapsto} \left(\begin{array}{c} g_o^+ g_u^+ g_o^+ g_u^- g_o^- g_u^+ g_o^+ g_u^+ \\ g_u^- g_o^- \end{array} \right)$$

This Fails! R2 implies that $g_o^\pm g_o^\mp = e = g_u^\pm g_u^\mp$ and then R3 implies that g_o^+ and g_u^+ commute, so the result is a simple counting invariant.

A Group Computer. Given G, can store group elements and perform operations on them:



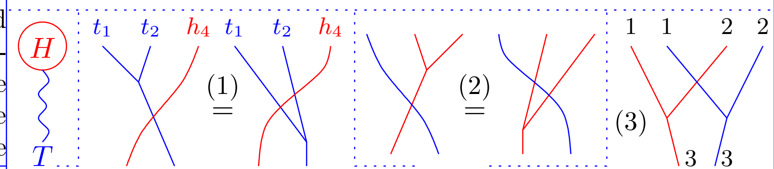
Also has S_x for inversion, e_x for unit insertion, d_x for register deletion, Δ_{xy}^z for element cloning, ρ_y^x for renamings, and $(D_1, D_2) \mapsto D_1 \cup D_2$ for merging, and many obvious composition axioms relating those. $P = \{x : g_1, y : g_2\} \Rightarrow P = \{d_y P\} \cup \{d_x P\}$

A Meta-Group. Is a similar "computer", only its internal structure is unknown to us. Namely it is a collection of sets $\{G_X\}$ indexed by all finite sets X, and a collection of operations m_z^{xy}, S_x, e_x, d_x, Δ_{xy}^z (sometimes), ρ_y^x, and $\cup$, satisfying the exact same *linear* properties.

Example 1. The non-meta example, $G_X := G^X$.

Example 2. $G_X := M_{X\times X}(\mathbb{Z})$, with simultaneous row and column operations, and "block diagonal" merges.

Bicrossed Products. If $G = HT$ is a group presented as a product of two of its subgroups, with $H \cap T = \{e\}$, then also $G = TH$ and G is determined by H, T, and the "swap" map $sw^{th} : (t,h) \mapsto (h', t')$ defined by $th = h't'$. The map sw satisfies (1) and (2) below; conversely, if $sw : T \times H \to H \times T$ satisfies (1) and (2) (+ lesser conditions), then (3) defines a group structure on $H \times T$, the "bicrossed product".



$tm_1^{12} /\!\!/ sw_{14} = sw_{24} /\!\!/ sw_{14} /\!\!/ tm_1^{12}$ $\qquad$ $gm_3^{12} := sw_{12} /\!\!/ tm_3^{12} /\!\!/ hm_3^{12}$

A **Meta-Bicrossed-Product** is a collection of sets $\beta(H,T)$ and operations tm_z^{xy}, hm_z^{xy} and sw_{xy}^{th} (and lesser ones), such that tm and hm are "associative" and (1) and (2) hold (+ lesser conditions). A meta-bicrossed-product defines a meta-group with $G_X := \beta(X,X)$ and gm as in (3).

β Calculus. Let $\beta(H,T)$ be

$$\left\{ \begin{array}{c|cccc} \omega & h_1 & h_2 & \cdots \\ \hline t_1 & \alpha_{11} & \alpha_{12} & \cdot \\ t_2 & \alpha_{21} & \alpha_{22} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \middle| \; \begin{array}{l} h_j \in H,\ t_i \in T, \text{ and } \omega \text{ and} \\ \text{the } \alpha_{ij} \text{ are rational func-} \\ \text{tions in variables } T_i, \text{ in bi-} \\ \text{jection with the } t_i\text{'s} \end{array} \right\},$$

$$tm_z^{xy} : \begin{array}{c|c} \omega & \cdots \\ \hline t_x & \alpha \\ t_y & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline t_z & \alpha+\beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & \alpha_1 & 0 \\ T_2 & 0 & \alpha_2 \end{array},$$

$$hm_z^{xy} : \begin{array}{c|ccc} \omega & h_x & h_y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & h_z & \cdots \\ \hline \vdots & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array},$$

$$sw_{xy}^{th} : \begin{array}{c|cc} \omega & h_y & \cdots \\ \hline t_x & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & h_y & \cdots \\ \hline t_x & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta-\gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1+\alpha$, $\langle\alpha\rangle := \sum_i \alpha_i$, and $\langle\gamma\rangle := \sum_{i\neq x} \gamma_i$, and let

$$R_{xy}^p := \begin{array}{c|cc} 1 & h_x & h_y \\ \hline t_x & 0 & T_x - 1 \\ t_y & 0 & 0 \end{array} \qquad R_{xy}^m := \begin{array}{c|cc} 1 & h_x & h_y \\ \hline t_x & 0 & T_x^{-1} - 1 \\ t_y & 0 & 0 \end{array}.$$

Theorem. Z^β is a tangle invariant (and more). Restricted to knots, the ω part is the Alexander polynomial. On links, it contains the multivariable Alexander polynomial. On braids, it is equivalent to the Burau representation.

Meta−Groups, Meta−Bicrossed−Products, and the Alexander Polynomial, 2

Why Happy? • Applications to w-knots.
• Everything that I know about the Alexander polynomial can be expressed cleanly in this language (even if without proof), except HF, but including genus, ribbonness, cabling, v-knots, knotted graphs, etc., and there's potential for vast generalizations.
• The least wasteful "Alexander for tangles" I'm aware of.
• Every step along the computation is the invariant of something.
• Fits on one sheet, including implementation.

I mean business!

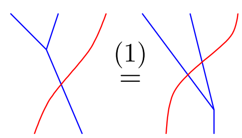
```
βSimp = Factor; SetAttributes[βCollect, Listable];
βCollect[B[ω_, Λ_]] := B[βSimp[ω],
   Collect[Λ, h_, Collect[#, t_, βSimp] &]];
βForm[B[ω_, Λ_]] := Module[{ts, hs, M},
   ts = Union[Cases[B[ω, Λ], (t | T)s_ ⧴ s, Infinity]];
   hs = Union[Cases[B[ω, Λ], hs_ ⧴ s, Infinity]];
   M = Outer[βSimp[Coefficient[Λ, h#1 t#2]] &, hs, ts];
   PrependTo[M, t# & /@ ts];
   M = Prepend[Transpose[M], Prepend[h# & /@ hs, ω]];
   MatrixForm[M]];
βForm[else_] := else /. β_B ⧴ βForm[β];
Format[β_B, StandardForm] := βForm[β];
```

```
⟨μ_⟩ := μ /. t_ → 1;
tm x_,y_→z_[β_] := β /. {t x|y → t z, T x|y → T z};
hm x_,y_→z_[B[ω_, Λ_]] := Module[
   {α = D[Λ, h x], β = D[Λ, h y], γ = Λ /. h x|y → 0},
   B[ω, (α + (1 + ⟨α⟩) β) h z + γ] // βCollect];
sw x_,y_[B[ω_, Λ_]] := Module[{α, β, γ, δ, ϵ},
   α = Coefficient[Λ, h y t x]; β = D[Λ, t x] /. h y → 0;
   γ = D[Λ, h y] /. t x → 0;   δ = Λ /. h y | t x → 0;
   ϵ = 1 + α;
   B[ω * ϵ, α (1 + ⟨γ⟩ / ϵ) h y t x + β (1 + ⟨γ⟩ / ϵ) t x
         + γ / ϵ h y                  + δ − γ * β / ϵ
    ] // βCollect];
gm x_,y_→z_[β_] := β // sw x,y // hm x,y→z // tm x,y→z;
B /: B[ω1_, Λ1_] B[ω2_, Λ2_] := B[ω1 * ω2, Λ1 + Λ2];
Rp x_,y_ := B[1, (T x − 1) t x h y];
Rm x_,y_ := B[1, (T x^-1 − 1) t x h y];
```

The key trick: $\begin{array}{c|c} \omega & h_j \\ \hline t_i & \alpha_{ij} \end{array} \longleftrightarrow B\left(\omega, \Lambda = \sum_{i,j} \alpha_{ij} t_i h_j\right).$

```
{β = B[ω, Sum[α10 i+j ti hj, {i, {1, 2, 3}}, {j, {4, 5}}]],
  β // tm1,2→1 // sw1,4,
  β // sw2,4 // sw1,4 // tm1,2→1
 } // ColumnForm
```

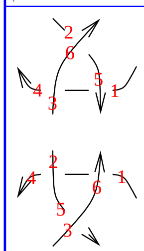
Some testing...



$$\begin{pmatrix} \omega & h_4 & h_5 \\ t_1 & \alpha_{14} & \alpha_{15} \\ t_2 & \alpha_{24} & \alpha_{25} \\ t_3 & \alpha_{34} & \alpha_{35} \end{pmatrix}$$

$$\begin{pmatrix} \omega\,(1+\alpha_{14}+\alpha_{24}) & h_4 & h_5 \\ t_1 & \frac{(\alpha_{14}+\alpha_{24})(1+\alpha_{14}+\alpha_{24}+\alpha_{34})}{1+\alpha_{14}+\alpha_{24}} & \frac{(\alpha_{15}+\alpha_{25})(1+\alpha_{14}+\alpha_{24}+\alpha_{34})}{1+\alpha_{14}+\alpha_{24}} \\ t_3 & \frac{\alpha_{34}}{1+\alpha_{14}+\alpha_{24}} & \frac{-\alpha_{15}\alpha_{34}-\alpha_{25}\alpha_{34}+\alpha_{35}+\alpha_{14}\alpha_{35}+\alpha_{24}\alpha_{35}}{1+\alpha_{14}+\alpha_{24}} \end{pmatrix}$$

$$\begin{pmatrix} \omega\,(1+\alpha_{14}+\alpha_{24}) & h_4 & h_5 \\ t_1 & \frac{(\alpha_{14}+\alpha_{24})(1+\alpha_{14}+\alpha_{24}+\alpha_{34})}{1+\alpha_{14}+\alpha_{24}} & \frac{(\alpha_{15}+\alpha_{25})(1+\alpha_{14}+\alpha_{24}+\alpha_{34})}{1+\alpha_{14}+\alpha_{24}} \\ t_3 & \frac{\alpha_{34}}{1+\alpha_{14}+\alpha_{24}} & \frac{-\alpha_{15}\alpha_{34}-\alpha_{25}\alpha_{34}+\alpha_{35}+\alpha_{14}\alpha_{35}+\alpha_{24}\alpha_{35}}{1+\alpha_{14}+\alpha_{24}} \end{pmatrix}$$




```
{Rm5,1 Rm6,2 Rp3,4 // gm1,4→1 // gm2,5→2 // gm3,6→3,
  Rp6,1 Rm2,4 Rm3,5 // gm1,4→1 // gm2,5→2 // gm3,6→3}
```

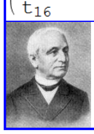
$$\left\{ \begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+T_2}{T_2} & 0 \\ t_3 & \frac{-1+T_3}{T_2} & -\frac{-1+T_3}{T_3} \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+T_2}{T_2} & 0 \\ t_3 & \frac{-1+T_3}{T_2} & -\frac{-1+T_3}{T_3} \end{pmatrix} \right\}$$

... divide and conquer!

```
β = Rm12,1 Rm2,7 Rm8,3 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15
```

8_{17}

1	h_1	h_3	h_5	h_7	h_9	h_{11}	h_{13}	h_{15}
t_2	0	0	0	$-\frac{-1+T_2}{T_2}$	0	0	0	0
t_4	0	0	0	0	0	$-\frac{-1+T_4}{T_4}$	0	0
t_6	0	0	0	0	0	0	$-1+T_6$	0
t_8	0	$-\frac{-1+T_8}{T_8}$	0	0	0	0	0	0
t_{10}	0	0	0	0	0	0	0	$-1+T_{10}$
t_{12}	$-\frac{-1+T_{12}}{T_{12}}$	0	0	0	0	0	0	0
t_{14}	0	0	0	0	$-1+T_{14}$	0	0	0
t_{16}	0	0	$-1+T_{16}$	0	0	0	0	0



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org

```
Do[β = β // gm1,k→1, {k, 2, 10}]; β
```

8_{17}, cont.

$$\begin{pmatrix} \frac{T_1^2+T_{16}-T_1 T_{16}}{T_1^2} & h_1 & h_{11} & h_{13} & h_{15} \\ t_1 & -\frac{(-1+T_1) T_{14} (T_1^3+T_{16}^2)}{T_1^2 T_{12} (T_1^2+T_{16}-T_1 T_{16})} & -\frac{(-1+T_1)(1-T_1+T_1^2) T_{14} T_{16}}{T_1 (T_1^2+T_{16}-T_1 T_{16})} & \frac{(-1+T_1)(1-T_1+T_1^2) T_{14}}{T_1^2+T_{16}-T_1 T_{16}} & -1+T_1 \\ t_{12} & -\frac{-1+T_{12}}{T_{12}} & 0 & 0 & 0 \\ t_{14} & \frac{(-1+T_{14})(-T_1+T_1^2+T_{16})}{T_{12}(T_1^2+T_{16}-T_1 T_{16})} & \frac{(-1+T_1)(1-T_1+T_1^2)(-1+T_{14}) T_{16}}{T_1 (T_1^2+T_{16}-T_1 T_{16})} & -\frac{(-1+T_1)(1-T_1+T_1^2)(-1+T_{14})}{T_1^2+T_{16}-T_1 T_{16}} & 0 \\ t_{16} & \frac{T_1(-1+T_{16})}{T_{12}(T_1^2+T_{16}-T_1 T_{16})} & \frac{(-1+T_1) T_1 (-1+T_{16})}{T_1^2+T_{16}-T_1 T_{16}} & -\frac{(-1+T_1)^2(-1+T_{16})}{T_1^2+T_{16}-T_1 T_{16}} & 0 \end{pmatrix}$$

James Waddell Alexander

```
Do[β = β // gm1,k→1, {k, 11, 16}]; β
```

$$\begin{pmatrix} -\frac{1-4T_1+8T_1^2-11T_1^3+8T_1^4-4T_1^5+T_1^6}{T_1^3} & h_1 \\ t_1 & 0 \end{pmatrix}$$

```
<< KnotTheory`
Alexander[Knot[8, 17]][T1] // Factor

Loading KnotTheory` version of August 22, 2010, 13:36:57.55.
Read more at http://katlas.org/wiki/KnotTheory.
KnotTheory::loading : Loading precomputed data in PD4Knots`.
```

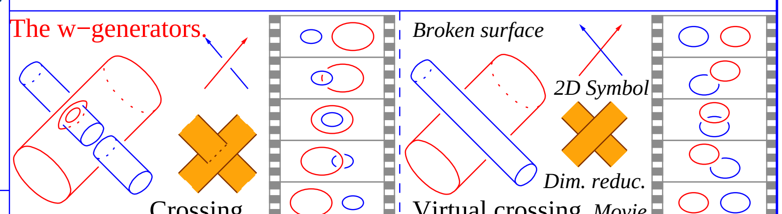
$$-\frac{1-4T_1+8T_1^2-11T_1^3+8T_1^4-4T_1^5+T_1^6}{T_1^3}$$

Where does it come from? The accidental[1] answer is that it is a symbolic calculus for a natural reduction[4] of the unique homomorphic expansion[2] of w-tangles[3].

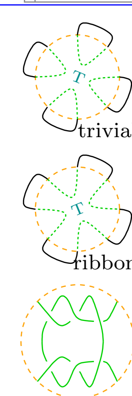
1. "Accidental" for it's only how I came about it. There ought to be a better answer.
2. A "homomorphic expansion", aka as a homomorphic universal finite type invariant, is a completely canonical construct whose presence implies that the objects in questions are susceptible to study using graded algebra.
3. "v-Tangles" are the meta-group generated by crossings modulo Reidemeister moves. "w-Tangles" are a natural quotient of v-tangles. They are at least related and perhaps identical to a certain class of 1D/2D knots in 4D.
4. To "only what is visible by the 2D Lie algebra".

A certain generalization will arise by not reducing as in 4. A vast generalization may arise when homomorphic expansions for v-tangles are understood, a task likely equivalent to the Etingof-Kazhdan quantization of Lie bialgebras.

The w−generators.



A Partial To Do List. 1. Where does it *more simply* come from?
2. Remove all the denominators.
3. How do determinants arise in this context (×2)?
4. Understand links.
5. Find the "reality condition".
6. Do some "Algebraic Knot Theory".
7. Categorify.
8. Do the same in other natural quotients of the v/w-story.



From Drorbn

In Progress

Finite Type Invariants of W-Knotted Objects: From Alexander to Kashiwara and Vergne

Joint with Zsuzsanna Dancso

Download WKO.pdf: last updated ≥ March 3, 2012. first edition: not yet.

Abstract. w-Knots, and more generally, w-knotted objects (w-braids, w-tangles, etc.) make a class of knotted objects which is wider but weaker than their "usual" counterparts. To get (say) w-knots from u-knots, one has to allow non-planar "virtual" knot diagrams, hence enlarging the the base set of knots. But then one imposes a new relation, the "overcrossings commute" relation, further beyond the ordinary collection of Reidemeister moves, making w-knotted objects a bit weaker once again.

The group of w-braids was studied (under the name "welded braids") by Fenn, Rimanyi and Rourke [FRR] and was shown to be isomorphic to the McCool group [Mc] of "basis-conjugating" automorphisms of a free group F_n - the smallest subgroup of $\mathrm{Aut}(F_n)$ that contains both braids and permutations. Brendle and Hatcher [BH], in work that traces back to Goldsmith [Gol], have shown this group to be a group of movies of flying rings in $\mathbb{R}^3$. Satoh [Sa] studied several classes of w-knotted objects (under the name "weakly-virtual") and has shown them to be closely related to certain classes of knotted surfaces in $\mathbb{R}^4$. So w-knotted objects are algebraically and topologically interesting.

In this article we study finite type invariants of several classes of w-knotted objects. Following Berceanu and Papadima [BP], we construct a homomorphic universal finite type invariant of w-braids, and hence show that the McCool group of automorphisms is "1-formal". We also construct a homomorphic universal finite type invariant of w-tangles. We find that the universal finite type invariant of w-knots is more or less the Alexander polynomial (details inside).

Much as the spaces $\mathcal{A}$ of chord diagrams for ordinary knotted objects are related to metrized Lie algebras, we find that the spaces $\mathcal{A}^w$ of "arrow diagrams" for w-knotted objects are related to not-necessarily-metrized Lie algebras. Many questions concerning w-knotted objects turn out to be equivalent to questions about Lie algebras. Most notably we find that a homomorphic universal finite type invariant of w-knotted trivalent graphs is essentially the same as a solution of the Kashiwara-Vergne [KV] conjecture and much of the Alekseev-Torrosian [AT] work on Drinfel'd associators and Kashiwara-Vergne can be re-interpreted as a study of w-knotted trivalent graphs.

The true value of w-knots, though, is likely to emerge later, for we expect them to serve as a warmup example for what we expect will be even more interesting - the study of virtual knots, or v-knots. We expect v-knotted objects to provide the global context whose projectivization (or "associated graded structure") will be the Etingof-Kazhdan theory of deformation quantization of Lie bialgebras [EK].

Wideo Companion

The **wClips Seminar** is a series of weekly wideotaped meetings at the University of Toronto, systematically going over the content of the WKO paper section by section.

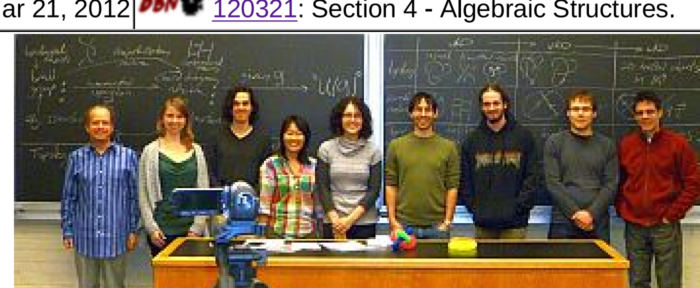
Next Meetings. On Wednesday March 28 we will have an out-of-sequence not-on-video meeting to watch and discuss the video of my talk at George Washington University (see Talks: GWU-1203). We will meet at 12 at my office (not the usual place!), and start watching the video shortly after that (less socializing!). On Wednesday April 4, 2012, 12-2, at Bahen 4010 we will return to the main sequence and talk about Section 3.7, "the Alexander polynomial".

Announcements. small circle, wide circle, UofT, LDT Blog (also here). Email Dror to **join our mailing list!**

Resources. How to use this site, Dror's notebook, blackboard shots.

The wClips

Date	Links
Jan 11, 2012	120111-1: Introduction. 120111-2: Section 2.1 - v-Braids.
Jan 18, 2012	120118-1: An introduction to this web site. 120118-2: Section 2.2 - w-Braids by generators and relations and as flying rings. 120118-3: Section 2.2 - w-Braids - other drawing conventions, "wens".
Jan 25, 2012	120125-1: Section 2.2.3 - basis conjugating automorphisms of F_n. 120125-2: A very quick introduction to finite type invariants in the "u" case.
Feb 1, 2012	120201: Section 2.3 - finite type invariants of v- and w-braids, arrow diagrams, 6T, TC and 4T relations, expansions / universal finite type invariants.
Feb 8, 2012	120208: Review of u,v, and w braids and of Section 2.3.
Feb 15, 2012	120215: Section 2.5 - mostly compatibilities of Z^w, also injectivity and uniqueness of Z^w.
Feb 22, 2012	120222: Section 2.5.5, $\alpha : \mathcal{A}^u \to \mathcal{A}^v$, and Section 3.1 (partially), the definition of v- and w-knots.
Feb 29, 2012	120229: Sections 3.1-3.4: v-Knots and w-Knots: Definitions, framings, finite type invariants, dimensions, and the expansion in the w case.
Mar 7, 2012	120307: Section 3.5: Jacobi diagrams and the bracket-rise theorem.
Mar 14, 2012	120314: Section 3.6 - the relation with Lie algebras.
Mar 21, 2012	120321: Section 4 - Algebraic Structures.



Group photo on January 11, 2012: DBN, ZD, Stephen Morgan, Lucy Zhang, Iva Halacheva, David Li-Bland, Sam Selmani, Oleg Chterental, Peter Lee.

The Most Important Missing Infrastructure Project in Knot Theory

January-23-12
10:12 AM

An "infrastructure project" is hard (and sometimes non-glorious) work that's done now and pays off later.

An example, and the most important one within knot theory, is the tabulation of knots up to 10 crossings. I think it precedes Rolfsen, yet the result is often called "the Rolfsen Table of Knots", as it is famously printed as an appendix to the famous book by Rolfsen. There is no doubt the production of the Rolfsen table was hard and non-glorious. Yet its impact was and is tremendous. Every new thought in knot theory is tested against the Rolfsen table, and it is hard to find a paper in knot theory that doesn't refer to the Rolfsen table in one way or another.

A second example is the Hoste-Thistlethwaite tabulation of knots with up to 17 crossings. Perhaps more fun to do as the real hard work was delegated to a machine, yet hard it certainly was: a proof is in the fact that nobody so far had tried to replicate their work, not even to a smaller crossing number. Yet again, it is hard to overestimate the value of that project: in many ways the Rolfsen table is "not yet generic", and many phenomena that appear to be rare when looking at the Rolfsen table become the rule when the view is expanded. Likewise, other phenomena only appear for the first time when looking at higher crossing numbers.

But as I like to say, knots are the wrong object to study in knot theory. Let me quote (with some variation) my own (with Dancso) "WKO" paper:

> Studying knots on their own is the parallel of studying cakes and pastries as they come out of the bakery - we sure want to make them our own, but the theory of desserts is more about the ingredients and how they are put together than about the end products. In algebraic knot theory this reflects through the fact that knots are not finitely generated in any sense (hence they must be made of some more basic ingredients), and through the fact that there are very few operations defined on knots (connected sums and satellite operations being the main exceptions), and thus most interesting properties of knots are transcendental, or non-algebraic, when viewed from within the algebra of knots and operations on knots (see [AKT-CFA]).
>
> The right objects for study in knot theory are thus the ingredients that make up knots and that permit a richer algebraic structure. These are braids (which are already well-studied and tabulated) and even more so tangles and tangled graphs.

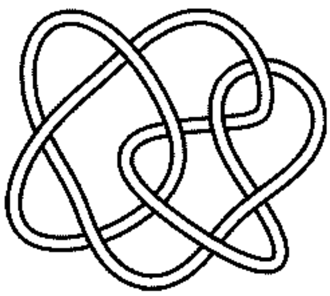
Thus in my mind the most important missing infrastructure project in knot theory is the tabulation of tangles to as high a crossing number as practical. This will enable a great amount of testing and experimentation for which the grounds are now still missing. The existence of such a tabulation will greatly impact the direction of knot theory, as many tangle theories and issues that are now ignored for the lack of scope, will suddenly become alive and relevant. The overall influence of such a tabulation, if done right, will be comparable to the influence of the Rolfsen table.

Aside. What are tangles? Are they embedded in a disk? A ball? Do they have an "up side" and a "down side"? Are the strands oriented? Do we mod out by some symmetries or figure out the action of some symmetries? Shouldn't we also calculate the affect of various tangle operations (strand doubling and deletion, juxtapositions, etc.)? Shouldn't we also enumerate virtual tangles? w-tangles? Tangled graphs?

In my mind it would be better to leave these questions to the tabulator. Anything is better than nothing, yet good tabulators would try to tabulate the more general things from which the more special ones can be sieved relatively easily, and would see that their programs already contain all that would be easy to implement within their frameworks. Counting legs is easy and can be left to the end user. Determining symmetries is better done along with the enumeration itself, and so it should.

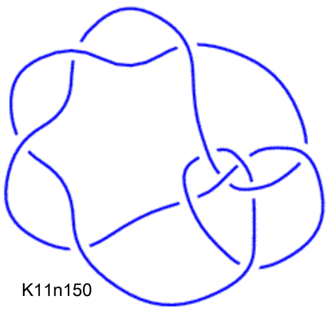
An even better tabulation should come with a modern front-end - a set of programs for basic manipulations of tangles, and a web-based "tangle atlas" for an even easier access.

Overall this would be a major project, well worthy of your time.

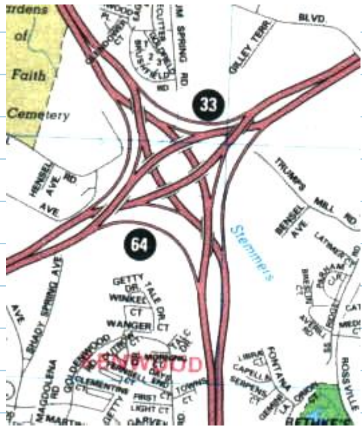


(KnotPlot image)

9_42 is Alexander Stoimenow's favourite

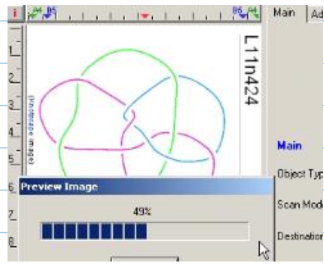


(Knotscape image)



The interchange of I-95 and I-695, northeast of Baltimore. (more)

From [AKT-CFA]



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