A Brief Introduction to Schemes and Sheaves

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1 Introduction

Scheme theory, perhaps more than any other subject, has a reputation for being extremely difficult and tedious to learn. One gets the impression that the subject involves many highly technical and difficult constructions, is exceedingly vast and abstract, and that it takes considerable time and energy before one is able to prove anything of value. Quite famously, the subject originated from Grothendieck's attempt to "simplify" an eighty page paper by Serre into the thousand page document that was to become *Les Éléments de géométrie algébrique* — a fact that is both oddly remarkable and offers little encouragement.

It is perhaps somewhat surprising, then, that there seems to be no shortage of graduate students and even undergraduates eager to devote time to understand schemes. The usual procedure is to sit down with a copy of Hartshorne, formally sift through a seemingly endless series of complex definitions, and then grudgingly admit defeat. Usually absent from these attempts at understanding schemes are good sources of intuition, motivation, and clear and identifiable goals. The result is that students learning the subject this way have difficulty explaining the "point" of a definition or a construction, and so don't know what it's related to, why it's there, and consequently can't use it.

The purpose of this article is to give the basic definitions of scheme theory in context. We will take the view that it is just as important, if not more so, to explain the *definitions themselves* as it is to explain the lemmas and the proofs. In doing so, we hope to remedy a common affliction that befalls those who read Hartshorne's book: not having any idea what is going on.

2 Schemes are like Manifolds

Our exposition of schemes will be according to the following mantra: "a scheme is to a variety as an abstract manifold is to an embedded submanifold of Euclidean space." As our goal is to explain the term "scheme", and not "variety", "abstract manifold" or "embedded submanifold of Euclidean space," we will assume that the reader understands and is comfortable with the latter three terms.[1] By assuming this background, we hope to explain and motivate many of the constructions in schemes *by analogy*, and thereby make the motivation for the subject clear.

Our mantra, as stated, already poses an interesting question: what is the difference between an abstract manifold (say smooth), and a submanifold of Euclidean space? The

[1] Here when we say "understand varieties" we mean in the ordinary sense of algebraic sets defined by polynomial equations, not in the sense of scheme theory.

Witney embedding theorem, which tells us that any smooth real manifold of dimension n can be embedded in a Euclidean space of dimension $2n$, would seem to suggest that there is none; with the usual notion of equivalence, diffeomorphism, smooth manifolds are the same as smooth submanifolds of $\mathbb{R}^m$. The answer, it turns out, is that the purpose of the smooth manifold construction is not to study objects which are *different* from submanifolds of Euclidean space, but to formulate the subject in a way which emphasizes the features of interest and avoids clouding the picture with artifacts corresponding to any particular embedding. That is, smooth manifold theory provides a *language* in which one can study manifolds, which once internalized, becomes more expressive than the more immediately accessible one which describes submanifolds of $\mathbb{R}^m$.

An illustrative example of this phenomenon is Maxwell's equations. Maxwell originally formulated electromagnetism with twenty equations. Modern vector notation brings this down to four. The language of smooth manifold theory requires just two: letting F be a 2-form (the electromagnetic field) and J a 3-form (corresponding to current density), the Maxwell equations on a smooth 4-dimensional Lorentzian manifold read

$$dF = 0 \qquad\qquad d \star F = \mu_0 J,$$

where d is the exterior derivative, and $\star$ is the Hodge star operator. Unlike the vector calculus version of the Maxwell equations, this modern formulation does not privilege any particular coordinate system, and thus despite being in some sense equivalent to the vector calculus version, it is actually more powerful. Of course, despite the apparent simplicity of these equations, the theory needed to formulate them is significantly more involved than what's needed to state the same equations in their original form. What has been achieved is a kind of simplicity at the expense of efficiency, and it is in precisely this sense that Grothendieck's development of scheme theory can be said to be a "simplification" of classical developments in geometry and algebra.

3 Core Definitions

3.1 Sheaves

The usual procedure when defining any kind of geometric space consists of two steps. In the first, one specifies a topological space, and in the second, one specifies a class of functions which give it its structure. For instance, one can speak of differentiable manifolds, smooth manifolds, analytic manifolds, or simply topological manifolds; each construction starts by specifying a second countable Hausdorff topological space, and the constructions differ due to the nature of the functions defined on the space. In this regard, the situation with schemes is no different. Just like a smooth manifold structure can be specified by giving a topological space and describing which functions are smooth, a scheme can be presented by giving an appropriate topological space and describing which functions are *regular*.

The tool we will use for managing the regular functions on the space is called a *sheaf*. Sheaves are general tools whose purpose is to define collections objects in some category (e.g. Sets, Groups, Rings, or Modules) which are stitched together topologically. In our case, we will have a space X, and to each open set $U \subset X$ we will have a ring $\mathcal{O}_X(U)$ giving the regular functions on U. We will want these rings to be related to each other by restriction maps and by the fact that we can "glue" families of functions defined over a cover to get a function on a larger open set; sheaves will encode this idea. More generally, sheaves can be used to define things like line and vector bundles by specifying their spaces of sections over any open set, and describing how those sections restrict to one another and glue together. Useful classes of sheaves also have the property that they form highly-structured categories: maps of sheaves can have kernels, images, and correspond to quotients. For this reason, they are also useful in developing various cohomology theories, and this application was also the subject of the aforementioned paper of Serre.

Before defining sheaves, we begin with the notion of a presheaf.

Definition 3.1. A *presheaf* $\mathcal{F}$ of *rings* associated to a topological space X consists of the following data:

(i) To each open set $U \subset X$, a ring $\mathcal{F}(U)$.

(ii) To each inclusion of open sets $U \hookrightarrow V$ a map $\mathrm{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ called the *restriction map* from $\mathcal{F}(V)$ to $\mathcal{F}(U)$. These maps satisfy the property that $\mathrm{res}_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$ and $\mathrm{res}_{V,U} \circ \mathrm{res}_{W,V} = \mathrm{res}_{W,U}$ where $U \subset V \subset W$ are open sets.

If $s \in \mathcal{F}(U)$, we call s a *section* of $\mathcal{F}$ over U.[2]

As an equivalent but more sophisticated formulation, one can consider the topological space X as a category, whose objects are open sets of X and whose maps are inclusions of open sets, and then define a presheaf on X as simply a contravariant functor from X to the category **Ring**. Replacing **Ring** with another category one can get presheaves of groups, modules, etc., but we will largely limit ourselves to rings in this article. When not otherwise specified, we will use the term "presheaf" (and later "sheaf") to refer to the case of rings.

Examples of presheaves can easily be found in manifold theory. If M is a C^k-manifold, the object $C^k(-)$ which associates to any open set $U \subset M$ the k-times differentiable functions $C^k(U)$ on U is a sheaf, with restriction maps corresponding to restrictions of functions. If E is a bundle on M, then the object $\Gamma(-, E)$, which associates to U the collection $\Gamma(U, E)$ of C^k-times differentiable sections of E over U is also a sheaf (of modules) with the obvious restriction maps. The reader can doubtless supply more examples by making the obvious changes to the manifold structure.

The preceding examples all have the property that sections over some open set U can be described as having been obtained by "gluing together" sections over some cover of U. In general, the presheaf axioms impose no such requirement. It is easy to cook up contrived examples by taking a two-point topological space with the discrete topology, and we will see some more interesting examples later. For now, it suffices to say that we need some additional properties on our presheaves, and presheaves with those properties will be called *sheaves*.

Definition 3.2. A *sheaf* $\mathcal{F}$ of rings on X is a presheaf of rings satisfying the following additional properties for any open cover $\{U_i\}_{i \in I}$ of any open set $U \subset X$:

(i) Suppose that $f_i \in \mathcal{F}(U_i)$ are a collection of sections which agree on overlaps (formally, $\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j$ whenever the intersection exists). Then they lift to a $f \in \mathcal{F}(U)$ which has the property that $\mathrm{res}_{U,U_i} f = f_i$ for all $i \in I$.

(ii) Suppose that $f, f' \in \mathcal{F}(U)$ and that $\mathrm{res}_{U,U_i} f = \mathrm{res}_{U,U_i} f'$ for all $i \in I$. Then $f = f'$.

[2]The terminology comes from thinking of $\mathcal{F}$ as corresponding to some sort of bundle.

Less formally, property (i) says that given collections of sections that agree on overlaps, we can glue them together to give sections over some larger set, and property (ii) says that sections are determined by their restrictions. These properties are commonly called the gluing and identity axioms respectively.

In usual mathematics style, we follow the description of a class of objects by a description of their morphisms. To first approximation, a morphism of sheaves is something that can be thought of as having been induced by a pullback map. For instance, if $F : M \to N$ is a map of smooth manifolds, we have a pullback map map $F^* : C^\infty(N) \to C^\infty(M)$ given by $F^*(g) = g \circ F$. By abuse of notation, we also have a map $F^* : C^\infty(V) \to C^\infty(F^{-1}(V))$ where $V \subset N$ is an open set, and defined in the same way. The map F^* (or rather maps, since there is in fact a different one for each open $V \subset N$) gives a morphism between the sheaf $C^\infty(-)$ of smooth functions on N and the pushforward sheaf (yes the name is confusing!) $C^\infty(F^{-1}(-))$, which is also a sheaf on N. Note that the ordinary pullback map F^* which induces this map of sheaves is simply the case when $V = N$. As usual with morphisms, we require that they preserve some extra structure. This next definition makes this precise.

Definition 3.3. Let X be a topological space, and let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. A *morphism of sheaves* $\eta : \mathcal{F} \to \mathcal{G}$ has the following properties:

(i) For each open set $U \subset X$, there is a morphism (of rings, in our case) $\eta_U : \mathcal{F}(U) \to \mathcal{G}(U)$.

(ii) If $U \subset V \subset X$, then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\eta_V} & \mathcal{G}(V) \\ {\scriptstyle \mathrm{res}_{V,U}} \downarrow & & \downarrow {\scriptstyle \mathrm{res}_{V,U}} \\ \mathcal{F}(U) & \xrightarrow{\eta_U} & \mathcal{G}(U) \end{array}$$

In plain language: restricting a section and then applying the map is the same as applying the map and then restricting.

One easily checks that the pullback map described above satisfies the above definition. If we use the functorial language (in which a presheaf was really a certain contravariant functor) then a morphism of sheaves is a natural transformation between the two sheaf functors.

A useful construction connected with sheaves is that of a *stalk*. A stalk is an object associated to any point p which captures the information of the sheaf which is defined in any neighbourhood of p. For instance, in the manifold case, the tangent space at a point is determined in any neighbourhood p, and we will find that there is a natural way of defining tangent spaces on appropriate schemes using stalks. Note that the behaviour of a function at a point can in some sense be regarded as something that's not really determined *at* the point but in any neighbourhood of it. For instance, if one wants to differentiate a function, it does not suffice to know its value at a point, but it suffices to know its values in any neighbourhood of that point, or its value *at the level of stalks*. It is also true that morphisms of sheaves are determined by the morphisms they induce on stalks, a fact which is often useful.

Definition 3.4. Let $p \in X$ be a point, and $\mathcal{F}$ a sheaf on X. The stalk $\mathcal{F}_p$ at p is the ring obtained by the following construction:

(i) The set of $\mathcal{F}_p$ is

$$\mathcal{F}_p := \{(f, U) : U \subset X \text{ open containing } p,\ f \in \mathcal{F}(U)\}/\sim,$$

where the equivalence relation $\sim$ is that $(f, U) \sim (g, V)$ if $\operatorname{res}_{U,U\cap V} f = \operatorname{res}_{V,U\cap V} g$.

(ii) The ring operations are defined via $(f, U)+(g, V) = (f+g, U\cap V)$, and $(f, U)\cdot(g, V) = (fg, U \cap V)$. One easily checks this is well-defined.

Note that the above intersections are always non-empty since all open sets under consideration contain p. We often omit the second entry of the ordered tuple and simply write $f \in \mathcal{F}_p$, with the understanding that we are only interested in the behaviour of f determined in any open neighbourhood of p. We call $f \in \mathcal{F}_p$ the *germ* of the function f at p.

It is not too difficult to see that one may restrict the allowed open subsets to just those that belong to a basis for the topology, since any open neighbourhood of p contains a basis neighbourhood.

In some sense, the above definition is really too concrete. The reason is that the importance of stalks stems largely from their relationship with the spaces of sections from which they are constructed; the exact construction of the stalk itself doesn't matter so much, provided that we have natural maps $\mathcal{F}(U) \to \mathcal{F}_p$ (in our case given by $f \mapsto (f, U)$) and that these maps behave nicely with the sheaf structure. This will become particularly evident when we define the structure sheaf of a scheme, where the associated stalks, although isomorphic to ones defined using the construction above, will typically have a much more natural description using localizations of rings. For this reason, the categorical definition of the same concept is often cleaner:

$$\mathcal{F}_p := \varinjlim_{p\in U} \mathcal{F}(U).$$

Given a morphism $\eta : \mathcal{F} \to \mathcal{G}$ of sheaves on X, one obtains a natural morphism of the stalks $\mathcal{F}_p \to \mathcal{G}_p$ induced by η. This is easy to see from the usual universal property nonsense, since $\mathcal{F}_p$ represents morphisms out of the spaces of sections $\mathcal{F}(U)$ lying over p satisfying the constraints imposed by the restriction maps, and the map $\eta : \mathcal{F} \to \mathcal{G}$ gives such a family of morphisms into $\mathcal{G}_p$ after post-composing with the maps $\mathcal{G}(U) \to \mathcal{G}_p$. Of course, this has the direct description of taking $(f, U) \in \mathcal{F}_p$ and mapping it to $(\eta_U(f), U) \in \mathcal{G}_p$, which can be shown to be well-defined.

We mentioned before that one often obtains sheaves on a space X from looking at bundles over X. In fact, there is a way of viewing *all* sheaves as coming from a kind of bundle. This construction uses the so-called *étalé space* of a sheaf, which is constructed (as a set) as the disjoint union $\bigsqcup_{p\in X} \mathcal{F}_p$. The "bundle map" is then the map $\pi : \bigsqcup_{p\in X} \mathcal{F}_p \to X$ which sends $f_p \in \mathcal{F}_p \mapsto p$, that is, which sends a germ at the point p to the point p in X. One then thinks of $\mathcal{F}_p$ as the fibre of π above p in the usual way. In his original eighty-page paper, Serre defined the notion of a sheaf using the étalé space construction. In his formulation, a sheaf was a collection of stalks $\mathcal{F}_p$ with a topology on the disjoint union $\bigsqcup_{p\in X} \mathcal{F}_p$ (subject to some restrictions). One would then define sections of $\mathcal{F}$ to be continuous maps from X to $\mathcal{F}$ whose composition with the projection π was the identity (i.e., as ordinary sections). Grothendieck, in his usual style, took Serre's definition and functorialized it, but the étalé space construction still plays an important role. The next definition illustrates this point.

Definition 3.5. Let $\mathcal{F}$ be a presheaf. The *sheafification* of a sheaf is in some sense the simplest sheaf that can be made from $\mathcal{F}$. We give two definitions:

(i) The *sheafification* of $\mathcal{F}$ is a sheaf $\mathcal{F}^{sh}$ and a map of presheaves[3] $\alpha : \mathcal{F} \to \mathcal{F}^{sh}$ such that for any other sheaf $\mathcal{G}$ and map of presheaves $\eta : \mathcal{F} \to \mathcal{G}$, there is a unique map $\beta : \mathcal{F}^{sh} \to \mathcal{G}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\alpha} & \mathcal{F}^{sh} \\ & \searrow^{\eta} & \downarrow{\scriptstyle \beta} \\ & & G \end{array}$$

(ii) Alternatively, we may give an explicit construction as follows. Define a topology on $\bigsqcup_{p \in X} \mathcal{F}_p$ as follows. For each $f \in \mathcal{F}(U)$, the set $\{f_p : p \in U\}$, where f_p is the germ of f at p, is open in $\bigsqcup_{p \in X} \mathcal{F}_p$. The topology is the one generated by these sets. We then define:

$$\mathcal{F}^{sh}(U) := \left\{ \text{continuous maps } s : U \to \bigsqcup_{p \in X} \mathcal{F}_p \right\}.$$

The map $\alpha : \mathcal{F} \to \mathcal{F}^{sh}$ is the one that sends $f \in \mathcal{F}(U)$ to the map $(p \mapsto f_p) \in \mathcal{F}(U)^{sh}$.

The idea of the sheafification is that it turns a presheaf into a sheaf by insisting that the elements of the sheaf be genuine "sections" of some bundle-like space, and thereby ensuring that the two sheaf axioms are satisfied (the sheaf axioms are easy to check for sections of bundles). To understand why this may be necessary, we consider an illustrative example. Let X be a Riemann Surface (a 2-dimensional real manifold whose charts are analytic functions to open subsets of $\mathbb{C}$), and let $\mathcal{O}^\times(-)$ be the sheaf which assigns to $U \subset X$ the set $\mathcal{O}^\times(U)$ of non-vanishing holomorphic functions on U. Then one can define a new sheaf $(\exp \mathcal{O}^\times)(U) := \{\exp \circ f : f \in \mathcal{O}^\times(U)\}$. We then wish to consider a quotient $\mathcal{O}^\times/(\exp \mathcal{O}^\times)$. The naïve thing to do is to define $(\mathcal{O}^\times/\exp \mathcal{O}^\times)(U) := \mathcal{O}^\times(U)/(\exp \mathcal{O}^\times)(U)$, but the result need only be a presheaf, and not a sheaf. To see why, consider a global section $f \in (\mathcal{O}^\times/\exp \mathcal{O}^\times)(X)$. Then given $p \in X$, one can find a small enough neighbourhood U of p such that it is possible to define a holomorphic branch of logarithm on $f(U)$, and hence express f as $\exp \circ (\log \circ f)$. Thus there is an open cover of X such that f restricts to 0 on that open cover, but f is not itself 0, violating the identity axiom. The solution is to take the naïve quotient, and then apply a sheafification.

In fact, the étalé space construction can do more than just turn a presheaf into a sheaf, it can also help us create a sheaf from partial data. For instance, as we will see when constructing affine schemes, it is often the case that we know what we want our sheaf to look like on a distinguished base for the topology on X, but we don't have a good description for the sections $\mathcal{F}(U)$ for arbitrary open sets $U \subset X$. The distinguished base will suffice to generate the topology on $\bigsqcup_{p \in X} \mathcal{F}_p$, and so we may define the sections of $\mathcal{F}(U)$ to be continuous maps $U \to \bigsqcup_{p \in X} \mathcal{F}_p$ as before. This can be viewed as a kind of "analytic continutation," in the sense that continuous sections of $\bigsqcup_{p \in X} \mathcal{F}_p$ are exactly ones which locally look like $p \mapsto f_p$ in some neighbourhood of p, and so one can get an element of $\mathcal{F}(U)$ by gluing together sections over the distinguished base. In fact, it suffices to specify a morphism of sheaves on a distinguished base too, since the induced map over the other open sets can be recovered from a similar gluing process.

[3]This is the same as a map of sheaves.

3.2 Affine Schemes

We begin with a reminder of our general strategy. We began our section on sheaves by observing that geometric constructions typically proceed via two steps: first one defines a topological space, and then one specifies a class of functions on the space which give it its structure. We then claimed that sheaves will be a necessary tool for specifying the functions on schemes, and spent some time discussing their properties. We now turn our attention to the first part of this construction, namely, the underlying topological space of a scheme.

There is a general principle of abstraction in mathematics which says the following: to abstract a concept, describe not what it is, but what it does. The simplest example of this phenomenon is the notion of a *vector*, which is commonly introduced in early mathematics education as something akin to an "arrow in space," and later abstracted as something belonging to a collection of objects satisfying the vector space axioms. As a related example, tangent vectors in abstract manifolds are commonly defined as directional derivative operators, where the directional derivative operator is to be thought of as a kind of proxy for the direction in which the functions are being differentiated.

When it comes to specifying the points of a scheme X, there is a similar sort of phenomenon at play. The points of X will turn out to correspond to prime ideals of rings. One interpretation of this is to simply think of a point of a space as something that can be used to evaluate a function on that space. In the case of algebraic varieties, functions of interest are polynomials $f(x_1, \ldots, x_n) \in \Bbbk[x_1, \ldots, x_n]$ in some variates $x_1, \ldots, x_n$. If one wants to evaluate f at a point $p = (p_1, \ldots, p_n)$, the computational procedure is to "set the variables $x_i = p_i$" and then compute $f(p_1, \ldots, p_n)$. The mathematical construction which "sets the variables," or imposes the relations $x_i = p_i$, is simply a quotient of rings, and the object being quotiented is the maximal ideal $\langle x_1 - p_1, \ldots, x_n - p_n \rangle$. Thus we see that the ideal $\langle x_1 - p_1, \ldots, x_n - p_n \rangle$ can be used as a proxy for the point p, as the map $\Bbbk[x_1, \ldots, x_n] \to \Bbbk[x_1, \ldots, x_n]/\langle x_1 - p_1, \ldots, x_n - p_n \rangle$ has a natural interpretation as a map which evaluates elements of $\Bbbk[x_1, \ldots, x_n]$ at p.

But thinking of ideals as points also gives additional advantages. For instance, a common difficulty when working with classical varieties is that it is very difficult to say anything interesting about varieties over finite fields, since the associated topological space gives very few points to work with. To take the simplest possible example, over $\mathbb{F}_2$, the polynomial $x^2 + x$ is indistinguishable from the 0 polynomial when one considers its values on the elements $0, 1 \in \mathbb{F}_2$, or on the ideals $\langle x - 0 \rangle$ and $\langle x - 1 \rangle$. But if one instead considers the generalized "point" $\langle x^2 \rangle$, one can detect the difference, since $x^2 + x \equiv x \pmod{x^2}$, and $0 \equiv 0 \pmod{x^2}$. The same phenomenon can be seen if one considers varieties over the reals, where the vanishing set of $x^4 - 1$ is enriched due to the presence of the generalized point $\langle x^2 + 1 \rangle$.

Much like the topological space of a manifold can be thought of as a union of open sets (homeomorphic to) open subsets of $\mathbb{R}^m$, the topological space of a scheme can be thought of as a union of sets homeomorphic to the topological space of an *affine scheme*. Affine schemes play the kind of primitive "building block" role that $\mathbb{R}^n$ plays in differential geometry. An affine scheme can be built from any ring R, and its topological space (as a set) is $\operatorname{Spec} R$, the collection of all prime ideals of R. One then gives this topological space a scheme structure by endowing it with a sheaf of functions, and defines general schemes as objects locally isomorphic to affine schemes. We now give a precise definition of affine schemes.

Definition 3.6. Let R be a ring. An affine scheme is a pair $(X, \mathcal{O}_X)$, where $X = \operatorname{Spec} R$, satisfying:

(i) X is a topological space generated by the open sets

$$D(f) := \{\mathfrak{p} \in \operatorname{Spec} R : f \not\equiv 0 \pmod{\mathfrak{p}}\}.$$

where $f \in R$.

(ii) $\mathcal{O}_X$ is a sheaf of rings, called the *structure sheaf* of X, determined on the base $D(f)$ for X by $\mathcal{O}_X(D(f)) = R_f$, with the restrictions $\operatorname{res}_{D(f),D(g)} : \mathcal{O}_X(D(f)) \to \mathcal{O}_X(D(g))$ given by the natural maps $R_f \to R_g$ given by further localizing R_f at g. For this to make sense, it should be the case that inverting g also inverts f whenever $D(g) \subset D(f)$; this follows from the fact that when $D(g) \subset D(f)$,

$$g \in \bigcap_{\substack{\mathfrak{p} \text{ prime} \\ f \in \mathfrak{p}}} \mathfrak{p} = \sqrt{\langle f \rangle}$$

since g vanishes wherever f does, and the equality holds by a theorem of commutative algebra. Recall from the previous section that specifying $\mathcal{O}_X$ on this base determines the sheaf structure.

The definition of the structure sheaf of an affine scheme may seem somewhat peculiar in two respects. Firstly, one needs to understand the role played by localization, and secondly, one needs to justify why the localizations can be thought of as restrictions. To understand the role of localization, we recall that if R is a ring, the localization $\iota : R \hookrightarrow R_f$ induces a map $\operatorname{Spec} R_f \hookrightarrow \operatorname{Spec} R$ given by $\mathfrak{p} \mapsto \iota^{-1}(\mathfrak{p})$. Thus, if R_f is thought of as again determining an affine scheme structure, the topological space $\operatorname{Spec} R_f$ will be naturally a subset of $\operatorname{Spec} R$. Under this identification, we also have $\operatorname{Spec} R_f = D(f)$. An intuitive reason this works is because by inverting f in R, one is forced to remove any ideals which set f to zero (since then an "evaluation" could result in division by zero), and so the prime ideals in $\operatorname{Spec} R_f$ are exactly those which don't contain f, or exactly those ideals $\mathfrak{p}$ such that $\mathfrak{p} \in D(f)$.

Secondly, we consider the role of the localization maps as restrictions. It may seem strange, for instance, that when R is (say) an integral domain, the restriction maps $R \hookrightarrow R_f$ are actually *inclusion* maps. This would seem to contradict the intuition that when one restricts a function to a smaller set, one ought to get less functions, since there are fewer possible points on which to define their values. The key insight here is that algebraic (or regular) functions really ought to be thought of as being "holomorphic". Holomorphic functions have the remarkable property that they are entirely determined on any open set – once you describe a holomorphic function on an open set, no matter how small, its values on any extension of its domain are determined. Therefore, restricting holomorphic functions to a smaller set does not identify any functions that are distinct on the larger set, but it can allow you to have "extra" functions which do not extend to the larger open set. To take a simple example, consider the affine plane $\mathbb{A}^2$. Over a field $\Bbbk$, the affine plane can be thought of as having the ring of regular functions $\Bbbk[x, y]$, where x is the function that assigns a point to its x-coordinate, and y is the function that assigns a point to its y-coordinate. Now suppose we remove the y-axis to obtain $\mathbb{A}^2 \setminus V(x)$, where $V(x) := \{\mathfrak{p} \in \operatorname{Spec} \Bbbk[x, y] : x \equiv 0 \pmod{\mathfrak{p}}\}$ is the vanishing set of x. Then the x-coordinate of every point in $\mathbb{A}^2 \setminus V(x)$ now has an inverse, so the function $\frac{1}{x}$ is defined everywhere (holomorphic), and so it should be no surprise that the resulting coordinate ring of $D(x) = \mathbb{A}^2 \setminus V(x)$ is $\Bbbk[x, y, \frac{1}{x}]$, which agrees with our definition above.

For affine schemes, there is a description of the stalks that is more natural than the usual construction. Recalling the categorical definition, we have

$$(\mathcal{O}_X)_\mathfrak{p} = \varinjlim_{\mathfrak{p}\in U} \mathcal{O}_X(U) = \varinjlim_{\mathfrak{p}\in D(f)} A_f = A_\mathfrak{p},$$

where we have used the fact that the limit is determined on a base, and $A_\mathfrak{p}$ is the ring obtained by inverting every element not lying in the ideal $\mathfrak{p}$.

As usual, given some objects (affine schemes), we proceed to define a notion of morphism. The usual way to do this for geometric spaces is to give a map of the underlying topological spaces which respects the structure of the functions on the space. In our case, the structure of the functions on the space is encoded in the form of the structure sheaf on the space, and so we can expect that a map between two affine schemes will also give a map between their respective structure sheaves. As we have alluded to earlier, a prototypical map of sheaves can be thought of as being induced by a pullback map, and we will see that this case is no exception.

Definition 3.7. A *morphism of affine schemes* $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of:

(i) A map $\pi : X \to Y$ of topological spaces (again denoted π by abuse of notation).

(ii) A morphism of sheaves $\pi^\sharp : \mathcal{O}_Y \to \pi_*\mathcal{O}_X$, where $\pi_*\mathcal{O}_X$ is the "pushforward" sheaf on Y given by $(\pi_*\mathcal{O}_X)(U) = \mathcal{O}_X(\pi^{-1}(U))$.

(iii) If $\mathfrak{p} \in X$, the morphism $\pi^\sharp$ of sheaves should send the unique maximal ideal in $(\mathcal{O}_Y)_{\pi(\mathfrak{p})}$ into the unique maximal ideal in $(\mathcal{O}_X)_\mathfrak{p}$. Informally, this says that "functions which vanishes at $\pi(\mathfrak{p})$ pullback to functions which vanish at $\mathfrak{p}$." See also the definition of *morphism of ringed space* in the next section.

We often write this morphism as $\pi : X \to Y$, where the morphism of sheaves is understood.

In many ways, this definition is exceedingly redundant. Denote by $\Gamma(U, \mathcal{F})$ the sections of $\mathcal{F}$ over the set U. Then given a morphism of $\pi : X \to Y$ of affine schemes, where $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, we get a morphism $\pi^\sharp : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(Y, \pi_*\mathcal{O}_X)$ which is simply a morphism $\pi^\sharp : B \to A$. This map $\pi^\sharp : B \to A$ then suffices to determine the entire sheaf morphism: over the open set $D(f) \subset Y$, we require the commutativity of the diagram

$$\begin{array}{ccc} B & \xrightarrow{\pi^\sharp} & A \\ \downarrow & & \downarrow \\ B_f & \xrightarrow{\pi^\sharp} & (\pi_*\mathcal{O}_X)(D(f)), \end{array}$$

and there is only one ring map that will do, since we must have $\pi^\sharp(1/f) = 1/\pi^\sharp(f)$. Since a morphism of sheaves is determined on a base, this tells us that the entire morphism $\pi^\sharp : \mathcal{O}_Y \to \pi_*\mathcal{O}_X$ is determined by the single "global" morphism $\pi^\sharp : B \to A$. The map $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ can also be recovered using condition (iii). We have verified:

Proposition 3.1. *Every morphism $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of affine schemes, where $X =$ SpecA and $Y =$ SpecB, is determined by a unique ring morphism $B \to A$, which is the morphism $\Gamma(Y, \mathcal{O}_Y) \to \Gamma(Y, \pi_*\mathcal{O}_X)$ determined by the morphism of structure sheaves.* □

Using similar reasoning, it is also not difficult to show that:

Proposition 3.2. *Every morphism $\pi^\sharp : B \to A$ of rings induces a unique morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ of affine schemes, with the map on topological spaces given by $\mathfrak{p} \mapsto (\pi^\sharp)^{-1}(\mathfrak{p})$.* □

These two propositions suggest that the category **Ring** and the category **AffSch** are really the "same." Using more work, which is no more difficult but simply requires some routine verifications, one can show that

Proposition 3.3. *The functors Spec : $\mathbf{Ring} \to \mathbf{AffSch}$ and $\Gamma(-, \mathcal{O}_{-}) : \mathbf{AffSch} \to \mathbf{Ring}$ give an equivalence of categories between* **Ring** *and* **AffSch***.* □

This is a very nice result, but it is almost disappointing! Could it be that the language of affine schemes is really just a glorified way of talking about rings? (Yes, in fact it is.) One could perhaps hope that maybe the notion of a general scheme is something more, and that once we are able to glue together affine schemes to make general schemes we will unleash scheme theory's full power. This hope is a bit naïve. Although it's true that general schemes are more than just rings *per se*, asking that schemes be more than "just rings" is a bit like asking that abstract smooth manifolds be more than just submanifolds of $\mathbb{R}^m$ — it's not the point. The point is that the language of schemes (and it really is a *language*) allows for the expression of concepts and notions that would be very difficult to express ordinarily, and in doing so, it can be used to clarify much of algebraic geometry (what concepts and notions? well, that's still some ways off...).

An important part of becoming familiar with the scheme-theoretic language is to become accustomed to translating algebraic facts in the ring-theoretic world to geometric facts in the scheme-theoretic world. As an example, suppose that R_1 and R_2 are rings and consider their direct product $R = R_1 \times R_2$. Let $e_1 = (1, 0) \in R_1$ and $e_2 = (0, 1) \in R_2$. Both e_1 and e_2 are idempotent, and so satisfy the equation $e_i^2 - e_i = e_i(e_i - 1) = 0$. If $\mathfrak{p} \in \operatorname{Spec} R$, then $R/\mathfrak{p}$ is an integral domain, and so evaluating the functions e_i at $\mathfrak{p}$ gives either 0 or 1. Moreover, since $1 = e_1 + e_2$, we see that e_1 is 0 at $\mathfrak{p}$ precisely when e_2 is 1, and vice versa. This shows that $\operatorname{Spec} R = V(e_1) \sqcup V(e_2) = D(e_1) \sqcup D(e_2)$. Now, the localization $R_{e_i} \cong R_i$, and hence $\operatorname{Spec} R = \operatorname{Spec} R_1 \sqcup \operatorname{Spec} R_2$. Thus we see that if a ring R may be expressed as a product $R = R_1 \times R_2$, its spectrum can be thought of as the disjoint union of the spectra of the factors. Another way of looking at the same fact is that Spec : $\mathbf{Ring} \to \mathbf{AffSch}$ gives a contravariant equivalence of categories, and so products in **Ring** become coproducts in **AffSch**, and a coproduct of topological spaces is exactly a disjoint union.

A list of correspondences, by no means complete, between **Ring** and **AffSch** is as follows:

(i) Connected components of $\operatorname{Spec} R$ correspond bijectively to idempotent elements in R.

(ii) Primary ideals in a Noetherian ring R correspond to irreducible components in $\operatorname{Spec} R$.

(iii) Affine schemes whose ring is a quotient of $\Bbbk[x_1, \ldots, x_n]$ correspond to affine varieties over $\Bbbk$.

(iv) Let R be a ring, and $\mathfrak{m}$ a maximal ideal (i.e., an "ordinary" point). The tangent space to $\operatorname{Spec} R$ at $\mathfrak{m}$ is the dual vector space $(\mathfrak{m}/\mathfrak{m}^2)^\vee$, which is a vector space over $R/\mathfrak{m}$, where the quotient $\mathfrak{m}/\mathfrak{m}^2$ may be thought of as occuring in the local ring $R_\mathfrak{m} = (\mathcal{O}_{\operatorname{Spec} R})_\mathfrak{m}$.

(v) The dimension of $\operatorname{Spec} R$ (as a topological space) is the length of a maximal chain (ordered by inclusion) of prime ideals in R.

(vi) Let R be an integral domain. Localization maps $R \hookrightarrow R[S^{-1}]$, where S is a multiplicative set correspond to open embeddings of affine schemes $\operatorname{Spec} R[S^{-1}] \hookrightarrow \operatorname{Spec} R$.

(vii) Let R be a ring and $I \subset R$ an ideal. Then the induced map $\operatorname{Spec} R/I \hookrightarrow \operatorname{Spec} R$ is a closed immersion of affine schemes (and all closed immersions of affine schemes look like this).

(viii) The notions of dominant, finite, regular, and rational morphisms from classical algebraic geometry correspond to equivalent notions for affine schemes.

(ix) A tensor product $A \otimes_C B$ of rings corresponds to the fibre product $X = \operatorname{Spec} A \otimes B = \operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B$ of affine schemes, which is to be thought of as an object with two morphisms $X \to \operatorname{Spec} A$ and $X \to \operatorname{Spec} B$ such that we have a commutative diagram of the form:

$$\begin{array}{ccc} X & \longrightarrow & \operatorname{Spec} B \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \longrightarrow & \operatorname{Spec} C. \end{array}$$

The two arrows on the bottom and the right are the ones that come from the maps $C \to A$ and $C \to B$ giving the tensor product. In fact, the fibre product diagram is just the image of Spec in **AffSch** of the tensor product diagram:

$$\begin{array}{ccc} A \otimes_C B & \longleftarrow & B \\ \uparrow & & \uparrow \\ A & \longleftarrow & C. \end{array}$$

(x) A special case of the previous example is the fibre of a morphism $\pi : \operatorname{Spec} A \to \operatorname{Spec} C$ over a subscheme $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} C$, which is $\operatorname{Spec} A \otimes_C B$, and corresponds to the usual notion of the fibre of a map in the case of varieties.

(xi) Suppose that M is a free R-module. Then M can be thought of (using a construction we do not give here!) as a vector bundle over $\operatorname{Spec} R$.

(xii) Local rings correspond to schemes whose topological space has a single "ordinary point" (also called a "closed point").

This list is a sampling of how scheme theory gives a geometric interpretation to constructions in algebra, in particular how facts about rings correspond to facts about affine schemes. One can create similar lists for the case of graded rings (which correspond to projective schemes), or various types of modules, which correspond to various types of sheaves. Part of learning scheme theory is to work enough with all these correspondences so that they become second nature.

3.3 Schemes

We've been promising for some time that we'll construct schemes as things which are locally affine schemes. It's time to make it happen! We will define schemes as topological spaces with a sheaf of rings such that the sheaf of rings is locally isomorphic to one coming from an affine scheme. To define what it means for a sheaf of rings to be "locally" something, we need the next three definitions.

Definition 3.8. Suppose that X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X. Further suppose that $U \subset X$ is open. Then define $\mathcal{O}_X\big|_U$ as a sheaf on U (with the subspace topology) on $V \subset U$ open via $\mathcal{O}_X\big|_U(V) = \mathcal{O}_X(V)$.

Definition 3.9. A pair $(X, \mathcal{O}_X)$ is a *ringed space* if X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X. A pair $(X, \mathcal{O}_X)$ is called a *locally ringed space* if in addition each stalk is a local ring, i.e., it has a unique maximal ideal.

Definition 3.10. A morphism of ringed spaces $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a map $\pi : X \to Y$ of topological spaces, and a morphism of sheaves $\pi^\sharp : \mathcal{O}_Y \to \pi_*\mathcal{O}_X$, where $\pi_*\mathcal{O}_X$ is the pushforward sheaf of Y, defined as before. A morphism of locally ringed space is a morphism of ringed spaces where in addition, whenever $q \in \pi^{-1}(p)$, the induced map[4] $(\mathcal{O}_Y)_p \to (\mathcal{O}_X)_q$ is a local ring homomorphism, i.e., it maps the unique maximal ideal of $(\mathcal{O}_Y)_p$ into the maximal ideal of $(\mathcal{O}_X)_q$.

The last two definitions essentially just formalize what we did for affine schemes in a more general context, since we will want to refer to morphisms of ringed spaces in situations where our ringed spaces may not be affine schemes. We are now ready for the notion of a scheme.

Definition 3.11. A *scheme* is a ringed space $(X, \mathcal{O}_X)$ such that for every $p \in X$ there is an open neighbourhood $U \subset X$ containing p such that $(U, \mathcal{O}_X\big|_U)$ is isomorphic as a ringed space to an affine scheme $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$.

We now give (the simplest possible!) example of a scheme which is not an affine scheme. Let $\Bbbk$ be a field, and consider the affine schemes $X = \operatorname{Spec} \Bbbk[x]$ and $Y = \operatorname{Spec} \Bbbk[y]$. We would like a scheme where the function x is the same as the function $\frac{1}{y}$. The common domain of these functions should be a set that excludes the origin (the point $\langle x - 0\rangle$) in X and the origin (the point $\langle y - 0\rangle$) in Y. We would like this domain to be $\operatorname{Spec} \Bbbk[x, 1/x] = \operatorname{Spec} \Bbbk[y, 1/y]$. To do this, we construct our scheme (which we will call $\mathbb{P}^1_\Bbbk$) in two stages: first as a topological space, and secondly we describe its sheaf.

For the topological space, we have a homeomorphism $\alpha : \operatorname{Spec} \Bbbk[x, 1/x] \to \operatorname{Spec} \Bbbk[y, 1/y]$ induced by the ring map $x \mapsto 1/y$ (which should be thought of as *identifying* x and $1/y$), and we may construct $\mathbb{P}^1_\Bbbk$ as the topological gluing of $\operatorname{Spec} \Bbbk[x]$ and $\operatorname{Spec} \Bbbk[y]$ along α. We may think of any primitive open $D(f) \subset X$ as being open in $\mathbb{P}^1_\Bbbk$ via the identification $X = \operatorname{Spec} \Bbbk[x] \hookrightarrow \mathbb{P}^1_\Bbbk$, and similarly for $D(g) \subset Y$. We claim that the primitive opens for X and Y generate the quotient topology on $\mathbb{P}^1_\Bbbk$. To see this, note that any open set in $\mathbb{P}^1_\Bbbk$ corresponds to an open set $U \sqcup V$ in $X \sqcup Y$ via the quotient map τ, and the open set $U \sqcup V$ is a union of primitive opens by the definition of the disjoint union topology. Since maps preserve unions, we see that $\tau(U \sqcup V)$ is also a union of primitive opens (this time with the identification induced by α), and so since $\tau(U \sqcup V)$ was an arbitrary open set, we are done.

To describe the sheaf, we describe it on a base. This is also easy: we define $\mathcal{O}_{\mathbb{P}^1_\Bbbk}$ to be equal to $\mathcal{O}_X(D(f))$ when $D(f) \subset X$, and $\mathcal{O}_Y(D(g))$ when $D(g) \subset Y$, noting that when $D(h)$ is a subset of both X and Y we have an identification of $\mathcal{O}_X(D(h))$ and $\mathcal{O}_Y(D(h))$ induced by the map α. This gives $\mathbb{P}^1_\Bbbk$ a sheaf of rings, and it is clear that $\mathcal{O}_{\mathbb{P}^1_\Bbbk}\Big|_X = \mathcal{O}_X$ and $\mathcal{O}_{\mathbb{P}^1_\Bbbk}\Big|_Y = \mathcal{O}_Y$, so the ringed space $(\mathbb{P}^1_\Bbbk, \mathcal{O}_{\mathbb{P}^1_\Bbbk})$ is in fact a scheme.

[4] We have not explicitly defined this map, but it's easy to figure out what it means.

We now compute the global sections of $\mathbb{P}^1_{\Bbbk}$ (i.e., the ring $\Gamma(\mathbb{P}^1_{\Bbbk}, \mathcal{O}_{\mathbb{P}^1_{\Bbbk}})$). By the identity axiom, global sections of $\mathbb{P}^1_{\Bbbk}$ can be described pairs of compatible (agreeing on overlaps) sections over the open cover $X \cup Y$. If (f_x, f_y) is such a pair, then $f_x \in \Bbbk[x]$, so f_x is a polynomial in $x = 1/y$, and $f_y \in \Bbbk[y]$, so f_y is a polynomial in $y = 1/x$. f_x and f_y agree on the overlap (that is, in the ring $\Bbbk[x, 1/x] = \Bbbk[y, 1/y]$, where $x = 1/y$), and so the polynomial f_x viewed as a polynomial in $y = 1/x$ must not have any negative-degree terms, and so must be constant, and similarly for f_y. Thus we see that both f_x and f_y are elements of $\Bbbk$, and they must be the same element since they agree in the ring $\Bbbk[x, 1/x] = \Bbbk[y, 1/y]$. Hence $\Gamma(\mathbb{P}^1_{\Bbbk}, \mathcal{O}_{\mathbb{P}^1_{\Bbbk}}) = \Bbbk$. Since the topological space of $\mathbb{P}^1_{\Bbbk}$ is not homeomorphic to $\operatorname{Spec} \Bbbk$, we see that $\mathbb{P}^1_{\Bbbk}$ is not affine.

This construction might seem tedious (and this was the simplest possible example!), but we will soon have a better way of describing $\mathbb{P}^1_{\Bbbk}$. This is via the *projective scheme* construction (that construction is altogether more tedious, but it is at least general in the sense that it describes a wide class of schemes). In much the same way as affine schemes are built out of rings, projective schemes are built out of graded rings. In the affine case, the construction mirrors the classical case of affine varieties, where the functions on the variety come from an affine coordinate ring R. In the projective case, the functions on a variety are instead described as ratios of homogeneous functions from a graded coordinate ring S. By *graded* coordinate ring, we mean that $S = \bigoplus_{i=0}^{\infty} S_i$, and that if $f_i \in S_i$ and $f_j \in S_j$, then $f_i f_j \in S_{i+j}$. The usual example is to think of $S = \Bbbk[x_0, x_1, \ldots, x_n]$, and to think of S_i as homogeneous polynomials of degree i. In general, if $f \in S_i$, we often say that f has degree i.

The purpose of the grading on S is to keep track of which quotients of polynomials make acceptable rational functions on the associated scheme. For instance, we can recall the case of ordinary projective space $\mathbb{P}^n_{\Bbbk}$ (in the sense of classical algebraic varieties). This is the space of lines through the origin of $\Bbbk^{n+1}$, where a point $p \in \mathbb{P}^n$ can be written in homogeneous coordinates $p = [p_0 : p_1 : \ldots : p_n]$, where $[p_0 : p_1 : \ldots : p_n]$ denotes the equivalence class $\{(\lambda p_0, \lambda p_1, \ldots, \lambda p_n) : \lambda \in \Bbbk\}$. If we want a rational function that we can evaluate on such a point, we need our process of evaluating the function to be independent of the choice of λ. If we take a ratio of two homogeneous polynomials $\frac{f(x_0, x_1, \ldots, x_n)}{g(x_0, x_1, \ldots, x_n)}$, then evaluating this expression at a projective point will be well-defined if and only if $\deg f = \deg g$, in which case scaling the input by a factor of λ will result in a scaling the output by a factor of $\lambda^{\deg f - \deg g} = 1$.

To capture this notion of "allowable quotients," we will need a projective version of localization, which is the purpose of the next definition.

Definition 3.12. Let S be a graded ring, and $\mathfrak{p}$ a homogeneous prime ideal[5] in S. We define

$$S_{(\mathfrak{p})} = \{f/g : \quad f, g \text{ homogeneous in } S,\ \deg f = \deg g,\ g \not\equiv 0 \pmod{\mathfrak{p}}\}$$

which is a ring with the obvious operations.

We then construct projective schemes as follows.

Definition 3.13. Let S be a graded ring, and define $X = \operatorname{Proj} S$ (as a set!) to be the set of all homogeneous prime ideals of S which do not contain the so-called "irrelevant ideal" $S_+ := \bigoplus_{i=1}^{\infty} S_i$ (the purpose of this is so that evaluating at $\mathfrak{p} \in \operatorname{Proj} S$ does not set all

[5] A homogeneous ideal is one generated by homogeneous elements.

non-constant functions to zero, which is necessary because $[0 : 0 : \ldots : 0]$ is not a point in projective space). We then define a scheme structure on X as follows:

(i) The topology of X is generated by the primitive open sets

$$D(f) := \{\mathfrak{p} \in X : f \not\equiv 0 \pmod{\mathfrak{p}}\},$$

where $f \in S$ is a homogeneous element.

(ii) The étalé space of the structure sheaf on X is the disjoint union $\bigsqcup_{\mathfrak{p} \in X} S_{(\mathfrak{p})}$, with the topology generated by the open sets $\{(f/g)_{\mathfrak{p}}\}_{\mathfrak{p} \in D(g)}$ where f and g are homogeneous elements of the same degree.

(iii) The sections $\mathcal{O}_X(U)$ where $U \subset X$ is open are the continuous maps from $U \to \bigsqcup_{\mathfrak{p} \in X} S_{(\mathfrak{p})}$.

Note that property (iii) essentially just says that the regular functions over an open set U locally look like homogeneous quotients f/g. Indeed, if $s : U \to \bigsqcup_{\mathfrak{p} \in X} S_{(\mathfrak{p})}$ is continuous, then if $V := \{(f/g)_{\mathfrak{p}}\}_{\mathfrak{p} \in D(g)}$ is open in the étalé space its inverse image under s must be open. This either means that the inverse image of V is empty, or that $D(g) \cap U$ is non-empty, in which case we must have $s(\mathfrak{p}) = (f/g)_{\mathfrak{p}}$ for all $\mathfrak{p} \in D(g) \cap U$.

The definition we have given should mirror the usual idea of projective varieties and the rational functions on them. It remains to show that this construction we have given satisfies the definition of a scheme.

Proposition 3.4. *With the construction as above, $X = \mathit{Proj}\, S$ is a scheme.*

Proof. It suffices to show that every point $\mathfrak{p} \in X$ contains an open neighbourhood isomorphic to an affine scheme. Since every $\mathfrak{p}$ is contained in some $D(g)$ (no homogeneous prime ideal kills all functions, otherwise it would contain the irrelevant ideal), it suffices to show that $(D(g), \mathcal{O}_X\big|_{D(g)})$ is affine. We will show it is isomorphic (as an affine scheme) to $\operatorname{Spec} S_{(g)}$, where $S_{(g)}$ means the degree 0 piece of the localization S_g (i.e., the subring of quotients where the numerator and denominator have the same degree).

The trickiest part of this exercise is to exhibit the homeomorphism between the set of homogeneous prime ideals in $D(g)$ and the ordinary prime ideals in $S_{(g)}$. Note that we have a localization map $\iota : S \to S_g$, and this gives a homeomorphism between prime ideals in $D(g)$ and homogeneous prime ideals in S_g, so it suffices to show that the homogeneous prime ideals in S_g are in homeomorphic bijection to the prime ideals in $S_{(g)}$.

Defining the map from homogeneous prime ideals in S_g to $\operatorname{Spec} S_{(g)}$ is easy: simply send $\mathfrak{p} \mapsto \mathfrak{p} \cap S_{(g)}$. The inverse map is much trickier. Given $\mathfrak{q} \in \operatorname{Spec} S_{(g)}$ we define the homogeneous sets $\mathfrak{p}_i = \{f : \ \deg f = i, \ \frac{f^{\deg g}}{g^{\deg f}} \in \mathfrak{q}\}$, and consider $\mathfrak{p}' := \bigoplus_{i \in \mathbb{Z}} \mathfrak{p}_i$. We make a series of claims:

(i) If $f_i \in \mathfrak{p}_i$ and $f_j \in \mathfrak{p}_j$, then $f_i f_j \in \mathfrak{p}_{i+j}$. This is because

$$\frac{(f_i f_j)^{\deg g}}{g^{\deg f_i + \deg f_j}} = \left(\frac{f_i^{\deg g}}{g^{\deg f_i}}\right)\left(\frac{f_j^{\deg g}}{g^{\deg f_j}}\right) \in \mathfrak{q}$$

(ii) If $f^2 \in \mathfrak{p}_{2i}$, then $f \in \mathfrak{p}_i$. This is because

$$\frac{(f^2)^{\deg g}}{g^{\deg f^2}} = \left(\frac{f^{\deg g}}{g^{\deg f}}\right)^2 \in \mathfrak{q}.$$

And so using the fact that $\mathfrak{q}$ is prime, we see that $f \in \mathfrak{p}_i$.

(iii) If $f, f' \in \mathfrak{p}_{\mathfrak{i}}$, then $f + f' \in \mathfrak{p}_i$. To see this, observe that

$$\frac{(f+f')^{2\deg g}}{g^{2\deg(f+f')}} \in \mathfrak{q},$$

since once the numerator is expanded each term will contain either a factor of $f^{\deg g}$ or $f'^{\deg g}$. Hence $(f+f')^2 \in \mathfrak{p}_{2\mathfrak{i}}$, and so $f + f' \in \mathfrak{p}_{\mathfrak{i}}$.

(iv) $\mathfrak{p}'$ is an ideal of S_g. We observe that if $r \in S_g$, then $r\mathfrak{p}_i \subset \mathfrak{p}_{i+\deg r}$. This is because if $f \in \mathfrak{p}_i$

$$\frac{(rf)^{\deg g}}{g^{\deg rf}} = \left(\frac{r^{\deg g}}{g^{\deg r}}\right)\left(\frac{f^{\deg g}}{g^{\deg f}}\right) \in \mathfrak{q}$$

since $\mathfrak{q}$ is an ideal and closed under multiplication from elements in $S_{(g)}$.

(v) $\mathfrak{p}'$ is a homogeneous ideal. This is because it is generated by the homogeneous elements in each $\mathfrak{p}_i$.

(vi) $\mathfrak{p}'$ is a prime ideal. To show a homogeneous ideal is prime, it suffices to consider homogeneous elements. Suppose that $f_i f_j \in \mathfrak{p}'$, where $\deg f_i = i$ and $\deg f_j = j$. Then we have as in (i) that

$$\left(\frac{f_i^{\deg g}}{g^{\deg f_i}}\right)\left(\frac{f_j^{\deg g}}{g^{\deg f_j}}\right) \in \mathfrak{q},$$

but then one of the factors is in $\mathfrak{q}$, and so either f_i or f_j is in $\mathfrak{p}'$.

(vii) The map $\mathfrak{p} \cap \operatorname{Spec} S_{(g)} =: \mathfrak{q} \mapsto \mathfrak{p}'$ gives an inverse to the map $\mathfrak{p} \mapsto \mathfrak{p} \cap \operatorname{Spec} S_{(g)}$. We do this by simply showing two containments. If $f = \sum_i f_i \in \mathfrak{p}$ where each f_i is homogeneous, then $\frac{f_i^{\deg g}}{g^{\deg f_i}} \in \mathfrak{q}$, and hence $f_i \in \mathfrak{p}'$, and thus $f \in \mathfrak{p}'$. Alternatively, if $f \in \mathfrak{p}'$, then each homogeneous element f_i of f satisfies $\frac{f_i^{\deg g}}{g^{\deg f_i}} \in \mathfrak{q} = \mathfrak{p} \cap \operatorname{Spec} S_{(g)}$, so in particular $\frac{f_i^{\deg g}}{g^{\deg f_i}} \in \mathfrak{p}$, hence $f_i \in \mathfrak{p}$ since $\mathfrak{p}$ is prime, hence $f \in \mathfrak{p}$.

(viii) The two exhibited maps give homeomorphisms between $D(g)$ and $\operatorname{Spec} S_{(g)}$. Rephrased, this says that they map open sets to open sets (it would be inverse images of open sets are open, but since they are bijections, one can say they map open sets to open sets), which can be rephrased as saying that $\mathfrak{p} \in D(f) \cap D(g)$ (a basis open in $D(g)$) if and only if $\mathfrak{p} \cap \operatorname{Spec} S_{(g)} \in D((f^{\deg g})/(g^{\deg f}))$. This is equivalent to saying that $f \in \mathfrak{p}$ if and only if $\frac{f^{\deg g}}{g^{\deg f}} \in \mathfrak{p} \cap \operatorname{Spec} S_{(g)}$, which is clear from the basic properties of prime ideals.

We have succeeded in showing that $D(g)$ is homeomorphic to $\operatorname{Spec} S_{(g)}$. To show that their spaces of sections are isomorphic, it suffices to observe that the stalk at $\mathfrak{p}$ of $\operatorname{Proj} S$ is naturally isomorphic to the stalk at $\mathfrak{p} \cap \operatorname{Spec} S_{(g)}$ of $\operatorname{Spec} S_{(g)}$. Indeed, they are both ratios of homogeneous elements of S, and by examining the correspondence between $\mathfrak{p}$ and $\mathfrak{p} \cap S_{(g)}$ one can show that the condition on the denominators is the same for both (essentially as we did in part (viii) above). One then gets an isomorphism between the structure sheaves based on the fact that both are determined on the basis sets (which we gave a bijection between) as continuous sections into their respective étalé spaces. We leave this verification to the reader[6]. □

It is worth taking time to convince yourself that the construction in the preceding proposition is really nothing more than a sophisticated version of the usual process of defining rational functions on projective varieties from classical algebraic geometry — all that has changed is the language.

4 Where to go from here?

In this article, which spans around fifteen pages, we have merely managed to define the core notions needed to discuss the classical examples of affine and projective varieties as schemes. This achievement is (so far) a linguistic one – nothing has really "happened". To seriously start using this theory, one then starts defining the sorts of things used in differential geometry – line and vector bundles, (co)homology theories – and algebraic geometry (like divisors), and uses the new abstract perspective to achieve results out of reach by ordinary methods. This is not light work: there's a reason why EGA is a thousand pages long, and Hartshorne's comparatively terse book is a few hundred. In order not to get lost, it's important to have a firm handle on the geometric intuition, since learning things just by understanding the formalism really is hopeless. In this article, we have done our best to provide the intuition and motivation for the definitions, with the hope that the reader can start to see the "bigger picture" that lies behind the abstraction.

[6]This may seem like somewhat of a cop out, but it could be worse! Hartshorne, in his book, replaces the entire eight-point verification of the bijection between the two sets of prime ideals with the sentence "The properties of localization show that [the map] is bijective as a map from $[D(g)$ to $\operatorname{Spec} S_{(g)}]$". Indeed, and the properties of elliptic curves show Fermat's Last Theorem!