# Pre-Talk: (Variations of) Hodge Structures

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  **Too Coarse:**

$$\mathrm{ProjVar}_k \to \mathrm{Vec}_k, \qquad X \mapsto \Gamma(X, \mathcal{O}_X).$$

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- (Co)homology functors occupy a middle ground through (co)homological algebra: constructions in $\mathcal{C}$ correspond to abelian category constructions in $\mathcal{D}$.

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**Solution**: "Remember" additional structure on the cohomology of complex manifolds. Obtain functors

$$\text{Algebraic Varieties}/\mathbb{C} \to \text{Hodge Structures over } (\mathbb{Z}, \mathbb{Q}, \mathbb{R} \ldots),$$

which are often injective on (subclasses) of objects, Hodge conjecture predicts fully-faithful on "motives", etc.

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$$c^{\mathrm{dR}} : H^n_{\mathrm{dR}}(X, \mathbb{R}) \otimes \mathbb{C} \to H^n_B(X, \mathbb{C}), \qquad [\omega] \mapsto \int_{(-)} \omega.$$

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$H^n_{\mathrm{dR}}(X, \mathbb{R}) \otimes \mathbb{C}$ has a natural filtration:

$$F^k(H^n_{\mathrm{dR}}(X, \mathbb{R}) \otimes \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}$$

$$H^{p,q} = \mathrm{span}\left\{ \underbrace{dz_1 \wedge \cdots \wedge dz_p}_{\text{holomorphic}} \wedge \underbrace{dz'_1 \wedge \cdots \wedge dz'_q}_{\text{anti-holomorphic}} \right\}$$

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As the the complex structure of $X$ varieties, the filtration $F^\bullet$ varies, but $H^n_B(X,\mathbb{C})$ stays fixed.

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**Definition**

*Given a subring $R \subset \mathbb{R}$, a $R$-Hodge structure of weight $n$ consists of a $R$-lattice $V$ and a filtration $F^\bullet$ of $V_{\mathbb{C}}$ such that whenever $p + q = n + 1$*

- *we have $F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}} = 0$;*
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Alternate definition (for those familiar with Shimura varieties): an $R$-lattice $V$ with a map

$$\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{R}} = \mathbb{S} \to \mathrm{GL}(V)_\mathbb{R}$$

with weight homomorphism $\mathbb{G}_{m,\mathbb{R}} \to \mathbb{S} \to \mathrm{GL}(V)_\mathbb{R}$ equal to multiplication by $t^n$.

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**Algebraic de Rham cohomology:**

- If $X/\mathbb{C}$ is affine, form the algebraic de Rham complex

$$0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_X \to 0,$$

  and define $H^i_{\mathrm{dR}}(X/\mathbb{C}) = \ker(\Omega^i_X \xrightarrow{d} \Omega^{i+1}_X)/\mathrm{im}(\Omega^{i-1}_X \xrightarrow{d} \Omega^i_X)$.

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Lattice structure from the comparison

$$H^i_{\mathrm{dR}}(X/\mathbb{C}) \simeq H^i(X^{\mathrm{an}}, \mathbb{Z}) \otimes \mathbb{C}.$$

# Hyperelliptic Example

$C = C^\circ \cup \{\infty\}$, where $C^\circ$ is given by the affine equation:

$$y^2 = x^{2g+1} + \lambda_{2g}x^{2g} + \cdots + \lambda_1 x + \lambda_0,$$

with non-vanishing discriminant.

(1) From the long exact sequence of a pair $H^1_{\mathrm{dR}}(C/\mathbb{Q}) \simeq H^1_{\mathrm{dR}}(C^\circ/\mathbb{Q})$.

(2) Algebraic de Rham cohomology is spanned by

$$\frac{dx}{y}, \frac{xdx}{y}, \cdots, \frac{x^{g-1}dx}{y}, \omega_{g+1}, \ldots, \omega_{2g},$$

where $\omega_{g+1}, \ldots, \omega_{2g}$ are forms of the "second kind".

(3)

$$F^1 H^1_{\mathrm{dR}}(C/\mathbb{Q}) = \mathrm{span}\left\{\frac{dx}{y}, \frac{xdx}{y}, \cdots, \frac{x^{g-1}dx}{y}\right\}.$$

(4) The integral lattice is given by the usual homology basis for the torus $C^{\mathrm{an}}$ transferred along $H^1_{\mathrm{dR}}(C/\mathbb{C}) \xrightarrow{\sim} H^1_B(C^{\mathrm{an}}, \mathbb{C})$.

# Reasons to Care about Hodge Theory as a Number Theorist

Most arithmetic consequences of Hodge theory come from understanding Hodge vectors; elements of $\mathrm{Hdg}(n) = H^n(X^{\mathrm{an}}, \mathbb{Q}) \cap H^{p,p}$ with $n = 2p$. Conjecturally, the following is true:

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- If $X$ is defined over $\overline{\mathbb{Q}}$, $\gamma \in \mathrm{Hdg}(n)$ is a Hodge vector, and $\omega$ is an algebraic form on $X$, then $\int_\gamma \omega \in 2\pi i \overline{\mathbb{Q}}$.

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- If $X$ is defined over $\mathbb{Q}$, elements of $\mathrm{Hdg}(n)$ constrain the image of

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell).$$

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$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell).$$
- Can study Shimura variety phenomena in more general settings (e.g. CM points and CM theory)

# Variations of Hodge structure (Idea)

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- Hodge filtration defines filtration $F^\bullet$ of $\mathcal{H} = \mathbb{V} \otimes \mathcal{O}_{S^{\mathrm{an}}}$ by vector subbundles.
- Variation of Hodge structure is a pair $(\mathbb{V}, F^\bullet)$, with fibres giving Hodge structures, plus transversality condition.

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E.g., the proofs of the Shafarevich and Tate conjectures for K3 surfaces (trivial canonical bundle, not a complex torus) both begin with the following step: construct an embedding

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of variations of Hodge structures, where $\mathbb{V}_{\mathrm{K3}}$ arises from a family of K3 surfaces, and $\mathbb{V}_{\mathrm{Ab}}$ arises from a family of abelian varieties.

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Transferring arithmetic properties from $\mathbb{V}_{\mathrm{Ab}}$ to $\mathbb{V}_{\mathrm{K3}}$ cannot be done without variational language.