An Effective Strategy for Shafarevich

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The Shafarevich Problem

Setup: Smooth projective family $f : X \to S$ defined over $\mathcal{O}_{K,N} := \mathcal{O}_K[N^{-1}]$, where $K \subset \mathbb{C}$ is a number field and $N \in \mathbb{Z}$.

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Examples: Solved for curves, abelian varieties, K3 surfaces, other sporadic examples

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Relationship between the two: Parshin families.

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For generalizations: steps (i) and (iii) are very hard for a general f, but (ii) is easy.

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Question reduces to: what is the dimension of $\overline{\psi^{-1}(\check{V}([\rho^{ss}]))}^{\mathrm{Zar}}$?

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The filtration can be interpreted in many ways. For instance:

$$F^k H^i(Y,\mathbb{C}) := \sum_{p \geqslant k} H^{p,i-p}$$

$$H^{p,q} := \text{span}\left\{ \begin{array}{c} C^\infty \text{ forms with p holomorphic} \\ \text{and q anti-holomorphic factors} \end{array} \right\}.$$

Variations of Hodge Structure

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Standard Example: Over a small neighbourhood $B \subset S(\mathbb{C})$, one can canonically identify the fibres of the map of smooth manifolds $X(\mathbb{C}) \to S(\mathbb{C})$ induced by $f : X \to S$, and hence their cohomology. (Notation: $\mathbb{V} = R^i f_*^{\mathrm{an}} \mathbb{Z}$.)

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Theorem

(Bakker-Tsimerman) Let $\mathbb{V}$ be a polarized integral variation of Hodge structure on $S(\mathbb{C})$, and $\psi : B \to \check{L}$ its period map on an analytic ball $B \subset S(\mathbb{C})$. Then if $\check{V} \subset \check{L}$ satisfies

$$\dim \overline{\psi(B)}^{Zar} - \dim \psi(B) \geqslant \dim \check{V},$$

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The theorem is ultimately about solutions to a system of K-algebraic differential equations satisfied by ψ, and can therefore be transfered to the p-adic setting.

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(1) Lawrence and Venkatesh have a strategy for solving Shafarevich problems for arbitrary smooth projective families $f : X \to S$ defined over rings $\mathcal{O}_{K,N} = \mathcal{O}_K[N^{-1}]$.

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(1) Lawrence and Venkatesh have a strategy for solving Shafarevich problems for arbitrary smooth projective families $f : X \to S$ defined over rings $\mathcal{O}_{K,N} = \mathcal{O}_K[N^{-1}]$.

(2) The most important step in this strategy is obtaining lower bounds for the quantities

$$\Delta_d = \min_{Z,\psi} \left[\dim \overline{\psi(B \cap Z)}^{\mathrm{Zar}} - \dim \psi(B \cap Z) \right],$$

where Z ranges over all dimension d irreducible subvarieties of $S_{\mathbb{C}}$, and ψ ranges over all "local period maps" $\psi : B \to \check{L}$.

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Remainder of this talk: An effective method for resolving (2) for arbitrary $f : X \to S$.

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Consider the relative hyperelliptic curve C/S with affine model:

$$y^2 = R(x) = 4\prod_{i=1}^{2g+1}(x - e_i) = \sum_{i=0}^{2g+1} \lambda_i x^i,$$

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Then letting $\mathcal{U}_i(x) = x^{i-1}, \mathcal{R}_i(x) = \sum_{k=i}^{2g+1-i}(k+1-i)\lambda_{k+1+i}x^k$

- $\mathcal{H} = \underbrace{\left(\bigoplus_{i=1}^{g} \mathcal{O}_S \frac{\mathcal{U}_i(x)dx}{y}\right)}_{F^1} \oplus \left(\bigoplus_{i=1}^{g} \mathcal{O}_S \frac{\mathcal{R}_i(x)dx}{4y}\right)$
- $\nabla : \mathcal{H} \to \Omega^1_S \otimes \mathcal{H}$ (formula on next slide)
- $\mathbb{V}_{\mathbb{C}}$ is the bundle of flat sections associated to $(\mathcal{H}, \nabla)$

Algebraic Models for Variations (2)

Proposition (Enolski-Richter)

$$\nabla_{\partial_{e_i}} = \begin{pmatrix} \boldsymbol{\alpha}_\ell^t & \boldsymbol{\gamma}_\ell \\ \boldsymbol{\beta}_\ell & -\boldsymbol{\alpha} \end{pmatrix}$$

where

$$\begin{aligned}
\boldsymbol{\alpha}_\ell &= \frac{-1}{2}\left(\frac{1}{R'(e_\ell)}\boldsymbol{\mathcal{U}}(e_\ell)\boldsymbol{\mathcal{R}}^t(e_\ell) - M_\ell\right), \\
\boldsymbol{\beta}_\ell &= -2\left(\frac{1}{R'(e_\ell)}\boldsymbol{\mathcal{U}}(e_\ell)\boldsymbol{\mathcal{U}}^t(e_\ell)\right), \\
\boldsymbol{\gamma}_\ell &= \frac{1}{8}\left(\frac{1}{R'(e_\ell)}\boldsymbol{\mathcal{R}}(e_\ell)\boldsymbol{\mathcal{R}}^t(e_\ell) - N_\ell\right),
\end{aligned}$$

with

$$M_\ell = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ e_\ell & 1 & 0 & \dots & 0 & 0 \\ e_\ell^2 & e_\ell & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ e_\ell^{g-2} & e_\ell^{g-3} & \dots & e_\ell & 1 & 0 \end{pmatrix},$$

and $N_\ell = e_\ell(M_\ell Q_\ell + Q_\ell M_\ell^t) + Q_\ell$, *where* Q_ℓ *is the diagonal matrix with* $(Q_\ell)_{k,k} = \mathcal{R}_k(e_\ell)/\mathcal{U}_{k+1}(e_\ell)$.

□

Local Period Maps

Suppose $\dim \mathbb{V} = m$, and let $\check{L}$ be the variety of flags on $\mathbb{Z}^m$ with the same Hodge numbers as $\mathbb{V}$. Fix a filtration compatible frame $v^1, \ldots, v^m$ for $\mathcal{H}$.

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Definition

A local period map is a map $\psi : B \to \check{L}^{an}_{\mathbb{C}}$ obtained as a composition $\psi = q^{an}_{\mathbb{C}} \circ f^{-1}$, where

- *(i) the map $f = [f_{ij}]$ is a varying change-of-basis matrix between $v^1, \ldots, v^m$ and a flat frame of $\mathbb{V}_{\mathbb{C}}$ defined on $B \subset S(\mathbb{C})$;*
- *(ii) the map $q : GL_m \to \check{L}$ is the canonical quotient, taking a basis $b^1, \ldots, b^m$ to the Hodge flag where each piece F^i is spanned by an inital segment of the sequence $b^1, \ldots, b^m$*

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Each germ of a local period map at a point $s \in S(\mathbb{C})$ is determined by the initial condition $f(s) \in \mathrm{GL}_m(\mathbb{C})$.

Local Period Maps (2)

Using the data $(\mathcal{H}, F^\bullet, \nabla)$, we can construct K-algebraic GL_m-invariant maps

$$\alpha : J_r^d S \times \mathrm{GL}_m \to J_r^d \check{L}, \qquad (j, M) \mapsto \psi_M \circ j,$$

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(1) for any algebraic variety X, $J_r^d X$ is the space

$$(J_r^d X)(T) = \mathrm{Hom}_T(T \times \mathbb{D}_r^d, X),$$

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(2) ψ_M is the local period map $\psi_M = q \circ f^{-1}$ determined by the property that $f(s) = M$, where is the basepoint of s of j.

Bounding (Δ_d)

Recall our goal is to get a lower bound for

$$\Delta_d = \min_{Z,\psi} \left[\dim \overline{\psi(B \cap Z)}^{\mathrm{Zar}} - \dim \psi(B \cap Z) \right].$$

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Theorem

Ranging over all Z, ψ, there are finitely many possibilities for $\overline{\psi(Z \cap B)}^{Zar}$ up to $\sim_{GL_m}$.

Bounding (Δ_d) (2)

If $\psi(B \cap Z) \subset \check{V} \subset \check{L}$ for some $\check{V}$, then one has a map (with $d = \dim Z$)

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So in particular the intersection (recall $\alpha : J_r^d S \times \mathrm{GL}_m \to J_r^d \check{L}$)

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Theorem

Suppose that (for simplicity) the maps ψ are quasi-finite. Then there exists finitely many choices of $\check{V}$ with $\dim \check{V} - d \leqslant k$ such that the following are equivalent:

(1) there exists Z, ψ with $\dim Z \geqslant d$ such that

$$\dim \overline{\psi(B \cap Z)}^{Zar} - \dim Z \leqslant k.$$

(2) For some $\check{V}$, the sets $\mathcal{F}_r(\check{V})$ are non-empty for all r.

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(3) If no Z exists with

$$\dim \overline{\psi(B \cap Z)}^{\mathrm{Zar}} - \dim Z \leqslant k,$$

eventually this is reflected in the sets $\mathcal{F}_r(\check{V})$ being empty for sufficiently large r and appropriate V.

Bounding (Δ_d) (3)

Algorithm for bounding Δ_d (simplified):

(1) Find all possible $\check{V}_1, \ldots, \check{V}_\ell \subset \check{L}$, up to equivalence by $\sim_{\mathrm{GL}}$, which can arise as $\overline{\psi(Z \cap B)}^{\mathrm{Zar}}$.

(2) For each such $\check{V}_i$, compute the sets $\mathcal{F}_r(\check{V}_i)$ for increasing r.

(3) If no Z exists with

$$\dim \overline{\psi(B \cap Z)}^{\mathrm{Zar}} - \dim Z \leqslant k,$$

eventually this is reflected in the sets $\mathcal{F}_r(\check{V})$ being empty for sufficiently large r and appropriate V.

(4) Varying k, one can compute an optimal lower bound for

$$\Delta_d = \min_{Z,\psi} \left[\dim \overline{\psi(B \cap Z)}^{\mathrm{Zar}} - \dim \psi(B \cap Z) \right].$$

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(3) General inefficiency.