Linear ODEs with constant coefficients

Let

$$\mathcal{L}(y) = y^{(n)} + \sum_{j=1}^{n} p_j y^{(n-j)},$$
$$Q(z) \stackrel{\text{def}}{=} z^n + \sum_{j=1}^{n} p_j z^{n-j}.$$

Claim 1.

$$\mathcal{L}\left(\frac{x^m}{m!} \cdot e^{rx}\right) = e^{rx}\left(\sum_{0 \le s \le \min(m,n)} \frac{x^{m-s}}{(m-s)!} \cdot \frac{Q^{(s)}(r)}{s!}\right)$$

Lemma 2.

$$(f \cdot g)^{(j)} = \sum_{s=o}^{j} \binom{j}{s} f^{(s)} \cdot g^{(j-s)}.$$

Proof. Proof by induction on j starting $j = 1$ and using

$$\binom{j}{s} + \binom{j}{s+1} = \frac{j!}{s!(j-s-1)!}\left\{\frac{1}{j-s} + \frac{1}{s+1}\right\} = \frac{(j+1)!}{(s+1)!(j-s)!} = \binom{j+1}{s+1}$$

$\square$

Proof of Claim 1. Lemma for $f = \frac{x^m}{m!}$ and $g = e^{rx}$ implies

$$\begin{aligned}
\mathcal{L}\left(\frac{x^m}{m!} \cdot e^{rx}\right) &= e^{rx}\left\{\sum_{j=0}^{n} p_j \sum_{0 \le s \le \min(j,m)} \left[\frac{x^{m-s}}{(m-s)!} \cdot r^{j-s}\right] \cdot \binom{j}{s}\right\} \\
&= e^{rx} \sum_{0 \le s \le \min(m,n)} \frac{x^{m-s}}{(m-s)!} \cdot \frac{1}{s!}\left[\sum_{j=0}^{n} p_j \cdot j \cdot (j-1) \cdot \ldots \cdot (j-s+1) \cdot r^{j-s}\right] \\
&= e^{rx} \sum_{0 \le s \le \min(m,n)} \frac{x^{m-s}}{(m-s)!} \cdot \frac{Q^{(s)}(r)}{s!}.
\end{aligned}$$

as required. $\square$

Corollary 3. *Assume that*

$$Q(z) = \prod_{1 \le j \le q} (z - r_j)^{mj}.$$

Then $\frac{x^k}{k!} e^{r_j x}$ *for* $k = 0, \ldots, m_j - 1, j = 1, \ldots, q$ *are solutions of equation* $\mathcal{L}(y) = 0$. *There are exactly* n *solutions; denote them* $Y_1, \ldots, Y_n$. *These solutions are linearly independent and therefore they "generate" by linear combination all solutions and* $W_{\{Y_1,\ldots,Y_n\}}(x_0) \neq 0$.

Proof. (i) Note that for $Y_{k,r} = x^k e^{rx}/k!$

$$(\frac{d}{dx} - \lambda)(Y_{k,r_j}) = (r_j - \lambda) \cdot Y_{k,r_j} + Y_{k-1,r_j}.$$

Hence, using

$$(\frac{d}{dx} - r_j)^s Y_{k,r_j} = \begin{cases} Y_{k-s,r_j} & s = 0, \ldots, k, \\ 0 & s = k+1 \end{cases}$$

we conclude that $\mathcal{L} Y_j = 0$.

(ii) Let us prove linear independence. Assume that

$$\sum_j \sum_{0 \le k \le m_j} C_{k,j} \cdot Y_{k,r_j}(x) \equiv 0$$

as $x \in I, I = \{x : \alpha < x < \beta\}$ where $C_{k_0-1,j_0} \neq 0$ and $C_{k,j_0} = 0$ for $k \geq k_0$.
Then

$$\prod_{j \neq j_0} (\frac{d}{dx} - r_j)^{m_j} (\frac{d}{dx} - r_{j_0})^{k_0-1} (Y_{k,r_j}) \equiv 0$$

for all $j \neq j_0, 1 \le k < m_j$ and for $j = j_0$ and $k = 0, 1, \ldots, k_0 - 2$.
Therefore,

$$\begin{aligned} 0 &= C_{k_0-1,j_0} \cdot \prod_{j \neq j_0} (\frac{d}{dx} - r_j)^{m_j} (\frac{d}{dx} - r_{j_0})^{k_0-1} (Y_{k_0-1,r_{j_0}}) \\ &= C_{k_0-1,j_0} \cdot \prod_{j \neq j_0} (\frac{d}{dx} - r_j)^{m_j} [e^{r_{j_0} x}] \\ &= C_{k_0-1,j_0} \cdot \prod_{j \neq j_0} (r_{j_0} - r_j)^{m_j} \cdot e^{r_{j_0} x}. \end{aligned}$$

Hence $C_{k_0-1,j_0} = 0$, as required.

$\square$