A simple “trick” to solve Euler ODEs

Consider equation

$$L_E(y) := \sum_{k=0}^{k=d1} c_k x^k y^{(k)} \ , \ solve\ ODE\ \ L_E(y)(x) = P(x) \ . \tag{1}$$

Step 1. Find characteristic polynomial

$$Q_{L_E}(r) := x^{-r} L_E(x^r) = \sum_{k=0}^{k=d} c_k r(r-1)\dots(r-k+1). \tag{2}$$

Step 2. Constant coefficient ODE in $z = z(t)$

$$L(z) = P(e^t). \tag{3}$$

is defined by its characteristic polynomial $Q_L(r) := Q_{L_E}(r)$.

Then solutions to (1) are $y(x) = z(\ln x)$, where $z(t)$ solves (3).

Proof of the above is based on the equality (which is proved below)

$$x^{k+1} f^{(k+1)}(x)\Big|_{x=e^t} = \Big|\Big[\Big(\frac{d}{dt} - k\Big)\Big(x^k f^{(k)}(x)\Big|_{x=e^t}\Big)\Big]\Big|_{t=\ln x} \tag{4}$$

from which it follows with $g(t) := f(e^t)$ that

$$x^{k+1} f^{(k+1)}(x)\Big|_{x=e^t} = \Big[\prod_{0\le j\le k} \Big|\Big(\frac{d}{dt} - j\Big) g\Big](t) \tag{5}$$

and therefore it also follows with $y(t) := z(\ln x)$ that

$$L(z)(t) := [L_E(y)](e^t) = \sum_{k=0}^{d} c_k \Big(\prod_{0\le j\le k-1} \Big(\frac{d}{dt} - j\Big) z\Big)(t) \ , \tag{6}$$

i.e $L(z)$ is a constant coefficients ODEs operator with the characteristic polynomial $Q_L(r) = e^{-rt} L(e^{rt}) = \Big[x^{-r} L_E(x^r)\Big]\Big|_{x=e^t} = Q_{L_E}(r)$, i.e. as in (3). Hence, solutions of $L_E(y)(x) = P(x)$ correspond to solutions of $L(z)(t) = P(e^t)$ via $y(x) = z(\ln x)$.

Proof of (4). By induction on $k \ge 0$:
It follows using chain rule that $xf'(x) = g_t'(\ln x)$, where $g(t) := f(e^t)$, and using product rule that $x^{k+1} f^{(k+1)}(x) = x\frac{d}{dx}\big(x^k f^{(k)}(x)\big) - kx^k f^{(k)}(x)$. Then, using the inductive assumption (4) follows. $\square$