MAT244, 2014F, Solutions to Term Test 2

Problem 1. *Solve the following initial value problem*

$$x^3y''' - 3x^2y'' + 6xy' - 6y = -24x^{-1} + 288\ln x \ ,$$
$$y(1) = 0, \quad y'(1) = -7, \quad y''(1) = 22 \ .$$

Solution. It is Euler's equaltion with the characteristic polynomial

$$r(r-1)(r-2) - 3r(r-1) + 6r - 6 = (r-1)(r^2 - 2r - 3r + 6) =$$
$$(r-1)(r^2 - 2r - 3r + 1) = (r-1)(r^2 - 5r + 6) = (r-1)(r-2)(r-3) \implies$$
$$r_1 = 1, r_2 = 2, r_3 = 3$$

and the general solution to homogeneous equation is

$$z = C_1x + C_2x^2 + C_3x^3. \tag{1.1}$$

Finding solution $y_{p1} = ax^{-1}$ for equation with the right-hand expression $-24x^{-1}$: $a(-2)(-3)(-4) = -24 \implies a = 1 \implies y_{p1} = x^{-1}$.

Finding solution $y_{p1} = a\ln x + b$ for equation with the right-hand expression $6\ln x$. Plugging $\ln x = t$ we arrive to a constant coefficient equation with the same characteristic polynomial $r^3 - 6r^2 + 11r - 6$:

$$y_t''' - 6y_t'' + 11y' - 6y = 288t.$$

Then $-24at + 2a - 24b = 288t \implies a = -12, b = -1 \implies y_{p2} = -12t - 1 = -12\ln x - 1$.

Then

$$y = z + y_{p1} + y_{p2} = C_1x + C_2x^2 + C_3x^3 + x^{-1} - 12\ln x - 1.$$

Satisfying initial conditions:

$$\begin{aligned} C_1 + C_2 + C_3 &= 3, \\ C_1 + 2C_2 + 3C_3 &= 6, \\ 2C_2 + 6C_3 &= 8 \end{aligned} \implies C_1 = C_2 = C_3 = 1$$

and

$$y = x + x^2 + x^3 + x^{-1} - 12\ln x - 1.$$

□

Problem 2. *(a) Determine the type of behavior (phase portrait) near the origin of the system of ODEs*

$$\begin{cases} x'_t = y \ , \\ y'_t = 2x + y. \end{cases}$$

(b) Solve for the system of ODEs from ***2a*** *the initial value problem with* $x(0) = 2 \ , \ y(0) = 1$ *.*

Solution. Looking for eigenvalues of the matrix $\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$:

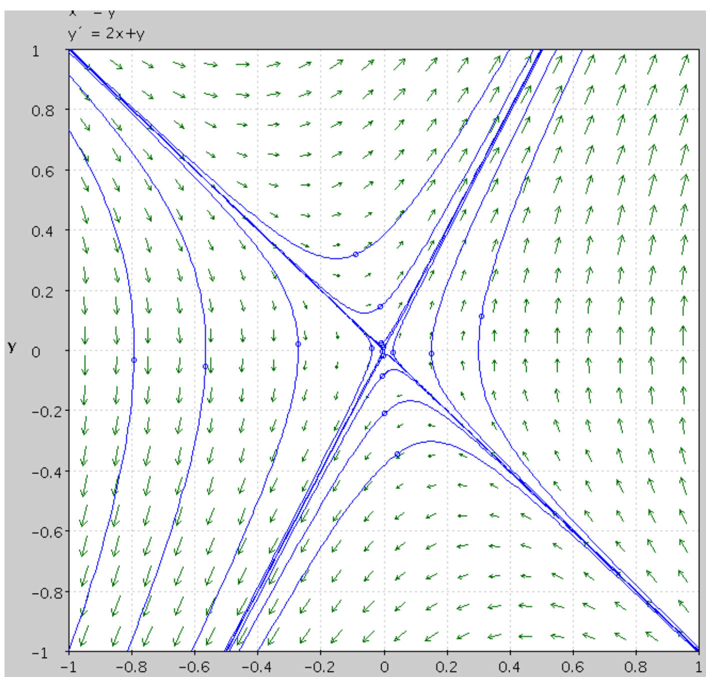
$$\begin{vmatrix} -r & 1 \\ 2 & 1-r \end{vmatrix} = r^2 - r - 2 = 0 \implies r_1 = -1, r_2 = 2.$$

Finding eigenvectors

(i) $r_1 = -1$, $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \implies \mathbf{e}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(ii) $r_2 = 2$, $\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \implies \mathbf{e}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Type of point: Saddle (unstable)



Then general solution to homogeneous system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}. \tag{2.1}$$

Using initial condition

$$C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \implies C_1 = C_2 = 1$$

and finally

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}.$$

□

Problem 3. *Find the general solution, sketch the phase portrait and determine the type of behavior near the origin of the the system of ODEs*

$$\begin{cases} x'_t = -x - 4y \ , \\ y'_t = \quad x - \ y. \end{cases}$$

Solution. Looking for eigenvalues of the matrix $\begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$:

$$\begin{vmatrix} -1-r & -4 \\ 1 & -1-r \end{vmatrix} = r^2 + 2r + 5 = 0 \implies r_{1,2} = -1 \pm 2i.$$

Finding eigenvectors

(i) $r_1 = -1 + 2i$, $\begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \implies \mathbf{e}_1 = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$.

(ii) $r_2 = -1 - 2i$, $\mathbf{e}_2 = \begin{pmatrix} -2i \\ 1 \end{pmatrix}$ (just conjugate).

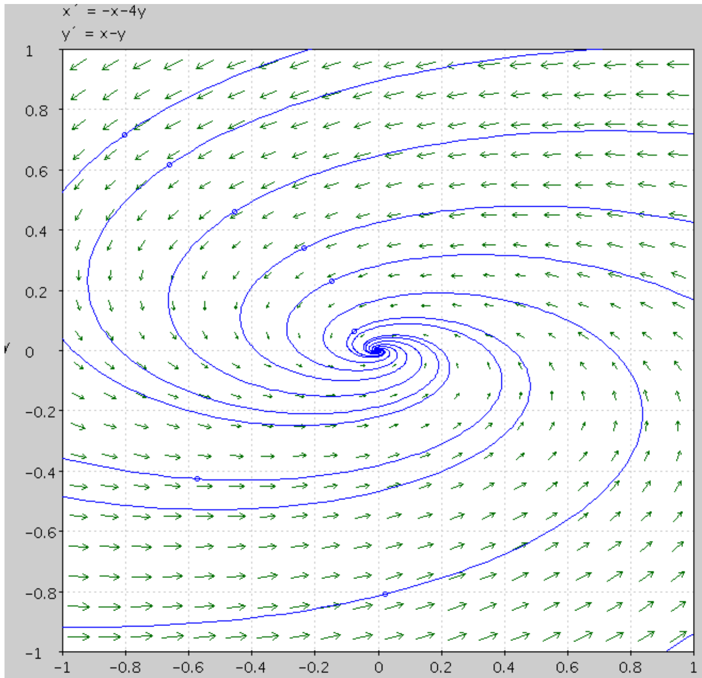
General solution in the complex form

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} -2i \\ 1 \end{pmatrix} e^{(-1+2i)t} + C_2 \begin{pmatrix} 2i \\ 1 \end{pmatrix} e^{(-1-2i)t}.$$

General solution in the real form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}(C_1 + iC_2) \begin{pmatrix} -2i \\ 1 \end{pmatrix} (\cos(2t) + i\sin(2t))e^{-t} =$$

$$e^{-t} \begin{pmatrix} 2C_1 \sin(2t) + 2C_2 \cos(2t) \\ \cos(2t) - C_2 \sin(2t) \end{pmatrix}$$



Type of point: Stable focus (since $-1 < 0$), counter-clockwise (since $-4 < 0$). $\square$

Problem 4. *Find the general solution and determine the type of behavior near the origin of the the system of ODEs*

$$\begin{cases} x_t' = -6x + 5y \ , \\ y_t' = -5x + 4y. \end{cases}$$

Solution. Looking for eigenvalues of the matrix $\begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}$:

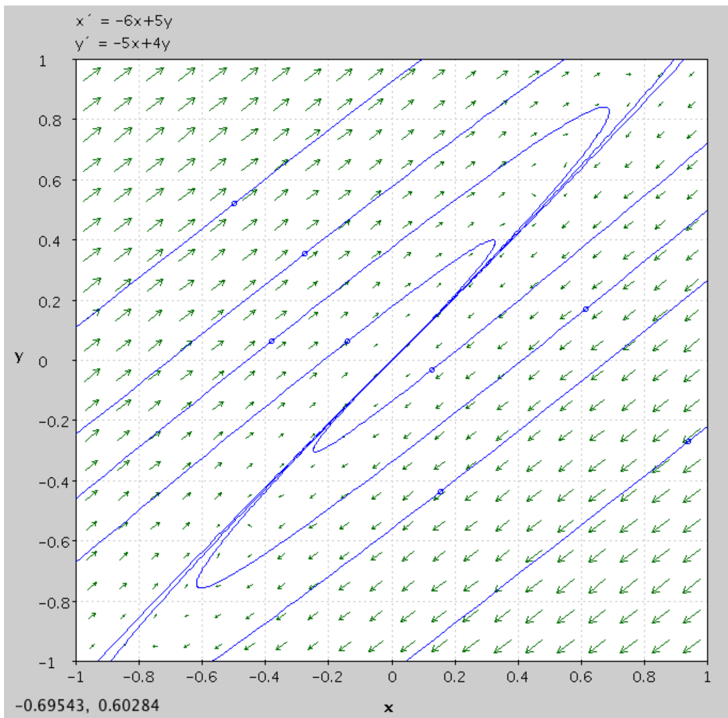
$$\begin{vmatrix} -6-r & 5 \\ -5 & 4-r \end{vmatrix} = r^2 + 2r + 1 = 0 \implies r_{1,2} = -1.$$

Finding eigenvectors $\begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \implies \alpha = \beta = 1$. So we have just one eigenvector and the associate eigenvector $\mathbf{e}_{1,1}$ is found from $\begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \mathbf{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and we can take $\mathbf{e}_{1,1} = \begin{pmatrix} -\frac{1}{5} \\ 0 \end{pmatrix}$.

Then the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + C_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} -\frac{1}{5} \\ 0 \end{pmatrix} \right] e^{-t}.$$

Type will be a stable (since $-1 < 0$) skewed node (see next page). $\square$



x´ = −6x+5y
y´ = −5x+4y
y
x
−0.69543, 0.60284