MAT244, 2014F, Solutions to Test 1

Problem 1. *Find integrating factor and then a general solution of ODE*

$$y + (2xy - e^{-2y})y' = 0 \ .$$

Also, find a solution satisfying $y(1) = -2$.

Solution 1. We seek μ such tha

$$(\mu M)_y = (\mu N)_x$$

where $M(x, y) = y$ and $N(x, y) = 2xy - e^{-2y}$. Notice that

$$-\frac{M_y - N_x}{M} = -\frac{1 - 2y}{y} = 2 - \frac{1}{y}$$

is independent of x. So in this case we have that

$$\log(\mu) = -\int \left(2 - \frac{1}{y}\right) dy = 2y - \ln(y)$$

hence

$$\mu = e^{2y - \ln(y)} = e^{2y}e^{-\ln(y)} = e^{2y}e^{\ln(\frac{1}{y})} = \frac{1}{y}e^{2y}.$$

Now after multiplying through by μ our equation becomes

$$e^{2y} + \left(2xe^{2y} - \frac{1}{y}\right) y' = 0$$

So

$$\Phi(x, y) = \int e^{2y} dx = xe^{2y} + g(y)$$

and hence if we want

$$2xe^{2y} - \frac{1}{y} = \frac{\partial \Phi}{\partial y}(x, y) = 2xe^{2y} + g'(y)$$

so

$$g'(y) = \frac{-1}{y}$$

so

$$g(y) = -\ln(|y|)$$

So the solution is of the form

$$xe^{2y} - \ln(|y|) = C$$

where c is a constant. We now use the initial conditions to determine c. Inputting this we get:

$$c = (1)e^{(-2)(-2)} - \ln(|-2|) = e^4 - \ln(2)$$

□

Solution 2. We begin manipulating as follows (by an application of the inverse function theorem):

$$\begin{aligned}
y + (2xy - e^{-2y})\frac{dy}{dx} &= 0 \\
(2xy - e^{-2y})\frac{dy}{dx} &= -y \\
\frac{dy}{dx} &= \frac{-y}{2xy - e^{-2y}} \\
\frac{dx}{dy} &= \frac{2xy - e^{-2y}}{-y} \\
\frac{dx}{dy} &= -2x + \frac{e^{-2y}}{y} \\
\frac{dx}{dy} + 2x &= \frac{e^{-2y}}{y}
\end{aligned}$$

We are viewing the above as a differential equation in x where y is now the independent variable. We solve the above by means of an integrating factor. So we wish to find μ so that;

$$\frac{d\mu}{dy} = 2\mu$$

hence,

$$\mu = e^{2y}$$

so the differential equation becomes

$$\frac{d(e^{2y}x)}{dy} = e^{2y} \cdot \frac{e^{-2y}}{y} = \frac{1}{y}$$

so

$$e^{2y}x = \ln(|y|) + C$$

so

$$xe^{2y} - \ln(|y|) = C$$

so inputting the initial conditions we arrive at the solution. $\square$

Problem 2. *(a) Find Wronskian $W(y_1, y_2)(x)$ of a fundamental set of solutions $y_1(x), y_2(x)$ for ODE*

$$x^3(\ln x + 1) \cdot y''(x) - (2\ln x + 3)x^2 \cdot y'(x) + (2\ln x + 3)xy(x) = 0, \qquad x > 1.$$

(b) Check that $y_1(x) = x$ is a solution and find another linearly independent solution.

Solution. (a) We wish to find the Wronskian of a fundamental set of solutions for the ODE

$$x^3(\ln(x)+1)\cdot y''(x) - (2\ln(x)+3)x^2 \cdot y'(x) + (2\ln(x)+3)x \cdot y(x) = 0, \qquad x > 1$$

so

$$y''(x) - \frac{2\ln(x)+3}{\ln(x)+1} \cdot \frac{1}{x} \cdot y'(x) + \frac{2\ln(x)+3}{\ln(x)+1} \cdot \frac{1}{x^2} \cdot \frac{1}{x^2} \cdot y(x) = 0$$

so $p(x) = -\frac{2\ln(x)+3}{\ln(x)+1} \cdot \frac{1}{x}$. Thus, be Abel's identity we can compute the Wronskian as follows:

$$\begin{aligned}
W(y_1, y_2)(x) &= e^{-\int p dx} \\
&= e^{\int \frac{2\ln(x)+3}{\ln(x)+1} \cdot \frac{1}{x} dx} \\
&= e^{\int \left(\frac{2(\ln(x)+1)+1}{\ln(x)+1} \cdot \frac{1}{x}\right) dx} = e^{\int \left(2 + \frac{1}{x(\ln(x)+1)}\right) dx} \\
&= e^{2\ln(x) + \int \frac{1}{x(\ln(x)+1)} dx}
\end{aligned}$$

By letting $u = \ln(x) + 1$ in the last integral we notice that $du = \frac{1}{x} dx$ so we get

$$e^{2\ln(x) + \int \frac{1}{u} du} = e^{2\ln(x) + \ln(u)} = e^{\ln(x^2) + \ln(\ln(x)+1)} = e^{\ln(x^2(\ln(x)+1))} = x^2(\ln(x)+1)$$

(b) We have that $y_1(x) = x$, $y_1'(x) = 1$, $y_1''(x) = 0$. In putting this into the differential equation we get:

$$\begin{aligned}
x^3(\ln(x)+1) \cdot y_1''(x) - &(2\ln(x)+3)x^2 \cdot y_1'(x) + (2\ln(x)+3)x \cdot y_1(x) \\
&= -(2\ln(x)+3)x^2 + (2\ln(x)+3)x^2 \\
&= 0
\end{aligned}$$

Thus, $y_1(x) = x$ is a solution to the differential equation. To find the other solution recall that the Wronskian satisfies (after using problem 2(a) and inputting $y_1(x) = x$):

$$xy_2' - y_2 = y_1 y_2' - y_2 y_1' - W(y_1, y_2) = x^2(\ln(x)+1)$$

so we get a differential equation for y_2. Notice that by dividingby x^2 we have that:

$$\left(\frac{y_2}{x}\right)' = \frac{xy_2' - y_2}{x^2} = \ln(x) + 1$$

so

$$\frac{y_2}{x} = \int (\ln(x) + 1)dx = x\ln(x) - x + x = x\ln(x)$$

where I have integrated by parts to solve the integral. This tells us that;

$$y_2(x) = x^2 \ln(x)$$

is another solution to the differential equation. □

Problem 3. *Find the general solution for equation*

$$z''(t) - z'(t) - 6z(t) = -6 + 10e^{-2t}.$$

Solution. To find the general solution to the equation

$$z'' - z' - 6z = -6 + 10e^{-2t}$$

we must first find the general solution to the homogeneous equation. Letting $z(t) = e^{rt}$ leads to the following equation:

$$0 = r^2 - r - 6 = (r-3)(r+2)$$

so the solutions are $r = 3$ and $r = -2$. Thus, the general solution to the homogeneous problem is given by $c_1e^{3t} + c_2e^{-2t}$. To complete the problem we simply have to find a particular problem to the inhomogeneous problem. Notice that if we let $z(t) = At + B$ where we wish to determine A and B so that $z'' - z' - 6z = -6$ then we get

$$-6At - (A+)[-A - 6BAt - 6B = -6$$

hence $A = 0$ and $B = 1$. Thus, we get that $z = 1$ is a particular solution to $z'' - z' - 6z = -6$. If we can find a solution to $z'' - z' - 6z = 10e^{2t}$ then we will have completed the question by adding the general solution to the previous two particular solutions. Notice that the Wronskian ig given by $W(y_1, y_2)(t) = -2e^t - 3e^t = -5e^t$ and so by the method of variation of parameters we have that;

$$u_1(t) = -\int \frac{e^{-2t}(10e^{-2t})}{-5e^t}dt = 2\int e^{-5t} = \frac{-2}{5}e^{-5t}$$

and

$$u_2(t) = \int \frac{e^{3t}(10e^{-2t})}{-5e^t}dt = -2\int 1dt = -2t$$

and hence the particular solution is given by

$$u_1y_1 + u_2y_2 = \frac{-2}{5}e^{-5t}e^{3t} + (-2t)e^{-2t} = -2e^{2t} - \frac{2e^{-2t}}{5}.$$

Since linear combinations of the solutions to the homogeneous equation remain solutions to the homogeneous equation then we may ignore the factor $-\frac{2e^{-2t}}{t}$. Hence, a particular solution to $z'' - z' - 6z = 10e^{-2t}$ is given by $-2te^{-2t}$. Thus, a the general solution to $z'' - z' - 6z = 1 - 10e^{-2t}$ is given by:

$$z(t) = c_1e^{3t} + c_2e^{-2t} + 1 - 2te^{2t}$$

□

Problem 4. *Find a particular solution of*

$$x^2y''(x) - 6y(x) = 10x^{-2} - 6, \qquad x > 0.$$

Solution. Let $t = \ln(x)$ then we convert the differential equation into one where the independent variable is t. Notice that:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt} = e^{-t}\frac{dy}{dt}$$

and

$$\frac{d^2y}{dt^2} = (-e^{-t})(e^{-t})\frac{dy}{dt} + (e^{-t})(e^{-t})\frac{d^2y}{dt^2} = e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}$$

Putting this information into the differential equation we get that

$$\begin{aligned} 10e^{-2t} - 6 = 10x^{-2} - 6 = x^2\frac{d^2y}{dx^2} - 6y \\ = e^{2t}\left(e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}\right) - 6y = \frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y \end{aligned}$$

Notice that htis is the same differential equation we encountered in Problem 3. Thus, a particular solution is given by:

$$1 - 2te^{-2t}$$

Converting back to x-coordinates we get:

$$1 - 2x^{-2}\ln(x)$$

□