MAT244, 2014F, Solutions to MidTerm

Problem 1. *If exists, find the integrating factor $\mu(x,y)$ depending only on x , only on y and on $x\cdot y$ justifying your answers and then solve the ODE*

$$\left(3x+\frac{6}{y}\right)+\left(\frac{x^2}{y}+\frac{3y}{x}\right)y'=0\ .$$

Also, find the solution satisfying $y(1)=2$.

Solution.

$$M=3x+\frac{6}{y},\ N=\frac{x^2}{y}+\frac{3y}{x}\implies P:=M_y-N_x=-\frac{6}{y^2}-\frac{2x}{y}+\frac{3y}{x^2}$$

and PM^{-1} depends on x, PN^{-1} depends on y, but

$$P/(xM-yN)=\left(-\frac{6}{y^2}-\frac{2x}{y}+\frac{3y}{x^2}\right):\left(2x^2+\frac{6x}{y}-\frac{3y^2}{x}\right)=-1/(xy)=:-h(xy)$$

with $h(z)=z^{-1}$ and therefore $\mu'/\mu=1/z \implies \mu=z$. So xy is an integrating factor. Using it we get

$$(3x^2y+6x)dx+(x^3+3y^2)dy=0\implies U_x=3x^2y+6x,\quad U_y=x^3+3y^2.$$

The first equation implies $U=x^3y+3x^2+\phi(y)$, plugging to the second equation we get $\phi'(y)=3y^2\implies\phi=y^3$. So

$$U:=x^3y+3x^2+y^3=C.$$

As $U(1,2)=13$ and $U:=x^3y+3x^2+y^3=13$ defines solution satisfying indicated initial condition. □

Problem 2. *Find the general solution of*

$$z''+2z'+z=e^{-x}\ln x\ ,\qquad x>0\ .$$

Also, find a solution satisfying $z(1)=-2$.

Solution. Caharacteristic equation is $r^2+2r+1=0 \implies r_1=r_2=-1 \implies z_1=e^{-x}, z_2=xe^{-x}$ is a fundamental system of solutions to homogeneous equation.

To solve inhomogeneous equation we employ method of variations:

$$\begin{cases} e^{-x}C_1'+xe^{-x}C_2'=0, \\ -e^{-x}C_1'-(x-1)e^{-x}C_2'=e^{-x}\ln x \end{cases} \implies C_1'=-x\ln x, C_2'=\ln x \implies$$

$$C_1=-\frac{1}{2}x^2\ln x+\frac{1}{4}x^2+c_1, C_2=x\ln x-x+c_2$$

and finally

$$y=\left(\frac{1}{2}x^2\ln x-\frac{3}{4}x^2+c_1+c_2x\right)e^{-x}.$$

□

Problem 3. *Find the general solution of*

$$x^3y'''+6x^2y''+5xy'-5y=x^2\ln x\ .$$

Solution. Characteristic equation is $r(r-1)(r-2)+6r(r-1)+5r-5=0$ which is $(r-1)(r^2+4r+5)=0$ and has roots $r_1=1$, $r_{2,3}=-2\pm i$. So general solution to homogeneous equation is

$$y^*=C_1x+x^{-2}\big(C_2\cos(\ln x)+C_3\sin(\ln x)\big)$$

and a particular solution to our equation is $y_p=Ax^2\ln x+Bx^2$ which results in $A=\frac{1}{17}$, $B=-\frac{25}{289}$ and

$$y=\frac{1}{17}x^2\ln x-\frac{25}{289}x^2+C_1x+x^{-2}\big(C_2\cos(\ln x)+C_3\sin(\ln x)\big).$$

□

Problem 4. *Find Wronskian* $W(y_1,y_2,y_3)(x)$ *of a fundamental set of solutions* $y_1(x)$, $y_2(x)$, $y_3(x)$ *without finding the* $y_j(x)$ *'s and then the general solution of the ODE*

$$(2-t)y'''+(2t-3)y''-ty'+y=0\ ,\ t<2\ .$$

Hint: e^t *solves the ODE.*

Solution. Equation to Wronskian is

$$\frac{W'}{W}=-\frac{2t-3}{2-t}=2+(t-2)^{-1} \implies \ln W=2t+\ln(t-2) \implies W=(t-2)e^{2t}.$$

Since we know that $y_1 = e^t$ is a solution we plug $y = ze^t$ resulting in

$$(2-t)z''' + (3-t)z'' = 0.$$

Instead of solving this equation which is a first order ODE with respect to z'' we observe that $z = t$ is a solution ($z = 1$ is trivial) and $y_2 = te^t$ is a solution to the original equation. Then to find the y_3 we write

$$W = \begin{vmatrix} y_1 & y_2 & y \\ y_1' & y_2' & y' \\ y_1'' & y_2'' & y'' \end{vmatrix} = \begin{vmatrix} e^t & te^t & y \\ e^t & (t+1)e^t & y' \\ e^t & (t+2)e^t & y'' \end{vmatrix} = e^{2t}(t-2)$$

which instantly simplifies to

$$\begin{vmatrix} 1 & t & y \\ 1 & t+1 & y' \\ 1 & t+2 & y'' \end{vmatrix} = t - 2$$

or

$$y'' - 2y' + y = t - 2$$

which has a particular solution $y_3 = -t$. Then

$$y = (C_1 + C_2 t)e^t + C_3 t$$

$\square$

Problem 5. *Find the general solution of the system of ODEs*

$$\begin{cases} x_t' = x + y + 2z \ , \\ y_t' = x + 2y + z \ , \\ z_t' = 2x + y + z \ . \end{cases}$$

Solution. Characteristic equation is

$$0 = \begin{vmatrix} 1-r & 1 & 2 \\ 1 & 2-r & 1 \\ 2 & 1 & 1-r \end{vmatrix} = \begin{vmatrix} 1-r & 1 & 2 \\ 1 & 2-r & 1 \\ 1+r & 0 & -(r+1) \end{vmatrix} =$$

$$(r+1)\begin{vmatrix} 1-r & 1 & 2 \\ 1 & 2-r & 1 \\ 1 & 0 & -1 \end{vmatrix} = (r+1)\begin{vmatrix} 3-r & 1 & 2 \\ 2 & 2-r & 1 \\ 0 & 0 & -1 \end{vmatrix} =$$

$$-(r+1)(r^2 - 5r + 4) = -(r+1)(r-1)(r-4)$$

where we subsequently subtracted the first row from the the third one, moved common factor $(r+1)$, added the third column to the first one and decomposed determinant by the last row.

So, eigenvalues are $r_1 = 4$, $r_2 = -1$, $r_3 = 1$.

Corresponding eigenvectors are $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C_1 e^{4t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + C_3 e^{t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

$\square$