MAT244, 2014F, Solutions to Final Exam

Problem 1. *Solve the initial value problem*

$$(3x + 2y)\,dx + \left(x + \frac{6y^2}{x}\right) dy = 0 \qquad y(0) = 3\,.$$

Solution. As $M = 3x + 2y$, $N = x + \frac{6y^2}{x}$, $M_y - N_x = 2 - \left(1 - \frac{6y^2}{x^2}\right)$ and $(M_y - N_x)/N = 1/x$ is a function of x only. So we can find integrating factor $\mu = \mu(x)$ from $\mu'/\mu = 1/x \implies \ln\mu = \ln x$ (modulo constant factor) and $\mu = x$. Therefore

$$(3x^2 + 2xy)dx + (x^2 + 6y^2)dy = 0.$$

then

$$U_x = 3x^2 + 2xy, \qquad U_y = x^2 + 6y^2$$

where the first equation implies that

$$U = x^3 + x^2y + \phi(y)$$

and plugging to the second equation we see that

$$\phi' = 6y^2 \implies y = 2y^3.$$

Then

$$U := x^3 + x^2y + 2y^3 = C$$

is a general solution and finding $C = 54$ from initial condition we arrive to

$$U := x^3 + x^2y + 2y^3 = 54.$$

$\square$

Problem 2. *Find the general solution of*

$$x^4y^{(4)} + 6x^3y^{(3)} + 7x^2y^{(2)} + xy' - y = 3\ln x + \cos(\ln x)\,.$$

Solution. It is Euler's equation. Its characteristic polynomial is

$$r(r-1)(r-2)(r-3)+6r(r-1)(r-2)+7r(r-1)+r-1= \\ r^4-6r^3+11r^2-6r+6r^3-18r^2+12r+7r^2-7r+r-1=r^4-1$$

with characteristic roots $r_{1,2}=\pm 1$, $r_{3,4}=\pm i$ and plugging $t=\ln x$ we arrive to

$$y_t^{(4)}-y=3t+\cos(t). \tag{2.1}$$

Solution to homogeneous equation is

$$z=C_1e^t+C_2e^{-t}+C_3\cos(t)+C_4\sin(t)= \\ C_1x+C_2x^{-1}+C_3\cos(\ln x)+C_4\sin(\ln x) \tag{2.2}$$

and the particular solution to inhomogeneous equation is $y_p=y_{p1}+y_{p2}$ with $y_{p1}=at+b$ and $y_{p2}=(c\cos(t)+d\sin(t))t$ solving equation with right hand expressions $f_1=3\ln x$ and $f_2=\cos(\ln x)$ respectively.

Plugging y_{p1} we get

$$-at-b=3t \implies a=-3, b=0 \implies y_{p1}=-3t=-3\ln x \tag{2.3}$$

and plugging y_{p2} we get

$$3(c\sin(t)-d\cos(t))=\cos(t) \implies c=0, d=-\frac{1}{3} \implies \\ y_{p2}=-\frac{1}{3}\sin(t)t=-\frac{1}{3}\sin(\ln x)\ln x. \tag{2.4}$$

Adding (2.2)–(2.4) we get

$$y=C_1x+C_2x^{-1}+C_3\cos(\ln x)+C_4\sin(\ln x)-3\ln x-\frac{1}{3}\sin(\ln x)\ln x.$$

$\square$

Problem 3. *Find the general solution of the system of ODEs*

$$\begin{cases} x'_t = -\frac{5}{4}x + \frac{3}{4}y + \frac{2}{1+e^t}\ , \\ y'_t = \quad \frac{3}{4}x - \frac{5}{4}y\ . \end{cases}$$

Solution. Characteristic equation is

$$\begin{vmatrix} -\frac{5}{4}-r & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4}-r \end{vmatrix} = (r+\frac{5}{4})^2 - \frac{9}{16} = 0$$

with characteristic roots $r_{1,2} = -\frac{5}{4} \pm \frac{3}{4}$, $r_1 = -\frac{1}{2}$, $r_2 = -2$.

Finding corresponding eigenvectors: (a) $r_1 = 1$,

$$\begin{pmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

and then $\alpha = \beta = 1$ and eigenvector is $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(b) $r_2 = 2$ and $\mathbf{e}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (since matrix is symmetric eigenvectors are orthogonal).

Therefore the general solution of the homogeneous system is

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-\frac{1}{2}t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}. \tag{3.1}$$

To solve inhomogeneous system we use method of variation of parameters leading to

$$\begin{pmatrix} e^{-\frac{t}{2}} & e^{-2t} \\ e^{-\frac{t}{2}} & -e^{-2t} \end{pmatrix} \begin{pmatrix} C_1' \\ C_2' \end{pmatrix} = \begin{pmatrix} \frac{2}{1+e^t} \\ 0 \end{pmatrix} \implies$$

$$C_1' = \frac{e^{\frac{t}{2}}}{1+e^t} \implies C_1 = \int \frac{e^{\frac{t}{2}}}{1+e^t}\,dt = 2\arctan(e^{\frac{t}{2}}) + c_1,$$

$$C_2' = \frac{e^{2t}}{1+e^{2t}} \implies C_2 = \int \frac{e^{2t}}{1+e^t}\,dt = \int \Bigl(e^t - \frac{e^t}{1+e^t}\Bigr)\,dt = e^t - \ln(1+e^t) + c_2$$

where the first integral is taken by substitution $u = e^{\frac{t}{2}}$ and the second by substitution $u = 1+e^t$.

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \bigl(2\arctan(e^{\frac{t}{2}}) + c_1\bigr) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-\frac{1}{2}t} + \bigl(e^t - \ln(1+e^t) + c_2\bigr) \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}.$$

□

Problem 4. *Find the general solution of the ODE*

$$xy' = y - xe^{y/x}$$

and solve the initial value problem $y(1) = -2$.

Solution. Since it is homogeneous equation we plug $y = ux$ and then

$$u'x^2 + ux = ux - xe^u \implies u' = -e^u \implies x^{-1}dx = -e^{-u}du \implies$$
$$\ln x = e^{-u} + \ln C \implies u = -\ln\ln(Cx) \implies y = -x\ln\ln(Cx).$$

As $x = 1$, $y = -2$, $u = -2$ we get $\ln\ln C = 2$, and $y = -x\ln(e^2 + \ln x)$. $\square$

Problem 5. *For the system of ODEs*

$$\begin{cases} x'_t = x(5 - 2x - 3y)\,, \\ y'_t = y(5 - 3x - 2y) \end{cases}$$

(a) describe the locations of all critical points,

(b) classify their types (including whatever relevant: stability, orientation, etc.),

(c) sketch the phase portraits near the critical points,

(d) sketch the phase portrait of this system of ODEs.

Solution. (a) Solving $x(5 - 2x - 3y = 0$, $y(5 - 3x - 2y)$ we have 4 cases $x = y = 0$, $x = 5 - 3x - 2y = 0$, $y = 5 - 2x - 3y = 0$ and $5 - 3x - 2y = 5 - 2x - 3 = 0$ giving us 4 points $(0, 0)$, $(0, \frac{5}{2})$, $(\frac{5}{2}, 0)$ and $(1, 1)$.

(b) Let $f = x(5-2x-3y) = 5x-2x^2-3xy$, $g = y(5-3x-2y) = 5y-2y^2-3xy$. Then $f_x = 5 - 4x - 3y$, $f_y = -3x$, $g_x = -3y$, $g_y = 5 - 4y - 3x$.

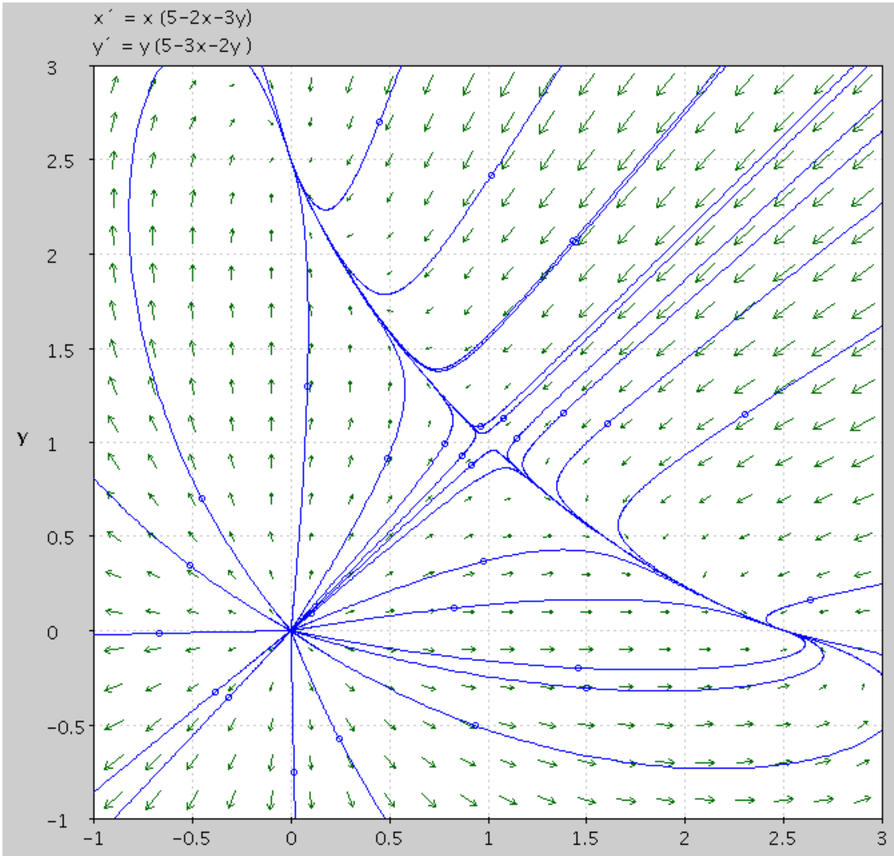
(i) $(0, 0)$; matrix $\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$ at this point equals $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ with eigenvalues $r_1 = r_2 = 5$; and eigenvectors $(1, 0)^T$ and $(0, 1)^T$; unstable node;

(ii) $(0, \frac{5}{2})$; matrix $\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$ at this point equals $\begin{pmatrix} -\frac{5}{2} & 0 \\ -\frac{15}{2} & -5 \end{pmatrix}$ with eigenvalues $r_1 = -\frac{5}{2}$, $r_2 = -5$ and eigenvectors $(1, -3)^T$ and $(1, 0)^T$ respectively; stable node;

(iii) $(\frac{5}{2}, 0)$; the same as in (ii);

(iv) $(1, 1)$; matrix $\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$ at this point equals $\begin{pmatrix} -2 & -3 \\ -3 & -2 \end{pmatrix}$ with eigenvalues $r_1 = -5$ and $r_2 = 1$ and eigenvectors $(1, 1)^T$ and $(1, -1)^T$ respectively; saddle.

(c-d) Plotting



Remark. This is "two competing species" system.

□

Problem 6. *For the system of ODEs*

$$\begin{cases} x_t' = \quad 4x^2y - 2x^2 - 4xy + 2y, \\ y_t' = -4xy^2 + 2y^2 + 4xy - 2x \end{cases}$$

(a) linearize the system at $x_0 = 1\,,\ y_0 = 1$ *and sketch the phase portrait of this linear system,*

(b) find the equation of the form $H(x, y) = C$ *satisfied by the trajectories of the nonlinear system,*

(c) describe the type of the critical point $x_0 = 1\,,\ y_0 = 1$ *of the nonlinear system.*

Solution. (a) Let $f = 4x^2y - 2x^2 - 4xy + 2y$, $g = -4xy^2 + 2y^2 + 4xy - 2x$; then $f_x(1,1) = 0$, $f_y(1,1) = 2$, $g_x(1,1) = -2$, $f_y(1,1) = 0$ and the linearized system is

$$\begin{cases} X'_t = 2Y, \\ Y'_t = -2X \end{cases}$$

with phase portrait consisting of clock-wise circles.

(b) Rewriting system as $f dx - g dy = 0$ we get

$$(4xy^2 - 2y^2 - 4xy + 2x)\,dx + (4x^2y - 2x^2 - 4xy + 2y)\,dy = 0$$

which is exact; then

$$H_x = 4xy^2 - 2y^2 - 4xy + 2x, \quad H_y = 4x^2y - 2x^2 - 4xy + 2y)$$

and the first equation implies that

$$H = 2x^2y^2 - 2xy^2 - 2x^2y + x^2 + \phi(y)$$

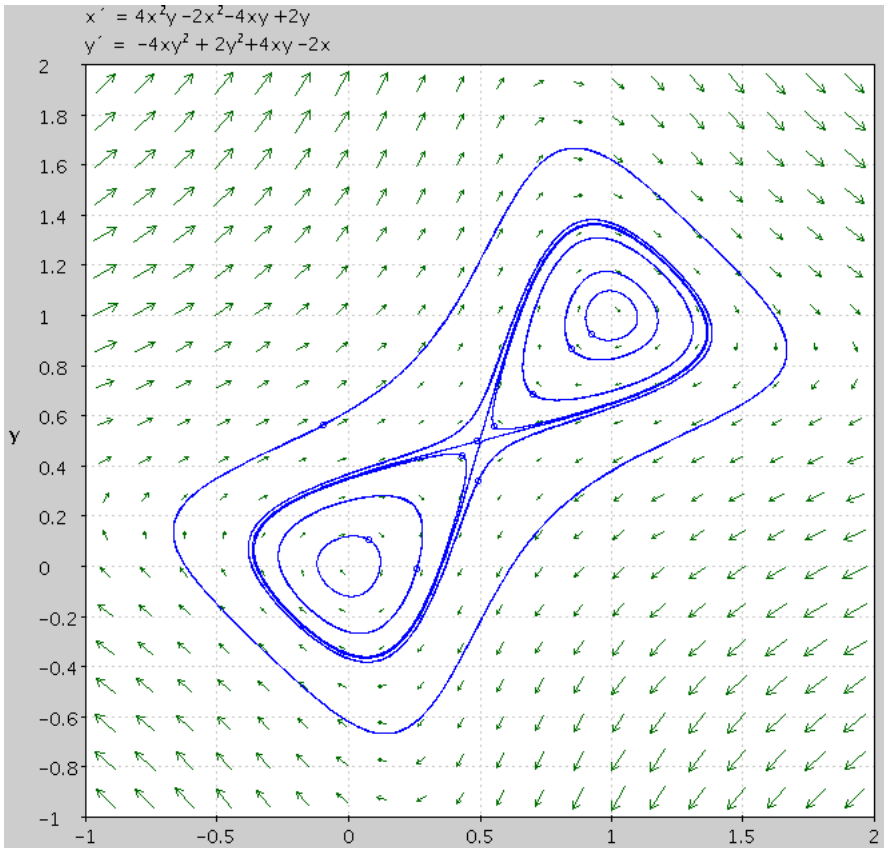
and the second equation implies that $\phi' = 2y$ and $y = y^2$ and then

$$H = 2x^2y^2 - 2xy^2 - 2x^2y + x^2 + y^2 = x^2(y-1)^2 + y^2(x-1)^2.$$

(c) Since linearized system has a center and original system has a solution $H(x,y) = C$ the type of the stationary point is a center.

Remark. In fact the system has also critical point $(0,0)$ of the type center, and critical point $(\frac{1}{2}, \frac{1}{2})$ of the type saddle (see next page)

□



x´ = 4x²y −2x²−4xy +2y
y´ = −4xy² + 2y²+4xy −2x
y
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
−1
−0.5
0
0.5
1
1.5
2