Methods of Integration

Reminder

There are two universal methods of integration:

- *Substitution*
$$\int f(\phi(x))\phi'(x)\,dx = \int f(y)\,dy\Big|_{y=\phi(x)};$$
- *By parts*
$$\int u\,dv = uv - \int v\,du.$$

Rational Functions

Elementary Fractions

The simplest are (we skip +const)

$$\int \frac{dx}{(x-a)} = \log|x-a|, \tag{1}$$

$$\int \frac{(x-b)\,dx}{\big((x-b)^2+c^2\big)} = \frac{1}{2}\log\big((x-b)^2+c^2\big), \tag{2}$$

$$\int \frac{dx}{\big((x-b)^2+c^2\big)} = \frac{1}{c}\arctan\Big(\frac{1}{c}(x-b)\Big). \tag{3}$$

The multiple (with $r \geq 2$):

$$\int \frac{dx}{(x-a)^r} = -\frac{1}{(r-1)}\frac{1}{(x-a)^{r-1}}, \tag{4}$$

$$\int \frac{(x-b)\,dx}{\big((x-b)^2+c^2\big)^r} = -\frac{1}{2(r-1)}\frac{1}{\big((x-b)^2+c^2\big)^{r-1}}, \tag{5}$$

$$\begin{aligned} I_r = \int \frac{dx}{\big((x-b)^2+c^2\big)^r} &= \frac{1}{c^2}\int \frac{dx}{\big((x-b)^2+c^2\big)^{r-1}} - \frac{1}{c^2}\int \frac{(x-b)^2\,dx}{\big((x-b)^2+c^2\big)^{r-1}} = \\ &\frac{1}{c^2}I_{r-1} + \frac{1}{2c^2(r-1)}\int (x-b)\,d\frac{1}{\big((x-b)^2+c^2\big)^{r-1}} = \\ &\frac{1}{c^2}I_{r-1} + \frac{1}{2c^2(r-1)}\frac{(x-b)}{\big((x-b)^2+c^2\big)^{r-1}} - \frac{1}{2c^2(r-1)}\int \frac{dx}{\big((x-b)^2+c^2\big)^{r-1}}. \end{aligned} \tag{6}$$

So

$$I_r = \frac{2r-3}{2c^2(r-1)}I_{r-1} + \frac{1}{2c^2(r-1)}\frac{(x-b)}{\big((x-b)^2+c^2\big)^{r-1}}$$

and we can calculate $I_2, I_3, \ldots$ recurrently.

General Rational Functions

Consider

$$\int R(x)\,dx = \int \frac{P_m(x)}{Q_n(x)}\,dx \tag{7}$$

where $P_m(x)$, $Q_n(x)$ are the polynomials of degrees m and $n \geq 1$ and the main coefficient in $Q_n(x)$ is 1.

Step 1 If $m < n$ go to *Step 2*. Else we divide P_m by Q_n with the remainder:

$$P_m(x) = S_{m-n}(x)Q_n(x) + T_{m'}(x) \tag{8}$$

with $m' \leq n-1$ and

$$\int R(x)\,dx = \int S_{m-n}(x)\,dx + \int \frac{T_{m'}(x)}{Q_n(x)}\,dx. \tag{9}$$

Step 2 So $m < n$. We can decompose polynomial $Q_n(x)$ into product

$$Q_n(x) = \prod_{j=1}^{p}(x-a_j)^{r_j} \times \prod_{j=p+1}^{q}\left((x-b_j)^2 + c_j^2\right)^{r_j} \tag{10}$$

where a_j are distinct real roots of Q_n, $b_j \pm ic_j$ are distinct non-real roots of Q_n, $c_j > 0$ and r_j is the multiplicity of the corresponding root.

Then one can decompose $\frac{P_m(x)}{Q_n(x)}$ with $m \leq n-1$ into elementary fractions:

$$\frac{P_m(x)}{Q_n(x)} = \sum_{j=1}^{p}\sum_{k=1}^{r_j}\frac{A_{j,k}}{(x-a_j)^k} + \sum_{j=p+1}^{q}\sum_{k=1}^{r_j}\frac{B_{j,k}(x-b_j)+C_{j,k}}{\left((x-b_j)^2+c_j^2\right)^k} \tag{11}$$

with unknown constant coefficients $A_{j,k}$, $B_{j,k}$, $C_{j,k}$.

Step 3 We find these coefficients from equation

$$P_m(x) = \sum_{j=1}^{p}\sum_{k=1}^{r_j}\frac{A_{j,k}\,Q_n(x)}{(x-a_j)^k} + \sum_{j=p+1}^{q}\sum_{k=1}^{r_j}\frac{\left(B_{j,k}(x-b_j)+C_{j,k}\right)Q_n(x)}{\left((x-b_j)^2+c_j^2\right)^k} \tag{12}$$

Note that $\dfrac{Q_n(x)}{(x-a_j)^k}$, $\dfrac{(x-b_j)Q_n(x)}{\left((x-b_j)^2+c_j^2\right)^k}$ and $\dfrac{Q_n(x)}{\left((x-b_j)^2+c_j^2\right)^k}$ are ploynomials of degrees $\leq n-1$.

Check that the total number of the coefficients is n.

Remember that substitution of a_j into equation (12) is a good idea.

Step 4 Now after coefficients are found we can integrate all the elementary fractions in (11).

Trigonometric Polynomials

- Trigonometric polynomial is $P(\cos x, \sin x) = \sum_{m,n} a_{m,n} \cos^m x \sin^n x$. To integrate trigonometric polynomial one needs to be able to integrate *trigonometric monomials* $\cos^m x \sin^n x$ with $m \geq 0$, $n \geq 0$.

  We however can consider also negative m, n but outputs is not necessarily good.

- $m = 2p, n = 2q$ Then $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$, $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$ and we get

$$\frac{1}{2^{p+q}} \bigl(1 + \cos(2x)\bigr)^p \bigl(1 - \cos(2x)\bigr)^q$$

  and lowered degree $m + n$.

- $m = 2p + 1$ Then

$$\int \cos^{2p+1} \sin^n x\, dx = \int \cos^{2p} x \sin^n x\, d \sin x = \int (1 - z^2)^p z^n\, dz$$

  after substitution $z = \sin x$.

- $n = 2q + 1$ Then

$$\int \cos^n \sin^{2q+1} x\, dx = -\int \cos^n x \sin^{2q} x\, d \cos x = -\int (1 - z^2)^q z^m\, dz$$

  after substitution $z = \cos x$.

- $m = 2m + 1$, $n = 2q + 1$ We can use any of theses substitutions.

Trigonometric Rational Functions

Trigonometric rational function is the ratio of two trigonometric polynomials: $R(\cos x, \sin x) = \dfrac{P(\cos x, \sin x)}{Q(\cos x, \sin x)}$.

Our purpose is to reduce it to integral of rational function.

- $R(u, v)$ is an even function with respect to u and v: $R(u, v) = R_1(u^2, v^2)$. We apply substitution $z = \tan x$. Then $\cos^2 x = \dfrac{1}{1 + z^2}$, $\sin^2 x = \dfrac{z^2}{1 + z^2}$ and $dx = \dfrac{dz}{1 + z^2}$, so we arrive to integral

$$\int R_1(\cos^2 x, \sin^2 x)\, dx = \int R_1\Bigl(\frac{1}{1 + z^2}, \frac{z^2}{1 + z^2}\Bigr) \frac{dz}{1 + z^2}.$$

- $R(u,v)$ is an odd function with respect to u: $R(u,v)=R_1(u^2,v)u$. We apply substitution $z=\sin x$ and we arrive to integral:
$$\int R_1(\cos^2 x, \sin x)\cos x\,dx = \int R_1\Bigl(1-z^2, z\Bigr)\,dz.$$
- $R(u,v)$ is an odd function with respect to v: $R(u,v)=R_1(u,v^2)v$. We apply substitution $z=\cos x$ and we arrive to integral:
$$\int R_1(\cos x, \sin^2 x)\sin x\,dx = -\int R_1\Bigl(z, 1-z^2\Bigr)\,dz.$$
- *General case* There is an universal substitution $z=\tan\frac{x}{2}$; however special substitutions above are better if applicable.

  As $z=\tan\dfrac{x}{2}$, $\cos x=\dfrac{1-z^2}{1+z^2}$, $\sin x=\dfrac{2z}{1+z^2}$ and $dx=\dfrac{2\,dz}{1+z^2}$, so we arrive to integral
$$\int R_1\Bigl(\frac{1-z^2}{1+z^2}, \frac{2z}{1+z^2}\Bigr)\,\frac{2dz}{1+z^2}.$$

Special Irrational Functions. I

We consider functions of the type
$$F(x)=R(x,\sqrt{\mathcal{Q}(x)}),\qquad \mathcal{Q}\quad \text{quadratic polynomial.}$$
Our purpose is to reduce to the integral of trigonometric rationalfunction.

By shift one can reduce $\mathcal{Q}$ to either x^2+c^2, or x^2-c^2, or c^2-x^2 with $c>0$ (excluding case x^2 when we get piece-wise rational function.

- $\mathcal{Q}=x^2+c^2$. Possile substitution: $x=c\tan z$, then we get
$$\int R\Bigl(x,\sqrt{x^2+c^2}\Bigr)\,dx = c\int R\Bigl(\frac{c\sin z}{\cos z}, \frac{c}{\cos z}\Bigr)\,\frac{dz}{cos^2 z}.$$
- $\mathcal{Q}=x^2+c^2$. Another possile substitution: $x=c\sinh z$, then we get
$$\int R\Bigl(x,\sqrt{x^2+c^2}\Bigr)\,dx = c\int R\Bigl(c\sinh z, c\cosh z\Bigr)\cosh z\,dz.$$
- $\mathcal{Q}=x^2-c^2$. Possile substitution: $x=c\sec z$, then we get
$$\int R\Bigl(x,\sqrt{x^2-c^2}\Bigr)\,dx = c\int R\Bigl(\frac{c}{\cos z}, \frac{c\sin z}{\cos z}\Bigr)\frac{\sin z}{\cos^3 z}$$

- $\mathcal{Q} = x^2 - c^2$. Another possile substitution: $x = c\cosh z$, then we get

$$\int R\Big(x, \sqrt{x^2 - c^2}\Big)\, dx = c \int R\Big(c\cosh z, c\sinh z\Big) \sinh z\, dz.$$

- $\mathcal{Q} = c^2 - x^2$. Possile substitution: $x = c\sin z$, then we get

$$\int R\Big(x, \sqrt{c^2 - x^2}\Big)\, dx = c \int R\Big(c\sin z, c\cos z\Big) \cos z\, dz$$

- $\mathcal{Q} = c^2 - x^2$. Another possile substitution: $x = c\tanh z$, then we get

$$\int R\Big(x, \sqrt{c^2 - x^2}\Big)\, dx = c \int R\Big(\frac{c\sinh z}{\cosh x}, \frac{c}{\cosh x}\Big) \frac{dz}{\cosh^2 z}.$$

Special Irrational Functions. II

Now we consider

$$\int R\Big(x, Z(x)^{1/m}\Big)\, dx, \qquad Z(x) = \frac{\alpha x + \beta}{\gamma x + \delta},$$

with $ad - bc \neq 0$ (otherwise $Z = \mathrm{const}$). This integral is calculated by substitution $Z(x) = z^m$. Then $x = -\dfrac{\delta z^m - \beta}{\gamma z^m - \alpha}$, $dx = \dfrac{\alpha\delta - \beta\gamma}{(\gamma z^m - \alpha)^2} \cdot m z^{m-1}\, dz$ and

$$\int R\Big(x, Z(x)^{1/m}\Big)\, dx = m(\alpha\delta - \beta\gamma) \int R\Big(-\frac{\delta z^m - \beta}{\gamma z^m - \alpha}, z\Big) z^{m-1}\, dz.$$