University of Toronto, Faculty of Arts and Science

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APM346 — Partial Differential Equations

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Duration — 3 hours

The 7 problems are independent. Total marks for this paper: 105.

A list of useful formulas is attached in page 4. No other aids allowed.

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1. (15 pts) Solve by the method of characteristics

$$\begin{cases} u_{tt} - 4u_{xx} = 0, \qquad x > 0,\ t > 0 \\ u_x(0,t) = 0, \\ u(x,0) = 0, u_t(x,0) = x. \end{cases}$$

2. (15 pts) Solve the heat equation with convection

$$\begin{cases} u_t - 2u_x - u_{xx} = 0, \qquad t > 0, \quad x \in \mathbb{R}, \\ u|_{t=0} = e^{-|x|} \end{cases}$$

3. (15 pts) Solve by the method of separation of variables

$$\begin{cases} u_{tt} - 4u_{xx} = 0, \qquad 0 < x < \pi,\ t > 0, \\ u_x(0,t) = u_x(\pi,t) = 0, \\ u(x,0) = 0, u_t(x,0) = \sin x. \end{cases}$$

Continued

4. (15 pts) Consider the Laplace equation in the half disk

$$u_{xx} + u_{yy} = 0 \qquad \text{in } \; r = \sqrt{x^2 + y^2} < a, \; y > 0, \tag{1}$$

with the boundary conditions

$$u = 1 \qquad \text{for } \; r = a, y > 0, \tag{2}$$

$$u = 0 \qquad \text{for } \; y = 0. \tag{3}$$

(a) Look for solutions u in the form of $u(r, \theta) = R(r)P(\theta)$ (in polar coordinates) and derive a set of ordinary differential equations for R and P. Write the correct boundary conditions for P.

(b) Solve the eigenvalue problem for P and find all eigenvalues.

(c) Solve the differential equation for R.

(d) Find the solution u of (1)–(3). Write the answer in terms of Fourier series.

5. (10 pts) Write the Full Fourier series of the function $u(x) = |\sin x|$ on $(-\pi, \pi)$. Calculate the coefficients.

6. (15 pts)

(a) Compute the Fourier transform of $f(x) = e^{-ax^2}$, where a is a strictly positive constant.

(b) Let $u(x, t)$ be the solution of the diffusion equation

$$u_t - u_{xx} = 0, \qquad x \in \mathbb{R}, \; u(x, 0) = g(x).$$

Write the equation satisfied by the Fourier transform $\hat{u}(k, t)$ of the solution $u(x, t)$. Find $\hat{u}(k, t)$.

(c) Using the properties of the Fourier transform, recover the general formula for the solution $u(x, t)$ of the diffusion equation on a line.

Continued

7. (20 pts) Let $f(x)$ be a continuous function on the line $(-\infty, +\infty)$ that vanishes for large $|x|$.

(a) Show that the function

$$g(x) = \sum_{n=-\infty}^{+\infty} f(x + 2\pi n)$$

is periodic of period 2π.

(b) Show that the complex Fourier coefficients c_m of $g(x)$ on the interval $(-\pi, \pi)$ are given by the formula

$$c_m = \frac{1}{2\pi} \hat{f}(m)$$

where $\hat{f}(k)$ is the Fourier transform of $f(x)$.

(c) In the complex Fourier series of $g(x)$ on $(-\pi, \pi)$, let $x = 0$ to obtain the *Poisson summation formula*

$$\sum_{n=-\infty}^{+\infty} f(2\pi n) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \hat{f}(n)$$

Continued

Appendix: Some useful formulas.
Not exam problems

1. The two dimensional Laplacian in polar coordinates:

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}.$$

2. The Stokes theorem

$$\int_D \frac{\partial f}{\partial x_i}\, dx = \int_{\partial D} f n_i \, d\sigma$$

where n (with components n_i) is the unit normal vector pointing outside.

3. The complex Fourier series of a periodic function $f(x)$ of period $2l$, defined on the interval $(-l, l)$ is

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{\pi i n x/l}$$

with the coefficients c_n given by the formula

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-\pi i n x/l} dx$$

4. The Fourier transform of a function $f(x)$ is defined by

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x) dx.$$

The inverse Fourier transform is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} \hat{f}(k) dk.$$

Here some of its properties:

(a) if $g(x) = f(ax)$, then $\hat{g}(k) = \frac{1}{|a|}\hat{f}(\frac{k}{a})$.

(b) if $g(x) = f(x-a)$, then $\hat{g}(k) = e^{-iak}\hat{f}(k)$.

(c) if $h = f * g$, then $\hat{h}(k) = \hat{f}(k)\hat{g}(k)$.

(d) if $f(x) = e^{-x^2/2}$, then $\hat{f}(k) = \sqrt{2\pi} e^{-k^2/2}$.