# Existence and uniqueness of minimal blow up solutions to an inhomogeneous mass critical NLS

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## The homogeneous focusing cubic NLS in $\mathbb{R}^2$

$$\begin{cases} iu_t = -\Delta u - |u|^2 u, \quad (t,x) \in [-1, T) \times \mathbb{R}^2 \\ u(-1, x) = u_0(x), \quad u_0 : \mathbb{R}^2 \to \mathbb{C} \end{cases}$$

$\forall (\lambda_0, t_0, x_0, \beta_0, \gamma_0) \in \mathbb{R}_*^+ \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}$, $v$ is also solution with:

$$v(t,x) = \lambda_0 u(\lambda_0^2 t + t_0, \lambda_0 x + x_0 - \beta_0 t) e^{i \frac{\beta_0}{2} \cdot (x - \frac{\beta_0}{2} t)} e^{i\gamma_0}$$

$Q$ unique positive radial solution to:

$$\Delta Q - Q + Q^3 = 0$$

Note that $Q(x)e^{it}$ is a solution

## Explicit minimal blow-up solutions

If $|u_0|_{L^2} < |Q|_{L^2}$, then $T = +\infty$

If $|u_0|_{L^2} > |Q|_{L^2}$, there are blow-up solutions (virial law)

Are there blow-up solutions with $|u_0|_{L^2} = |Q|_{L^2}$?

Conformal transform: $v(t,x) = \frac{1}{|t|}\overline{u}\left(\frac{1}{t}, \frac{x}{t}\right) e^{i\frac{|x|^2}{4t}}$

Explicit finite time blow-up solution $S(t,x)$:

$$S(t,x) = \frac{1}{|t|} Q\left(\frac{x}{t}\right) e^{i\frac{|x|^2}{4t} - \frac{i}{t}}, \quad |S(t)|_{L^2} = |Q|_{L^2}$$

Blow up speed: $|\nabla S(t)|_{L^2} \sim \frac{1}{|t|}$

# The uniqueness result of Frank Merle

Theorem [(Merle 93')]: A solution with $u_0 \in H^1$, $\|u_0\|_{L^2} = \|Q\|_{L^2}$ and blowing up at $t = 0$ is equal to $S(t)$ up to the symmetries of the flow

Step 1: Show $\lim_{t \to 0} \int |x|^2 |u(t,x)|^2 dx = 0$

Step 2: Define $v(t,x) = \frac{1}{|t|} \overline{u} \left( \frac{1}{t}, \frac{x}{t} \right) e^{i \frac{|x|^2}{4t}}$

Then: $\|v\|_{L^2} = \|Q\|_{L^2}$ and $E(v) = \frac{1}{8} \lim_{t \to 0} \int |x|^2 |u(t,x)|^2 dx = 0$

$\Rightarrow v = Q$ up to the symmetries of the flow

Proof fundamentally relies on the conformal transform

Goal: Find a proof which works without conformal transform

## An inhomogeneous focusing cubic NLS in $\mathbb{R}^2$

$$\begin{cases} iu_t = -\Delta u - k(x)|u|^2 u, & (t,x) \in [-1,T) \times \mathbb{R}^2 \\ u(-1,x) = u_0(x), & u_0 : \mathbb{R}^2 \to \mathbb{C} \end{cases}$$

$$L^2\text{norm}: \quad \int |u(t,x)|^2 = \int |u_0(x)|^2,$$

$$\text{Energy}: \quad E(u(t,x)) = \frac{1}{2}\int |\nabla u(t,x)|^2 - \frac{1}{4}\int k(x)|u(t,x)|^4 = E(u_0)$$

Only symmetry: phase invariance

Toy model to analyze the properties of cubic NLS in the absence of conformal invariance

## First results in the inhomogeneous case (Merle 96')

Assume $\sup_{x\in\mathbb{R}^2} k(x) < +\infty$. For simplicity, $\sup_{x\in\mathbb{R}^2} k(x) = 1$

If $\|u_0\|_{L^2} < \|Q\|_{L^2} \Rightarrow$ global existence

If $\|u_0\|_{L^2} > \|Q\|_{L^2}$, there are blow-up solutions

If a critical mass blow-up elements exists, then there is $x_0 \in \mathbb{R}^2$ such that $u$ blows-up at $x_0$

Furthermore, $k$ must be at least $C^2$ at $x_0$, and $k(x_0) = 1$

Assume $\sup_{x \in \mathbb{R}^2} k(x) = 1$ is attained

Necessary condition for the existence: let $u$ with $\|u\|_{L^2} = \|Q\|_{L^2}$ be a solution which blows up at $t = 0$ and $x = x_0$. Then:

$$E_0 + \frac{1}{8} \int \nabla^2 k(x_0)(y, y) Q^4 > 0$$

We pick $x_0 \in \mathbb{R}^2$ with $k(x_0) = 1$ and focus on the nondegenerate case:

$$\nabla^2 k(x_0) < 0$$

Existence part in the degenerate case $\nabla^2 k(x_0) = 0$ has been obtained by Banica, Carles, Duyckaerts 09'

# The existence and uniqueness result

Theorem [Raphaël, S. 10']: Let $x_0 \in \mathbb{R}^2$ with $k(x_0) = 1$ and $\nabla^2 k(x_0) < 0$. Then for all $E_0$ satisfying

$$E_0 + \frac{1}{8}\int \nabla^2 k(x_0)(y,y)Q^4 > 0$$

there exists a unique up to phase shift critical mass blow up solution which blows up at $t = 0$ and $x = x_0$ with energy $E_0$.

The blow up speed is given by the conformal law:

$$|\nabla u(t)|_{L^2} \sim \frac{1}{|t|}$$

Same speed as in the homogeneous case

## Strategy of the proof

- Construction of an approximate solution at high order
- Modulation theory, conservation laws, and first consequences
- The localized virial identity, the control of the blow-up speed and existence of a critical element
- Uniqueness of the critical element

## Rescaled variables

$$x_0 = 0,\ T = 0$$

$$u(t,x) = \frac{1}{[k(\alpha(t))]^{\frac{1}{2}}} \frac{1}{\lambda(t)} v\left(s, \frac{x-\alpha(t)}{\lambda(t)}\right) e^{i\gamma(t)}, \quad \frac{ds}{dt} = \frac{1}{\lambda^2}$$

$$i\partial_s v + \Delta v - v + \frac{k(\lambda y + \alpha)}{k(\alpha)} v|v|^2 = i\frac{\lambda_s}{\lambda}\Lambda v + i\frac{\alpha_s}{\lambda}\cdot\left(\nabla v + \frac{\lambda}{2}\frac{\nabla k(\alpha)}{k(\alpha)} v\right) + (\gamma_s - 1)v$$

$$\Lambda f = f + y \cdot \nabla f$$

When $k = 1$, the minimal blow-up solution is given by $v = Q(y) e^{-i\frac{b|y|^2}{4}} e^{i\beta\cdot y}$ with $b, \lambda, \beta, \alpha, \gamma$ satisfying:

$$b_s = -b^2, \quad -\frac{\lambda_s}{\lambda} = b, \quad \frac{\alpha_s}{\lambda} = 2\beta, \quad \beta_s = -b\beta, \quad \gamma_s = 1 + |\beta|^2$$

## Approximate solution at order 5

Let $\mathcal{P} = (b, \lambda, \beta, \alpha)$

In view of the homogeneous case, let $v = P_{\mathcal{P}}(y) e^{-i\frac{b|y|^2}{4}} e^{i\beta\cdot y}$ with:

$$b_s = -b^2 + \mathcal{B}_1, \quad -\frac{\lambda_s}{\lambda} = b, \quad \frac{\alpha_s}{\lambda} = 2\beta, \quad \beta_s = -b\beta + \mathcal{B}_2, \quad \gamma_s = 1 + |\beta|^2$$

where $\mathcal{B}_1$, $\mathcal{B}_2$ will be chosen later

$$i(-b^2 + \mathcal{B}_1)\partial_b P_{\mathcal{P}} - i\lambda b \partial_\lambda P_{\mathcal{P}} + 2i\beta\lambda\partial_\alpha P_{\mathcal{P}} + i(-b\beta + \mathcal{B}_2)\partial_\beta P_{\mathcal{P}} + \mathcal{B}_1 \frac{|y|^2}{4} P_{\mathcal{P}}$$
$$- \left\{ \mathcal{B}_2 \cdot y + i\lambda\beta \cdot \frac{\nabla k(\alpha(t))}{k(\alpha(t))} \right\} P_{\mathcal{P}} + \Delta P_{\mathcal{P}} - P_{\mathcal{P}} + \frac{k(\lambda(t)y + \alpha(t))}{k(\alpha(t))} P_{\mathcal{P}} |P_{\mathcal{P}}|^2 = 0$$

$$P_{\mathcal{P}} = Q + T_1 + iS_1 + T_2 + iS_2 + T_3 + iS_3 + T_4 + iS_4$$

# Computation of $T_1, S_1, T_2, S_2$

$$L_+ = -\Delta + 1 - 3Q^2, \quad L_- = -\Delta + 1 - Q^2$$

$$Ker\{L_+\} = \text{span}\{\nabla Q\}, \quad Ker\{L_-\} = \text{span}\{Q\}$$

$$L_+(T_j) = -\mathcal{B}_{2\,j} yQ + \cdots \text{ and } L_-(S_j) = \mathcal{L}(\mathcal{B}_{1\,j-1})\rho + \cdots$$

where $L_+\rho = |y|^2 Q$ and $\mathcal{L}$ is an explicit linear operator

We may choose $T_1 = S_1 = S_2 = 0$ and $\mathcal{B}_{1\,1} = \mathcal{B}_{2\,1} = 0$

To solve $T_2 \Rightarrow \mathcal{B}_{2\,2} = c_0(\alpha)\lambda$ with

$$(c_0(\alpha))_j := \frac{\left(\int Q^4\right)}{2\left(\int Q^2\right)} \nabla^2 k(0)(e_j, \alpha) \text{ for } j = 1, 2$$

# Computation of $S_3$

$$L_-(S_3) = -\lambda b \partial_\lambda T_2 + 2\beta\lambda\partial_\alpha T_2 + \mathcal{L}(\mathcal{B}_{1\,2})\rho$$

We may solve for $S_3$ iff:

$$(-\lambda b \partial_\lambda T_2 + 2\beta\lambda\partial_\alpha T_2 + \mathcal{L}(\mathcal{B}_{1\,2})\rho, Q) = 0$$

$$(T_2, Q) = -\frac{\lambda^2}{4}(\nabla^2 k(0)(y,y)Q^3, \Lambda Q) = 0$$

We may take $\mathcal{B}_{1\,2} = 0$. This cancellation is the reason why we do not need to adjust the law for $b$ at the order 2 and hence why the critical blow up will still display the conformal speed

## End of the construction of the approximate solution

$$Q_{\mathcal{P}}(y) = P_{\mathcal{P}}(y) e^{-i\frac{b|y|^2}{4}} e^{i\beta \cdot y} \text{ with}$$

$$P_{\mathcal{P}} = Q + T_2 + T_3 + iS_3 + T_4 + iS_4$$

$$b_s = -b^2, \quad -\frac{\lambda_s}{\lambda} = b, \quad \frac{\alpha_s}{\lambda} = 2\beta, \quad \beta_s = -b\beta + c_0(\alpha)\lambda + \cdots, \quad \gamma_s = 1 + |\beta|^2$$

$Q_{\mathcal{P}}$ satisfies the equation of $v$ up to a remainder $O(\mathcal{P}^5)$

$$E(Q_{\mathcal{P}}) = \frac{b^2}{8} \int |y|^2 Q^2 + \frac{|\beta|^2}{2} \int Q^2 - \frac{\lambda^2}{8} \int \nabla^2 k(0)(y,y) Q^4 + O(\mathcal{P}^4)$$

## Modulation theory

$$u(t,x) = \frac{1}{[k(\alpha(t))]^{\frac{1}{2}}} \frac{1}{\lambda(t)} (Q_{\mathcal{P}(t)} + \varepsilon) \left( s, \frac{x-\alpha(t)}{\lambda(t)} \right) e^{i\gamma(t)}$$

$$\Im(\varepsilon, \nabla Q_{\mathcal{P}}) = 0$$

$$\Re(\varepsilon, y Q_{\mathcal{P}}) = 0$$

$$\Im(\varepsilon, \Lambda Q_{\mathcal{P}}) = 0$$

$$\Re(\varepsilon, |y|^2 Q_{\mathcal{P}}) = 0$$

$$-(\varepsilon_1, \rho_2) + (\varepsilon_2, \rho_1) = 0$$

where $\varepsilon = \varepsilon_1 + i\varepsilon_2$, $L_+\rho = |y|^2 Q$ and $\rho_1 + i\rho_2 = \rho(y) e^{-ib\frac{|y|^2}{4} + i\beta\cdot y}$

## First consequences

$$(L_+\varepsilon_1,\varepsilon_1)+(L_-\varepsilon_2,\varepsilon_2)\geq c_0\|\varepsilon\|^2_{H^1}$$
$$-\quad\frac{1}{c_0}\left\{(\varepsilon_1,Q)^2+(\varepsilon_1,|y|^2Q)^2+(\varepsilon_1,yQ)^2+(\varepsilon_2,\rho)^2\right\}$$

Sum of conservation of mass and energy + orthogonality conditions:

$$b^2+|\beta|^2+|\alpha|^2+\|\varepsilon\|^2_{H^1}\lesssim\lambda^2\left(E_0+\frac{1}{8}\int\nabla^2k(0)(y,y)Q^4\right)+\cdots$$

$$\Rightarrow\;E_0+\frac{1}{8}\int\nabla^2k(0)(y,y)Q^4>0$$

$$\frac{\lambda_s}{\lambda}=-b+\cdots\Rightarrow\lambda_t=\frac{\lambda_s}{\lambda^2}=O(1)\Rightarrow 0<\lambda\lesssim|t|$$

Conformal blow-up speed if $b\sim\lambda$

# The localized virial identity

Let $A > 0$ large enough and $\phi : \mathbb{R}^2 \to \mathbb{R}$ smooth cut off function with
$\nabla\phi(y) = y$ for $|y| \leq 1$

$$\left\{ -\left(\frac{b}{\lambda}\right)\frac{|yQ|_{L^2}^2}{4} + \frac{1}{2\lambda}\Im\left(\int A\nabla\phi\left(\frac{y}{A}\right)\cdot\nabla\varepsilon\overline{\varepsilon}\right)\right\}_s$$
$$\geq \frac{c}{\lambda}\left\{|\alpha|^2 + \|\varepsilon\|_{L^2}^2\right\} + \cdots$$

Since $b + \|\varepsilon\|_{H^1} \lesssim \lambda$, we obtain after integration in $s$:

$$\int_{t_0}^0 \frac{1}{\lambda}\left\{\frac{|\alpha|^2}{\lambda^2} + \frac{\|\varepsilon\|_{L^2}^2}{\lambda^2}\right\} dt < +\infty$$

# Control of the blow-up speed

Consequences of the localized viriel bound:

- $\alpha/\lambda$ converges to 0 in time average
- $\|\varepsilon\|_{H^1}/\lambda$ converges to 0 in time average
- $\beta/\lambda$ converges to 0 in time average

Conservation of mass and energy + localized virial identity:

$$\lim_{t\to 0} \frac{b}{\lambda} = \frac{1}{C_0} \text{ with } C_0 = \frac{\|yQ\|_{L^2}}{\sqrt{8\tilde{E}_0}}$$

$$b \sim \lambda \text{ and } \frac{\lambda_s}{\lambda} + b = \cdots \;\Rightarrow\; \lambda(t) = \frac{|t|}{C_0}(1 + o(1))$$

## Existence of a critical blow-up element

Sum of mass and energy divided by $\lambda^2$ yields:

$$\frac{|\beta|^2}{\lambda^2}+\frac{|\alpha|^2}{\lambda^2}+\frac{\|\varepsilon\|_{H^1}^2}{\lambda^2}\lesssim\frac{1}{C_0^2}-\frac{b^2}{\lambda^2}+\cdots$$

Integration is $s$ of the localized virial identity and $\lim_{t\to 0}\frac{b}{\lambda}=\frac{1}{C_0}$:

$$\int_s^{+\infty}\frac{1}{\lambda}\left\{|\alpha|^2+\|\varepsilon\|_{L^2}^2\right\}d\sigma\lesssim\frac{b}{\lambda}-\frac{1}{C_0}+\cdots$$

$$\Rightarrow |\beta|+|\alpha|+\|\varepsilon\|_{H^1}\lesssim\lambda^2,\ \left|\frac{b}{\lambda}-\frac{1}{C_0}\right|\lesssim\lambda^2,\ \lambda(t)=-\frac{t}{C_0}+O(|t|^3)$$

Previous estimates + compactness argument

$\Rightarrow$ existence of a critical blow-up element

## Uniqueness of the critical blow-up element

$w = u - u_c$ with $u_c$ the critical blow-up element we have constructed

Goal: we need to show $w \equiv 0$

We first improve our estimates:

$$|\gamma - \gamma_c| + \frac{|\alpha - \alpha_c|}{|t|} + \frac{|\lambda - \lambda_c|}{|t|} + |b - b_c| + |\beta - \beta_c| + \|\varepsilon\|_{H^1} + \|\varepsilon_c\|_{H^1} \lesssim t^4$$

$\Rightarrow$ control of the instabilities generated by the null space of the linearized operator around $u_c$

$\Rightarrow$ One may integrate the flow close to $u_c$ backwards from $t = 0$

Since $\lim_{t \to 0} \|w\|_{H^1} = 0$, we deduce $w \equiv 0$