THE ENERGY-CRITICAL QUANTUM HARMONIC OSCILLATOR

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ABSTRACT. We consider the energy critical nonlinear Schrödinger equation in dimensions 3 and above with a harmonic oscillator potential. In the defocusing situation, we prove global wellposedness for all initial data in the energy space Σ. This extends a result of Killip-Visan-Zhang, who treated the radial case. For the focusing nonlinearity, we obtain wellposedness for data in Σ satisfying an analogue of the usual size restriction in terms of the ground state W. We implement the concentration compactness variant of the induction on energy paradigm and, in particular, develop profile decompositions adapted to the harmonic oscillator.

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1. INTRODUCTION

We study the initial value problem for the energy-critical nonlinear Schrödinger equation on $\mathbf{R}^d$, $d \geq 3$, with a harmonic oscillator potential:

$$\begin{cases} i\partial_t u = (-\frac{1}{2}\Delta + \frac{1}{2}|x|^2)u + \mu|u|^{\frac{4}{d-2}}u, \quad \mu = \pm 1, \\ u(0) = u_0 \in \Sigma(\mathbf{R}^d). \end{cases} \tag{1.1}$$

The equation is *defocusing* if $\mu = 1$ and *focusing* if $\mu = -1$. Solutions to this PDE conserve energy, defined as

$$E(u(t)) = \int_{\mathbf{R}^d} \left[\tfrac{1}{2}|\nabla u(t)|^2 + \tfrac{1}{2}|x|^2|u(t)|^2 + \tfrac{d-2}{d}\mu|u(t)|^{\frac{2d}{d-2}}\right] dx = E(u(0)). \tag{1.2}$$

The term "energy-critical" refers to the fact that if we ignore the $|x|^2/2$ term in the equation and the energy, the scaling

$$u(t,x) \mapsto u^\lambda(t,x) := \lambda^{-\frac{2}{d-2}}u(\lambda^{-2}t, \lambda^{-1}x) \tag{1.3}$$

preserves both the equation and the energy. We take initial data in the weighted Sobolev space Σ, which is the natural space of functions associated with the energy functional. This space is equipped with the norm

$$\|f\|_\Sigma^2 = \|\nabla f\|_{L^2}^2 + \|xf\|_{L^2}^2 = \|f\|_{\dot{H}^1}^2 + \|f\|_{L^2(|x|^2\,dx)}^2 \tag{1.4}$$

We will frequently employ the notation

$$H = -\tfrac{1}{2}\Delta + \tfrac{1}{2}|x|^2, \quad F(z) = \mu|z|^{\frac{4}{d-2}}z.$$

Definition. A (strong) *solution* to (1.1) is a function $u : I \times \mathbf{R}^d \to \mathbf{C}$ that belongs to $C^0_t(K;\Sigma)$ for every compact interval $K \subset I$, and that satisfies the Duhamel formula

$$u(t) = e^{-itH}u(0) - i\int_0^t e^{-i(t-s)H}F(u(s))\,ds \quad \text{for all} \quad t \in I. \tag{1.5}$$

The hypothesis on u implies that $F(u) \in C^0_{t,loc}L_x^{\frac{2d}{d+2}}(I \times \mathbf{R}^d)$. Consequently, the right side above is well-defined, at least as a weak integral of tempered distributions.

Equation (1.1) and its variants

$$i\partial_t u = (-\tfrac{1}{2}\Delta + V)u + F(u), \quad V = \pm\tfrac{1}{2}|x|^2, \quad F(u) = \pm|u|^p u, \quad p > 0$$

have received considerable attention, especially in the energy-subcritical regime $p < 4/(d-2)$. The equation with a confining potential $V = |x|^2/2$ has been used to model Bose-Einstein condensates in a trap (see [33], for example). Let us briefly review the mathematical literature.

Carles [4], [5] proved global wellposedness for a defocusing nonlinearity $F(u) = |u|^p u,\ p < 4/(d-2)$ when the potential $V(x) = |x|^2/2$ is either confining or repulsive, and obtained various wellposedness and blowup results for a focusing nonlinearity $F(u) = -|u|^p u$. In [6], he also studied the case of an anisotropic harmonic oscillator with $V(x) = \sum_j \delta_j x_j^2/2,\ \delta_j \in \{1, 0, -1\}$.

There has also been interest in more general potentials. The paper [25] proves long-time existence in the presence of a focusing, mass-subcritical nonlinearity $F(u) = -|u|^p u,\ p < 4/d$ when $V(x)$ is merely assumed to grow subquadratically (by which we mean $\partial^\alpha V \in L^\infty$ for all $|\alpha| \geq 2$). More recently, Carles [7] considered *time-dependent* subquadratic potentials $V(t,x)$. Taking initial data in Σ, he established global existence and uniqueness when $4/d \leq p < 4/(d-2)$ for the defocusing nonlinearity and $0 < p < 4/d$ in the focusing case.

This paper studies the energy-critical problem $p = 4/(d-2)$. While the critical equation still admits a local theory, the duration of local existence obtained by the usual fixed-point argument depends on the profile and not merely on the norm of the initial data u_0. Therefore, one cannot pass directly from local wellposedness to global wellposedness using conservation laws as in the subcritical case. This issue is most evident if we temporarily discard the potential and consider the equation

$$i\partial_t u = -\tfrac{1}{2}\Delta u + \mu|u|^{\frac{4}{d-2}}u, \quad u(0) = u_0 \in \dot{H}^1(\mathbf{R}^d), \quad d \geq 3, \tag{1.6}$$

which has the Hamiltonian

$$E_\Delta(u) = \int \tfrac{1}{2}|\nabla u|^2 + \mu\tfrac{d-2}{d}|u|^{\frac{2d}{d-2}}\,dx.$$

We shall refer to this equation in the sequel as the "potential-free", "translation-invariant", or "scale-invariant" problem. Since the spacetime scaling (1.3) preserves both the equation and the $\dot{H}^1$ norm of the initial data, the lifespan guaranteed by the local wellposedness theory cannot depend merely on $\|u_0\|_{\dot{H}^1}$. One cannot iterate the local existence argument to obtain global existence because with each iteration the solution could conceivably become more concentrated in space while remaining bounded in $\dot{H}^1$; the lifespans might therefore shrink to zero too quickly to cover all of $\mathbf{R}$. The scale invariance makes the analysis of (1.6) highly nontrivial.

We mention equation (1.6) because the original equation increasingly resembles (1.6) as the initial data concentrates at a point; see sections 4.2 and 5 for more precise statements concerning this limit. Hence, one would expect the essential difficulties in the energy-critical NLS to also manifest themselves in the energy-critical harmonic oscillator. Understanding the scale-invariant problem is therefore an important step toward understanding the harmonic oscillator. The last fifteen years have witnessed intensive study of the former, and the following conjecture has been verified in all but a few cases:

Conjecture 1.1. *When $\mu = 1$, solutions to* (1.6) *exist globally and scatter. That is, for any $u_0 \in \dot{H}^1(\mathbf{R}^d)$, there exists a unique global solution $u : \mathbf{R} \times \mathbf{R}^d \to \mathbf{C}$ to* (1.6) *with $u(0) = u_0$, and this solution satisfies a spacetime bound*

$$S_{\mathbf{R}}(u) := \int_{\mathbf{R}}\int_{\mathbf{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}}\,dx\,dt \leq C(E_\Delta(u_0)) < \infty. \tag{1.7}$$

Moreover, there exist functions $u_\pm \in \dot{H}^1(\mathbf{R}^d)$ such that

$$\lim_{t \to \pm\infty} \|u(t) - e^{\pm\frac{it\Delta}{2}}u_\pm\|_{\dot{H}^1} = 0,$$

and the correspondences $u_0 \mapsto u_\pm(u_0)$ *are homeomorphisms of* $\dot{H}^1$.

When $\mu = -1$, *one also has global wellposedness and scattering provided that*

$$E_\Delta(u_0) < E_\Delta(W), \quad \|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2},$$

where the ground state

$$W(x) = \big(1 + \frac{2|x|^2}{d(d-2)}\big)^{-\frac{d-2}{2}} \in \dot{H}^1(\mathbf{R}^d)$$

solves the elliptic equation $\frac{1}{2}\Delta + |W|^{\frac{4}{d-2}} W = 0$.

Theorem 1.1. *Conjecture 1.1 holds for the defocusing equation. For the focusing equation, the conjecture holds for radial initial data when* $d \geq 3$, *and for all initial data when* $d \geq 5$.

Proof. See [2, 9, 26, 31] for the defocusing case and [16, 21] for the focusing case. □

One can formulate a similar conjecture for (1.1); however, as the linear propagator is periodic in time, one only expects uniform local-in-time spacetime bounds.

Conjecture 1.2. *When* $\mu = 1$, *equation* (1.1) *is globally wellposed. That is, for each* $u_0 \in \Sigma$ *there is a unique global solution* $u : \mathbf{R} \times \mathbf{R}^d \to \mathbf{C}$ *with* $u(0) = u_0$. *This solution obeys the spacetime bound*

$$S_I(u) := \int_I \int_{\mathbf{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}} \, dx \, dt \leq C(|I|, \|u_0\|_\Sigma) \tag{1.8}$$

for any compact interval $I \subset \mathbf{R}$.

If $\mu = -1$, *then the same is true provided also that*

$$E(u_0) < E_\Delta(W) \quad \text{and} \quad \|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}.$$

In [24], Killip-Visan-Zhang verified this conjecture for $\mu = 1$ and spherically symmetric initial data. By adapting an argument of Bourgain-Tao for the equation without potential (1.6), they proved that the defocusing problem (1.1) is globally wellposed, and also obtained scattering for the repulsive potential. We consider only the confining potential. In this paper, we remove the assumption of spherical symmetry for the defocusing harmonic oscillator. In addition, we establish global wellposedness for the focusing problem under the assumption that Conjecture 1.1 holds for all dimensions.

Theorem 1.2. *Assume that Conjecture 1.1 holds. Then Conjecture 1.2 holds.*

By Theorem 1.1, this result is conditional only in the focusing situation for nonradial data in dimensions 3 and 4. Moreover, in the focusing case we have essentially the same blowup result as for the potential-free NLS with the same proof as in that case; see [21]. We recall the argument in Section 7.

Theorem 1.3 (Blowup)**.** *Suppose* $\mu = -1$ *and* $d \geq 3$. *If* $u_0 \in \Sigma$ *satisfies* $E(u_0) < E_\Delta(W)$ *and* $\|\nabla u_0\|_2 > \|\nabla W\|_2$, *then the solution to* (1.1) *blows up in finite time.*

Remark. By Lemma 7.1, $E(u_0) < E_\Delta(W)$ implies that either $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$ or $\|\nabla u\|_{L^2} > \|\nabla W\|_{L^2}$.

Mathematically, the energy-critical NLS with quadratic potential has several interesting properties. On one hand, it is a nontrivial variant of the potential-free equation. If the quadratic potential is replaced by a weaker potential, the proof of global wellposedness can sometimes ride on the coat tails of Theorem 1.1. For example, we show in Section 8 that for smooth, bounded potentials with bounded derivative, one obtains global wellposedness by treating the potential as a perturbation to (1.6). Further, the Avron-Herbst formula given in [7] reduces the problem with a linear potential $V(x) = Ex$ to (1.6). On the other hand, the linear propagator e^{-itH} for the harmonic oscillator does admit an explicit formula. In view of the preceding remarks, we believe that (1.1) is the most accessible generalization of (1.6) which does not come for free.

Proof outline. The local theory for (1.1) shows that global existence is equivalent to the uniform *a priori* spacetime bound (1.8). To prove this bound for all solutions, we apply the general strategy of induction on energy pioneered by Bourgain [2] and refined over the years by Colliander-Keel-Staffilani-Takaoka-Tao [9], Keraani [18], Kenig-Merle [16], and others. These arguments proceed roughly as follows.

(1) Show that the failure of Theorem 1.2 would imply the existence of a minimal-energy counterexample.
(2) Show that the counterexample cannot actually exist.

By the local theory, uniform spacetime bounds hold for all solutions with sufficiently small energy $E(u)$. Assuming that Theorem 1.2 fails, we obtain a positive threshold $0 < E_c < \infty$ such that (1.8) holds whenever $E(u) < E_c$ and fails when $E(u) > E_c$.

As the spacetime estimates of interest are local-in-time, it suffices to prevent the blowup of spacetime norm on unit-length time intervals. This will be achieved by a Palais-Smale compactness theorem (Proposition 6.1), from which one deduces that failure of Theorem 1.2 would imply the existence of a solution u_c with $E(u_c) = E_c$, which blows up on a unit time interval, and which also has an impossibly strong compactness property (namely, its orbit $\{u_c(t)\}$ must be precompact in Σ). Put differently, we shall discover that the only scenario where blowup could possibly occur is when the solution is highly concentrated at a point and behaves like a solution to the potential-free equation (1.6); but that equation is already known to be wellposed.

This paradigm of recovering the potential-free NLS in certain limiting regimes has been applied to various other equations. See [19, 20, 14, 13, 12, 23] for adaptations to gKdV, Klein-Gordon, and NLS in various domains and manifolds. While the particulars are unique to each case, a common key step is to prove an appropriate compactness theorem in the style of Proposition 6.1. As in the previous work, our proof of that proposition uses three main ingredients.

The first prerequisite is a local wellposedness theory that gives local existence and uniqueness as well as stability of solutions with respect to perturbations of the initial data or the equation itself. In our case, local wellposedness will follow from familiar arguments employing the dispersive estimate satisfied by the linear propagator e^{-itH}, as well the fractional product and chain rules for the operators H^γ, $\gamma \geq 0$. We review the relevant results in Section 3.

We also need a linear profile decomposition for the Strichartz inequality

$$\|e^{-itH} f\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \lesssim \|H^{\frac{1}{2}} f\|_{L_x^2}. \tag{1.9}$$

Such a decomposition in the context of energy-critical Schrödinger equations was first proved by Keraani [17] in the translation-invariant setting for the free particle Hamiltonian $H = -\Delta$, and quantifies the manner in which a sequence of functions f_n with $\|H^{1/2} f_n\|_{L^2}$ bounded may fail to produce a subsequence of $e^{-itH} f_n$ converging in the spacetime norm. The defect of compactness arises in Keraani's case from a noncompact group of symmetries of the inequality (1.9), which includes spatial translations and scaling. In our setting, there are no obvious symmetries of (1.9); nonetheless, compactness can fail and in Section 4 we formulate a profile decomposition for (1.9) when H is the Hamiltonian of the harmonic oscillator.

Finally, we need to study (1.1) when the initial data is highly concentrated in space, corresponding to a single profile in the linear profile decomposition just discussed. In Section 5, we show that blowup cannot occur in this regime. The basic idea is that while the solution to (1.1) remains highly localized in space, it can be well-approximated up to a phase factor by the corresponding solution to the scale-invariant energy-critical NLS

$$(i\partial_t + \tfrac{1}{2}\Delta) u = \pm |u|^{\frac{4}{d-2}} u. \tag{1.10}$$

By the time this approximation breaks down, the solution to the original equation will have dispersed and can instead be approximated by a solution to the linear equation $(i\partial_t - H)u = 0$. We use as a black box the nontrivial fact (which is still a conjecture in a few cases) that solutions to (1.6) obey global spacetime bounds. By stability theory, the spacetime bounds for the approximations will be transferred to the solution for the original equation and will therefore preclude blowup.

While this paper considers the potential $V(x) = \frac{1}{2}|x|^2$, the argument can be adapted to a wider class of subquadratic potentials defined by the following hypotheses:

- $\partial^k V \in L^\infty$ for all $k \geq 2$.
- $V(x) \geq \delta |x|^2$ for some $\delta > 0$.

Fujiwara showed [10] that the linear propagator for such potentials has a nice oscillatory integral rerpresentation, which can be used as a substitute for the Mehler formula (2.2) for the harmonic oscillator. We focus on the harmonic oscillator because this concrete case already contains the main ideas.

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2. Preliminaries

2.1. **Notation and basic estimates.** We write $X \lesssim Y$ to mean $X \le CY$ for some constant C, and $X \sim Y$ if both $X \lesssim Y$ and $Y \lesssim X$. If $I \subset \mathbf{R}$ is an interval, the mixed Lebesgue norms on $I \times \mathbf{R}^d$ are defined by

$$\|f\|_{L_t^q L_x^r(I\times\mathbf{R}^d)} = \left(\int_I \left(\int_{\mathbf{R}^d} |f(t,x)|^r\,dx\right)^{\frac{q}{r}} dt\right)^{\frac{1}{q}} = \|f(t)\|_{L_t^q(I;L_x^r(\mathbf{R}^d))},$$

The operator $H = -\frac{1}{2}\Delta + \frac{1}{2}|x|^2$ is positive on $L^2(\mathbf{R}^d)$. Its associated heat kernel is given by Mehler's formula

$$e^{-tH}(x,y) = e^{\tilde{\gamma}(t)(x^2+y^2)} e^{\frac{\sinh(t)\Delta}{2}}(x,y), \tag{2.1}$$

where

$$\tilde{\gamma}(t) = \frac{1-\cosh t}{2\sinh t} = -\frac{t}{4} + O(t^3) \quad \text{as} \quad t \to 0.$$

By analytic continuation, the associated one-parameter unitary group has the integral kernel

$$e^{-itH}f(x) = \frac{1}{(2\pi i \sin t)^{\frac{d}{2}}} \int e^{\frac{i}{\sin t}\left(\frac{x^2+y^2}{2}\cos t - xy\right)} f(y)\,dy. \tag{2.2}$$

Comparing this to the well-known free propagator

$$e^{\frac{it\Delta}{2}}f(x) = \frac{1}{(2\pi i t)^{\frac{d}{2}}} \int e^{\frac{i|x-y|^2}{2t}} f(y)\,dy, \tag{2.3}$$

we obtain the relation

$$e^{-itH}f = e^{i\gamma(t)|x|^2} e^{\frac{i\sin(t)\Delta}{2}}(e^{i\gamma(t)|x|^2}f), \tag{2.4}$$

where

$$\gamma(t) = \frac{\cos t - 1}{2\sin t} = -\frac{t}{4} + O(t^3) \quad \text{as} \quad t \to 0.$$

Mehler's formula immediately implies the local-in-time dispersive estimate

$$\|e^{-itH}f\|_{L_x^\infty} \lesssim |\sin t|^{-\frac{d}{2}} \|f\|_{L^1}. \tag{2.5}$$

For $d \ge 3$, call a pair of exponents (q,r) *admissible* if $q \ge 2$ and $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$. Write

$$\|f\|_{S(I)} = \|f\|_{L_t^\infty L_x^2} + \|f\|_{L_t^2 L_x^{\frac{2d}{d-2}}}$$

with all norms taken over the spacetime slab $I \times \mathbf{R}^d$. By interpolation, this norm controls the $L_t^q L_x^r$ norm for all other admissible pairs. Let

$$\|F\|_{N(I)} = \inf\{\|F_1\|_{L_t^{q_1'} L_x^{r_1'}} + \|F_2\|_{L_t^{q_2'} L_x^{r_2'}} : (q_k, r_k) \text{ admissible},\ F = F_1 + F_2\},$$

where (q_k', r_k') is the Hölder dual to (q_k, r_k).

Lemma 2.1 (Strichartz estimates)**.** *Let I be a compact time interval containing t_0, and let $u : I \times \mathbf{R}^d \to \mathbf{C}$ be a solution to the inhomogeneous Schrödinger equation*

$$(i\partial_t - H)u = F.$$

Then there is a constant $C = C(|I|)$, depending only on the length of the interval, such that

$$\|u\|_{S(I)} \le C(\|u(t_0)\|_{L^2} + \|F\|_{N(I)}).$$

Proof. This follows from the dispersive estimate (2.5), the unitarity of e^{-itH} on L^2, and general considerations; see [15]. By partitioning time into unit intervals, we see that the constant C grows at worst like $|I|^{\frac{1}{2}}$ (which corresponds to the time exponent $q = 2$). □

We use the fractional powers H^γ of the operator H, defined via the Borel functional calculus, as a substitute for the usual derivative $(-\Delta)^\gamma$. The former has the advantage of commuting with the linear propagator e^{-itH}. Trivially

$$\|H^{\frac12} f\|_{L^2} \sim \|(-\Delta)^{\frac12} f\|_{L^2} + \||x|f\|_{L^2} \sim \|f\|_\Sigma.$$

Using complex interpolation, Killip, Visan, and Zhang extended this equivalence to other L^p norms and other powers of H.

Lemma 2.2 ([24, Lemma 2.7]). *For $0 \le \gamma \le 1$ and $1 < p < \infty$, one has*

$$\|H^\gamma f\|_{L^p(\mathbf{R}^d)} \sim \|(-\Delta)^\gamma f\|_{L^p(\mathbf{R}^d)} + \||x|^{2\gamma} f\|_{L^p(\mathbf{R}^d)}.$$

As a consequence, H^γ inherits many properties of $(-\Delta)^\gamma$, including Sobolev embedding:

Lemma 2.3 ([24, Lemma 2.8]). *Suppose $\gamma \in [0,1]$ and $1 < p < \frac{d}{2\gamma}$, and define p^* by $\frac{1}{p^*} = \frac1p - \frac{2\gamma}{d}$. Then*

$$\|f\|_{L^{p^*}(\mathbf{R}^d)} \lesssim \|H^\gamma f\|_{L^p(\mathbf{R}^d)}.$$

Similarly, the fractional chain and product rules carry over to the current setting:

Corollary 2.4 ([24, Proposition 2.10]). *Let $F(z) = |z|^{\frac{4}{d-2}} z$. For any $0 \le \gamma \le \frac12$ and $1 < p < \infty$,*

$$\|H^\gamma F(u)\|_{L^p(\mathbf{R}^d)} \lesssim \|F'(u)\|_{L^{p_0}(\mathbf{R}^d)} \|H^\gamma f\|_{L^{p_1}(\mathbf{R}^d)}$$

for all $p_0, p_1 \in (1,\infty)$ with $p^{-1} = p_0^{-1} + p_1^{-1}$.

Using Lemma 2.2 and the Christ-Weinstein fractional product rule for $(-\Delta)^\gamma$ (e.g. [30]), we obtain

Corollary 2.5. *For $\gamma \in (0,1]$, $r, p_i, q_i \in (1,\infty)$ with $r^{-1} = p_i^{-1} + q_i^{-1}$, $i = 1,2$, we have*

$$\|H^\gamma(fg)\|_r \lesssim \|H^\gamma f\|_{p_1} \|g\|_{q_1} + \|f\|_{p_2} \|H^\gamma g\|_{q_2}.$$

The exponent $\gamma = \frac12$ is particularly relevant to us, and it will be convenient to use the notation

$$\|f\|_{L_t^q \Sigma_x^r (I \times \mathbf{R}^d)} = \|H^{\frac12} f\|_{L_t^q L_x^r (I\times\mathbf{R}^d)}.$$

The superscript of Σ is assumed to be 2 if omitted. We shall need the following refinement of Fatou's Lemma due to Brézis and Lieb:

Lemma 2.6 (Refined Fatou [3]). *Fix $1 \le p < \infty$, and suppose f_n is a sequence of functions in $L^p(\mathbf{R}^d)$ such that $\sup_n \|f_n\|_p < \infty$ and $f_n \to f$ pointwise. Then*

$$\lim_{n\to\infty} \int_{\mathbf{R}^d} \big| |f_n|^p - |f_n - f|^p - |f|^p \big| \, dx = 0.$$

Finally, we record a Mikhlin-type spectral multiplier theorem.

Theorem 2.7 (Hebisch [11]). *If $F : \mathbf{R} \to \mathbf{C}$ is a bounded function which obeys the derivative estimates*

$$|\partial^k F(\lambda)| \lesssim_k |\lambda|^{-k} \quad \text{for all} \quad 0 \le k \le \tfrac{d}{2} + 1,$$

then the operator $F(H)$, defined initially on L^2 via the Borel functional calculus, is bounded on L^p for all $1 < p < \infty$.

2.2. **Littlewood-Paley theory.** Using Theorem 2.7 as a substitute for the Mikhlin multiplier theorem, we obtain a Littlewood-Paley theory adapted to H by mimicking the classical development for Fourier multipliers. We define Littlewood-Paley projections using both compactly supported bump functions and also the heat kernel of H. The parabolic maximum principle implies that

$$0 \le e^{-tH}(x,y) \le e^{\frac{t\Delta}{2}}(x,y) = \tfrac{1}{(2\pi t)^{d/2}} e^{-\frac{|x-y|^2}{2t}}. \tag{2.6}$$

Fix a smooth function φ supported in $|\lambda| \le 2$ with $\varphi(\lambda) = 1$ for $|\lambda| \le 1$, and let $\psi(\lambda) = \varphi(\lambda) - \varphi(2\lambda)$. For each dyadic number $N \in 2^{\mathbf{Z}}$, which we will often refer to as "frequency," define

$$P^H_{\le N} = \varphi(\sqrt{H/N^2}), \quad P^H_N = \psi(\sqrt{H/N^2}),$$
$$\tilde P^H_{\le N} = e^{-H/N^2}, \qquad \tilde P^H_N = e^{-H/N^2} - e^{-4H/N^2}.$$

The associated operators $P^H_{<N}, P^H_{>N}$, etc. are defined in the usual manner.

Remark. As the spectrum of H is bounded away from 0, by choosing φ appropriately we can arrange for $P_{<1} = 0$; thus we will only consider frequencies $N \geq 1$.

Later on we shall need the classical Littlewood-Paley projectors

$$P^{\Delta}_{\leq N} = \varphi(\sqrt{-\Delta/N^2}) \quad P^{\Delta}_N = \psi(\sqrt{-\Delta/N^2}), \tag{2.7}$$

$$\tilde{P}^{\Delta}_{\leq N} = e^{\Delta/2N^2} \qquad \tilde{P}^{\Delta}_N = e^{\Delta/2N^2} - e^{2\Delta/N^2}. \tag{2.8}$$

The maximum principle implies the pointwise bound

$$|\tilde{P}^H_N f(x)| + |\tilde{P}^H_{\leq N} f(x)| \lesssim \tilde{P}^{\Delta}_{\leq N}|f|(x) + \tilde{P}^{\Delta}_{\leq N/2}|f|(x). \tag{2.9}$$

To reduce clutter we usually suppress the superscripts H and Δ unless both types of projectors arise in the same context. For the rest of this section, $P_{\leq N}$ and P_N denote $P^H_{\leq N}$ and P^H_N, respectively.

Lemma 2.8 (Bernstein estimates). *For $f \in C^\infty_c(\mathbf{R}^d)$, $1 < p \leq q < \infty$, $s \geq 0$, one has the Bernstein inequalities*

$$\|P_{\leq N} f\|_p \lesssim \|\tilde{P}_{\leq N} f\|_p, \quad \|P_N f\|_p \lesssim \|\tilde{P}_N f\|_p \tag{2.10}$$

$$\|P_{\leq N} f\|_p + \|P_N f\|_p + \|\tilde{P}_{\leq N} f\|_p + \|\tilde{P}_N f\|_p \lesssim \|f\|_p \tag{2.11}$$

$$\|P_{\leq N} f\|_q + \|P_N f\|_q + \|\tilde{P}_{\leq N} f\|_q + \|\tilde{P}_N f\|_q \lesssim N^{\frac{d}{p} - \frac{d}{q}} \|f\|_p \tag{2.12}$$

$$N^{2s}\|P_N f\|_p \sim \|H^s P_N f\|_p \tag{2.13}$$

$$\|P_{>N} f\|_p \lesssim N^{-2s}\|H^s P_{>N} f\|_p. \tag{2.14}$$

In (2.12)*, the estimates for $\tilde{P}_{\leq N} f$ and $\tilde{P}_N f$ also hold when $p = 1$, $q = \infty$. Further,*

$$f = \sum_N P_N f = \sum_N \tilde{P}_N f \tag{2.15}$$

where the series converge in L^p, $1 < p < \infty$. Finally, we have the square function estimate

$$\|f\|_p \sim \|(\sum_N |P_N f|^2)^{1/2}\|_p. \tag{2.16}$$

Proof. The estimates (2.10) follow immediately from Theorem 2.7. To see (2.11), observe that the functions $\varphi(\sqrt{\cdot/N^2})$, $e^{-\cdot/N^2}$ satisfy the hypotheses of Theorem 2.7 uniformly in N. Next use (2.6) together with Young's convolution inequality to get

$$\|\tilde{P}_{\leq N} f\|_q + \|\tilde{P}_N f\|_q \lesssim N^{\frac{d}{q} - \frac{d}{p}} \|f\|_p \quad \text{for } 1 \leq p \leq q \leq \infty. \tag{2.17}$$

From (2.10) we obtain the rest of (2.12). Now consider (2.13). Let $\tilde{\psi}$ be a fattened version of ψ so that $\tilde{\psi} = 1$ on the support of ψ. Put $F(\lambda) = \lambda^s \tilde{\psi}(\sqrt{\lambda})$. By Theorem 2.7, the relation $\psi = \tilde{\psi}\psi$, and the functional calculus,

$$\|N^{-2s} H^s P_N f\|_p = \|F(H/N^2) P_N f\|_p \lesssim \|P_N f\|_p.$$

The reverse inequality follows by considering $F(x) = \lambda^{-s}\tilde{\psi}(\lambda)$.

We turn to (2.15). The equality holds in L^2 by the functional calculus and the fact that the spectrum of H is bounded away from 0. For $p \neq 2$, choose q and $0 < \theta < 1$ so that $p^{-1} = 2^{-1}(1-\theta) + q^{-1}\theta$. By (2.11), the partial sum operators

$$S_{N_0, N_1} = \sum_{N_0 < N \leq N_1} P_N, \qquad \tilde{S}_{N_0, N_1} = \sum_{N_0 < N \leq N_1} \tilde{P}_N$$

are bounded on every L^p, $1 < p < \infty$, uniformly in N_0, N_1. Thus by Hölder's inequality,

$$\|f - S_{N_0,N_1} f\|_p \leq \|f - S_{N_0,N_1} f\|_2^{1-\theta} \|f - S_{N_0,N_1} f\|_q^{\theta} \to 0 \text{ as } N_0 \to 0, \ N_1 \to \infty,$$

and similarly for the partial sums $\tilde{S}_{N_0,N_1} f$. The estimate (2.14) follows from (2.11), (2.13), and the decomposition $P_{>N} f = \sum_{M>N} P_M f$.

To prove the square function estimate, run the usual Khintchine's inequality argument using Theorem 2.7 in place of the Mikhlin multiplier theorem. □

2.3. **Local smoothing.** The following local smoothing lemma and its corollary will be needed when proving properties of the nonlinear profile decomposition in Section 6.

Lemma 2.9. *If $u = e^{-itH}\phi,\ \phi \in \Sigma(\mathbf{R}^d)$, then*

$$\int_I \int_{\mathbf{R}^d} |\nabla u(x)|^2 \langle R^{-1}(x-z)\rangle^{-3}\,dx\,dt \lesssim R(1+|I|)\|u\|_{L_t^\infty L_x^2} \|H^{1/2}u\|_{L_t^\infty L_x^2}.$$

with the constant independent of $z \in \mathbf{R}^d$ and $R > 0$.

Proof. We recall the Morawetz identity. Let a be a sufficiently smooth function of x; then for any u satisfying the linear equation $i\partial_t u = (-\frac{1}{2}\Delta + V)u$, one has

$$\begin{aligned}\partial_t \int \nabla a \cdot \operatorname{Im}(\overline{u}\nabla u)\,dx &= \int a_{jk}\operatorname{Re}(u_j \overline{u}_k)\,dx - \tfrac{1}{4}\int |u|^2 a_{jjkk}\,dx \\ &\quad - \tfrac{1}{2}\int |u|^2 \nabla a\cdot\nabla V\,dx\end{aligned} \tag{2.18}$$

We use this identity with $a(x) = \langle R^{-1}(x-z)\rangle$ and $V = \frac{1}{2}|x|^2$, and compute

$$a_j(x) = \frac{R^{-2}(x_j - z_j)}{\langle R^{-1}(x-z)\rangle},\ a_{jk}(x) = R^{-2}\left[\frac{\delta_{jk}}{\langle R^{-1}(x-z)\rangle} - \frac{R^{-2}(x_j-z_j)(x_k-z_k)}{\langle R^{-1}(x-z)\rangle^3}\right]$$

$$\Delta^2 a(x) \le -\frac{15R^{-4}}{\langle R^{-1}(x-z)\rangle^7}.$$

As $\Delta^2 a \le 0$, the right side of (2.18) is bounded below by

$$\begin{aligned}&R^{-2}\int \langle R^{-1}(x-z)\rangle^{-1}\left[|\nabla u|^2 - |\tfrac{R^{-1}(x-z)}{\langle R^{-1}(x-z)\rangle}\cdot\nabla u|^2\right]dx - \tfrac{1}{2R}\int |u|^2 \tfrac{R^{-1}(x-z)}{\langle R^{-1}(x-z)\rangle}\cdot x\,dx \\ &\ge R^{-2}\int |\nabla u(x)|^2 \langle R^{-1}(x-z)\rangle^{-3}\,dx - \tfrac{R^{-1}}{2}\int |u|^2|x|\,dx.\end{aligned}$$

Integrating in time and applying Cauchy-Schwarz, we get

$$\begin{aligned}&R^{-2}\int_I\int_{\mathbf{R}^d} \langle R^{-1}(x-z)\rangle^{-3}|\nabla u(t,x)|^2\,dxdt \\ &\qquad \lesssim \sup_{t\in I} R^{-1}\int \tfrac{R^{-1}(x-z)}{\langle R^{-1}(x-z)\rangle}|u(t,x)||\nabla u(t,x)|\,dx + \tfrac{1}{2R}\int_I\int_{\mathbf{R}^d}|x||u|^2\,dxdt \\ &\qquad \lesssim R^{-1}(1+|I|)\|u\|_{L_t^\infty L_x^2}\|H^{1/2}u\|_{L_t^\infty L_x^2}.\end{aligned}$$

This completes the proof of the lemma. □

Corollary 2.10. *Fix $\phi \in \Sigma(\mathbf{R}^d)$. Then for all $T, R \le 1$, we have*

$$\|\nabla e^{-itH}\phi\|_{L^2_{t,x}(|t-t_0|\le T,\ |x-x_0|\le R)} \lesssim T^{\frac{2}{3(d+2)}} R^{\frac{3d+2}{3(d+2)}} \|\phi\|_\Sigma^{\frac{2}{3}} \|e^{-itH}\phi\|^{\frac{1}{3}}_{L_{t,x}^{\frac{2(d+2)}{d-2}}}.$$

When $d = 3$, we also have

$$\|\nabla e^{-itH}\phi\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}}(|t-t_0|\le T,\ |x-x_0|\le R)} \lesssim T^{\frac{23}{180}} R^{\frac{11}{45}} \|e^{-itH}\phi\|^{\frac{5}{48}}_{L^{10}_{t,x}} \|\phi\|_\Sigma^{\frac{43}{48}}$$

Proof. The proofs are fairly standard (see [32] or [23]); we present the details for the second claim, which is slightly more involved. Let E the region $\{|t-t_0| \le T,\ |x-x_0|\le R\}$. Norms which do not specify the region of integration are taken over the spacetime slab $\{|t-t_0|\le T\}\times\mathbf{R}^3$. By Hölder,

$$\|\nabla e^{-itH}\phi\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}}(E)} \le \|\nabla e^{-itH}\phi\|^{\frac{1}{3}}_{L^2_{t,x}(E)} \|\nabla e^{-itH}\phi\|^{\frac{2}{3}}_{L_t^5 L_x^{\frac{20}{9}}(E)}.$$

By Hölder and Strichartz,

$$\|\nabla e^{-itH}\phi\|_{L_t^5 L_x^{\frac{20}{9}}(E)} \lesssim T^{\frac{1}{8}} \|\nabla e^{-itH}\phi\|_{L_t^{\frac{40}{3}} L_x^{\frac{20}{9}}} \lesssim T^{\frac{1}{8}}\|\phi\|_\Sigma. \tag{2.19}$$

We now estimate $\|\nabla e^{-itH}\phi\|_{L^2_{t,x}}$. Let $N \in 2^{\mathbf{N}}$ be a dyadic number to be chosen later, and decompose

$$\|\nabla e^{-itH}\phi\|_{L^2_{t,x}(E)} \le \|\nabla e^{-itH}P^H_{\le N}\phi\|_{L^2_{t,x}(E)} + \|\nabla e^{-itH}P^H_{>N}\phi\|_{L^2_{t,x}(E)}.$$

For the low frequency piece, apply Hölder and the Bernstein inequalities to obtain

$$\|\nabla e^{-itH} P^H_{\le N}\phi\|_{L^2_{t,x}} \lesssim T^{\frac{2}{5}} R^{\frac{6}{5}} \|\nabla e^{-itH} P^H_{\le N}\phi\|_{L^{10}_{t,x}} \lesssim T^{\frac{2}{5}} R^{\frac{6}{5}} N \|e^{-itH}\phi\|_{L^{10}_{t,x}}.$$

For the high-frequency piece, apply local smoothing and Bernstein:

$$\|\nabla e^{-itH} P^H_{>N}\phi\|_{L^2_{t,x}} \lesssim R^{\frac{1}{2}} \|P^H_{>N}\phi\|^{\frac{1}{2}}_{L^2} \|H^{\frac{1}{2}}\phi\|^{\frac{1}{2}}_{\Sigma} \lesssim R^{\frac{1}{2}} N^{-\frac{1}{2}} \|\phi\|_{\Sigma}.$$

Optimizing in N, we obtain

$$\|\nabla e^{-itH}\phi\|_{L^2_{t,x}} \lesssim T^{\frac{2}{15}} R^{\frac{11}{15}} \|e^{-itH}\phi\|^{\frac{1}{3}}_{L^{10}_{t,x}} \|\phi\|^{\frac{2}{3}}_{\Sigma}.$$

Combining this estimate with (2.19) yields the conclusion of the corollary. □

3. Local theory

We record some standard results concerning local-wellposedness for (1.1). These are direct analogues of the theory for the scale-invariant equation. By Lemma 2.3 and Corollaries 2.4 and 2.5, we can use essentially the same proofs as in that case. The reader should consult [22] for those proofs.

Proposition 3.1 (Local wellposedness)**.** *Let $u_0 \in \Sigma(\mathbf{R}^d)$ and fix a compact time interval $0 \in I \subset \mathbf{R}$. Then there exists a constant $\eta_0 = \eta_0(d, |I|)$ such that whenever $\eta < \eta_0$ and*

$$\|H^{\frac{1}{2}} e^{-itH} u_0\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}} (I \times \mathbf{R}^d)} \le \eta,$$

there exists a unique solution $u : I \times \mathbf{R}^d \to \mathbf{C}$ to (1.1) which satisfies the bounds

$$\|H^{\frac{1}{2}} u\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}} (I \times \mathbf{R}^d)} \le 2\eta \quad \text{and} \quad \|H^{\frac{1}{2}} u\|_{S(I)} \lesssim \|u_0\|_{\Sigma} + \eta^{\frac{d+2}{d-2}}.$$

Corollary 3.2 (Blowup criterion)**.** *Suppose $u : (T_{min}, T_{max}) \times \mathbf{R}^d \to \mathbf{C}$ is a maximal lifespan solution to (1.1), and fix $T_{min} < t_0 < T_{max}$. If $T_{max} < \infty$, then*

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}} ([t_0, T_{max}))} = \infty.$$

If $T_{min} > -\infty$, then

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}} ((T_{min}, t_0])} = \infty.$$

Proposition 3.3 (Stability)**.** *Fix $t_0 \in I \subset \mathbf{R}$ an interval of unit length and let $\tilde{u} : I \times \mathbf{R}^d \to \mathbf{C}$ be an approximate solution to (1.1) in the sense that*

$$i\partial_t \tilde{u} = H\tilde{u} \pm |\tilde{u}|^{\frac{4}{d-2}} \tilde{u} + e$$

for some function e. Assume that

$$\|\tilde{u}\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \le L, \quad \|H^{\frac{1}{2}} \tilde{u}\|_{L^\infty_t L^2_x} \le E, \tag{3.1}$$

and that for some $0 < \varepsilon < \varepsilon_0(E, L)$ one has

$$\|\tilde{u}(t_0) - u_0\|_{\Sigma} + \|H^{\frac{1}{2}} e\|_{N(I)} \le \varepsilon, \tag{3.2}$$

Then there exists a unique solution $u : I \times \mathbf{R}^d \to \mathbf{C}$ to (1.1) with $u(t_0) = u_0$ and which further satisfies the estimates

$$\|\tilde{u} - u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} + \|H^{\frac{1}{2}}(\tilde{u} - u)\|_{S(I)} \le C(E, L)\varepsilon^c \tag{3.3}$$

where $0 < c = c(d) < 1$ and $C(E, L)$ is a function which is nondecreasing in each variable.

4. Concentration compactness

The purpose of this section is to prove a linear profile decomposition for the Strichartz inequality

$$\|e^{-itH}f\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbf{R}^d)} \le C(|I|,d)\|f\|_\Sigma.$$

Our decomposition resembles that of Keraani [17] in the sense that each profile has a characteristic length scale and location in spacetime. But since the space Σ lacks both translation and scaling symmetry, the precise definitions of our profiles are more complicated.

Keraani considered the analogous Strichartz estimate

$$\|e^{it\Delta}f\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(\mathbf{R}\times\mathbf{R}^d)} \lesssim \|f\|_{\dot{H}^1(\mathbf{R}^d)}.$$

Recall that in that situation, if f_n is a bounded sequence in $\dot{H}^1$ with nontrivial linear evolution, then one has a decomposition $f_n = \phi_n + r_n$ where $\phi_n = e^{it_n\Delta}G_n\phi$, G_n are certain unitary scaling and translation operators on $\dot{H}^1$ (defined as in (4.1)), and ϕ is a weak limit of $G_n^{-1}e^{-it_n\Delta}f_n$ in $\dot{H}^1$. The "bubble" ϕ_n is nontrivial and decouples from the remainder r_n in various norms. By applying this decomposition inductively to the remainder r_n, one obtains the full collection of profiles constituting f_n.

We follow the general presentation in [22, 32]. Let $f_n \in \Sigma$ be a bounded sequence. Using a variant of Keraani's argument, we seek an $\dot{H}^1$-weak limit ϕ in terms of f_n and write $f_n = \phi_n + r_n$ where ϕ_n is defined analogously as before by "moving the operators onto f_n." However, two main issues arise.

The first is that while f_n belong to Σ, an $\dot{H}^1$ weak limit of a sequence like $G_n^{-1}e^{it_nH}f_n$ need only belong to $\dot{H}^1$. Indeed, the $\dot{H}^1$ isometries G_n^{-1} will in general have unbounded norm as operators on Σ because of the $|x|^2$ weight. To define ϕ_n, we need to introduce spatial cutoffs to obtain functions in Σ.

Secondly, to establish the various orthogonality assertions one must understand how the linear propagator e^{-itH} interacts with the $\dot{H}^1$ symmetries of translation and scaling in certain limits. This interaction is studied in Section 4.2. In particular, the convergence results obtained there serve as a substitute for the scaling relation

$$e^{it\Delta}G_n = G_n e^{iN_n^2 t\Delta} \quad \text{where} \quad G_n\phi = N_n^{\frac{d-2}{2}}\phi(N_n(\cdot - x_n)).$$

They can also be regarded as a precise form of the heuristic that as the initial data concentrates at a point x_0, the potential $V(x) = |x|^2/2$ can be regarded over short time intervals as essentially equal to the constant potential $V(x_0)$; hence for short times the linear propagator e^{-itH} can be approximated up to a phase factor by the free particle propagator. Section 5 addresses a nonlinear version of this statement.

4.1. An inverse Strichartz inequality.

Unless indicated otherwise, $0 \in I$ in this section will denote a fixed interval of length at most 1, and all spacetime norms will be taken over $I \times \mathbf{R}^d$.

Suppose f_n is a sequence of functions in Σ with nontrivial linear evolution $e^{-itH}f_n$. The following refined Strichartz estimate shows that there must be a "frequency" N_n which makes a nontrivial contribution to the evolution.

Proposition 4.1 (Refined Strichartz)**.**

$$\|e^{-itH}f\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \lesssim \|f\|_\Sigma^{\frac{4}{d+2}} \sup_N \|e^{-itH}P_Nf\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{d-2}{d+2}}$$

Proof. Using the Littlewood-Paley theory, we may quote essentially verbatim the proof of refined Strichartz for the free particle propagator ([32] Lemma 3.1). Write f_N for P_Nf, where $P_N = P_N^H$ unless indicated otherwise. When $d \ge 6$, apply the square function estimate (2.16), Hölder, Bernstein, and Strichartz to get

$$\begin{aligned}
&\|e^{-itH}f\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{2(d+2)}{d-2}} \sim \Big\|(\sum_N |e^{-itH}f_N|^2)^{1/2}\Big\|_{\frac{2(d+2)}{d-2}}^{\frac{2(d+2)}{d-2}} = \iint (\sum_N |e^{-itH}f_N|^2)^{\frac{d+2}{d-2}}\,dx\,dt \\
&\lesssim \sum_{M\le N} \iint |e^{-itH}f_M|^{\frac{d+2}{d-2}} |e^{-itH}f_N|^{\frac{d+2}{d-2}}\,dx\,dt \\
&\lesssim \sum_{M\le N} \|e^{-itH}f_M\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \|e^{-itH}f_M\|_{L_{t,x}^{\frac{2(d+2)}{d-4}}} \|e^{-itH}f_N\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \|e^{-itH}f_N\|_{L_{t,x}^{\frac{2(d+2)}{d}}}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \sup_N \|e^{-itH} f_N\|^{\frac{8}{d-2}}_{L^{\frac{2(d+2)}{d-2}}_{t,x}} \sum_{M \le N} M^2 \|e^{-itH} f_M\|_{L_t^{\frac{2(d+2)}{d-4}} L_x^{\frac{2d(d+2)}{d^2+8}}} \|f_N\|_{L^2} \\
&\lesssim \sup_N \|e^{-itH} f_N\|^{\frac{8}{d-2}}_{L^{\frac{2(d+2)}{d-2}}_{t,x}} \sum_{M \le N} M^2 \|f_M\|_{L^2_x} \|f_N\|_{L^2_x} \\
&\lesssim \sup_N \|e^{-itH} f_N\|^{\frac{8}{d-2}}_{L^{\frac{2(d+2)}{d-2}}_{t,x}} \sum_{M \le N} \frac{M}{N} \|H^{1/2} f_M\|_{L^2} \|H^{1/2} f_N\|_{L^2_x} \\
&\lesssim \sup_N \|e^{-itH} f_N\|^{\frac{8}{d-2}}_{L^{\frac{2(d+2)}{d-2}}_{t,x}} \|f\|^2_{\Sigma}.
\end{aligned}$$

The cases $d = 3, 4, 5$ are handled similarly with some minor modifications in the applications of Hölder's inequality. □

The next proposition goes one step further and asserts that the sequence $e^{-itH} f_n$ with nontrivial spacetime norm must in fact contain a bubble centered at some (t_n, x_n) with spatial scale N_n^{-1}. First we introduce some vocabulary and notation which are common to concentration compactness arguments.

Definition 4.1. A *frame* is a sequence $(t_n, x_n, N_n) \in I \times \mathbf{R}^d \times 2^{\mathbf{N}}$ conforming to one of the following scenarios:

(1) $N_n \equiv 1,\ t_n \equiv 0$, and $x_n \equiv 0$.
(2) $N_n \to \infty$ and $N_n^{-1}|x_n| \to r_\infty \in [0, \infty)$.

The parameters $t_n,\ x_n,\ N_n$ will specify the temporal center, spatial center, and inverse length scale of a function. The condition that $|x_n| \lesssim N_n$ reflects the fact that we only consider functions obeying some uniform bound in Σ, and such functions cannot be centered arbitrarily far from the origin. We need to augment the frame $\{(t_n, x_n, N_n)\}$ by an auxiliary parameter N_n', which corresponds to a sequence of spatial cutoffs adapted to the frame.

Definition 4.2. An *augmented frame* is a sequence $(t_n, x_n, N_n, N_n') \in I \times \mathbf{R}^d \times 2^{\mathbf{N}} \times \mathbf{R}$ belonging to one of the following types:

(1) $N_n \equiv 1,\ t_n \equiv 0,\ x_n \equiv 0,\ N_n' \equiv 1$.
(2) $N_n \to \infty,\ N_n^{-1}|x_n| \to r_\infty \in [0, \infty)$, and either
   (2a) $N_n' \equiv 1$ if $r_\infty > 0$, or
   (2b) $N_n^{1/2} \le N_n' \le N_n,\ N_n^{-1}|x_n|(\frac{N_n}{N_n'}) \to 0$, and $\frac{N_n}{N_n'} \to \infty$ if $r_\infty = 0$.

Associated to an augmented frame (t_n, x_n, N_n, N_n') is a family of scaling and translation operators

$$\begin{aligned}
(G_n \phi)(x) &= N_n^{\frac{d-2}{2}} \phi(N_n(x - x_n)) \\
(\tilde{G}_n f)(t, x) &= N_n^{\frac{d-2}{2}} f(N_n^2(t - t_n), N_n(x - x_n)),
\end{aligned} \tag{4.1}$$

as well as spatial cutoff operators

$$S_n \phi = \begin{cases} \phi, & \text{for frames of type 1} \quad (\text{i.e. } N_n \equiv 1), \\ \chi(\frac{N_n}{N_n'} \cdot)\phi, & \text{for frames of type 2} \quad (\text{i.e. } N_n \to \infty), \end{cases} \tag{4.2}$$

where χ is a smooth compactly supported function equal to 1 on the ball $\{|x| \le 1\}$. An easy computation yields the following mapping properties:

$$\begin{aligned}
&\lim_{n \to \infty} S_n = I \text{ strongly in } \dot{H}^1 \text{and in } \Sigma, \\
&\limsup_{n \to \infty} \|G_n\|_{\Sigma \to \Sigma} < \infty.
\end{aligned} \tag{4.3}$$

For future reference, we record a technical lemma that, as a special case, asserts that the Σ norm is controlled almost entirely by the $\dot{H}^1$ norm for functions concentrating near the origin.

Lemma 4.2 (Approximation)**.** *Let (q, r) be an admissible pair of exponents with $2 \le r < d$, and let $\mathcal{F} = \{(t_n, x_n, N_n, N_n')\}$ be an augmented frame of type 2.*

(1) *Suppose $\mathcal{F}$ is of type 2a in Definition 4.2. Then for $\{f_n\} \subseteq L_t^q H_x^{1,r}(\mathbf{R}\times\mathbf{R}^d)$, we have*
$$\limsup_n \|\tilde{G}_n S_n f_n\|_{L_t^q \Sigma_x^r} \lesssim \limsup_n \|f_n\|_{L_t^q H_x^{1,r}}.$$
(2) *Suppose $\mathcal{F}$ is of type 2b and $f_n \in L_t^q \dot{H}_x^{1,r}(\mathbf{R}\times\mathbf{R}^d)$. Then*
$$\limsup_n \|\tilde{G}_n S_n f_n\|_{L_t^q \Sigma_x^r} \lesssim \limsup_n \|f_n\|_{L_t^q \dot{H}_x^{1,r}}.$$

Here $H^{1,r}(\mathbf{R}^d)$ and $\dot{H}^{1,r}(\mathbf{R}^d)$ denote the Sobolev spaces equipped with the norms
$$\|f\|_{H^{1,r}} = \|\langle\nabla\rangle\|_{L^r(\mathbf{R}^d)}, \quad \|f\|_{\dot{H}^{1,r}} = \||\nabla|f\|_{L^r(\mathbf{R}^d)}.$$

Proof. By time translation invariance we may assume $t_n \equiv 0$. By Lemma 2.2, it suffices to separately bound $\|\nabla\tilde{G}_n S_n f_n\|_{L_t^q L_x^r}$ and $\||x|\tilde{G}_n S_n f_n\|_{L_t^q L_x^r}$. Using a change of variables, the admissibility of (q,r), Hölder, and Sobolev embedding (hence the restriction $r<d$), we have
$$\begin{aligned}
&\|\nabla\tilde{G}_n S_n f_n\|_{L_t^q L_x^r} = \|\nabla[N_n^{\frac{d-2}{2}} f_n(N_n^2 t, N_n(x-x_n))\chi(N_n'(x-x_n))]\|_{L_t^q L_x^r}\\
&\lesssim \|(\nabla f_n)(t,x)\|_{L_t^q L_x^r} + \tfrac{N_n'}{N_n}\|f_n(t,x)\|_{L_t^q L_x^r(\mathbf{R}\times\{|x|\sim\frac{N_n}{N_n'}\})}\\
&\lesssim \|\nabla f_n\|_{L_t^q \dot{H}_x^{1,r}}.
\end{aligned}$$
To estimate $\||x|\tilde{G}_n S_n f_n\|_{L_t^q L_x^r}$ we distinguish the two cases. Consider first the case where $f_n \in L_t^q H_x^{1,r}$. Using the bound $|x_n| \lesssim N_n$ and a change of variables, we obtain
$$\||x|\tilde{G}_n S_n f_n\|_{L_t^q L^r} \lesssim N_n^{\frac{d}{2}}\|f_n(N_n^2 t, N_n(x-x_n))\|_{L^r} \lesssim \|f_n\|_{L_t^q L^r} \lesssim \|f_n\|_{L_t^q H_x^{1,r}}.$$

Next, consider the case where f_n are merely assumed to lie in $L_t^q \dot{H}_x^{1,r}$. For each t, we apply Hölder and Sobolev embedding to get
$$\begin{aligned}
&\||x|\tilde{G}_n S_n f_n\|_{L_x^r}^r = N_n^{\frac{dr}{2}-d-r}\int_{|x|\lesssim\frac{N_n}{N_n'}} |x_n + N_n^{-1}x|^r |f_n(N_n^2 t, x)|^r\,dx\\
&\lesssim N_n^{\frac{dr}{2}-d}\left[N_n^{-r}|x_n|^r + N_n^{-2r}(\tfrac{N_n}{N_n'})^r\right]\int_{|x|\lesssim\frac{N_n}{N_n'}} |f_n(N_n^2 t, x)|^r\,dx\\
&\lesssim N_n^{\frac{dr}{2}-d}\left[N_n^{-r}|x_n|^r(\tfrac{N_n}{N_n'})^r + (N_n')^{-2r}\right]\|\nabla f_n(N_n^2 t)\|_{L_x^r}^r.
\end{aligned}$$
By the hypotheses on the parameter N_n' in Definition 4.2, the expression inside the brackets goes to 0 as $n\to\infty$. After integrating in t and changing variables, we conclude
$$\||x|\tilde{G}_n S_n f_n\|_{L_t^q L_x^r} \lesssim c_n \|f_n\|_{L_t^q \dot{H}_x^{1,r}}$$
where $c_n = o(1)$ as $n\to\infty$. This completes the proof of the lemma. □

Proposition 4.3 (Inverse Strichartz). *Let I be a compact interval containing 0 of length at most 1, and suppose f_n is a sequence of functions in $\Sigma(\mathbf{R}^d)$ satisfying*
$$0 < \varepsilon \le \|e^{-itH}f_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbf{R}^d)} \lesssim \|f_n\|_\Sigma \le A < \infty.$$
Then, after passing to a subsequence, there exists an augmented frame
$$\mathcal{F} = \{(t_n, x_n, N_n, N_n')\}$$
and a sequence of functions $\phi_n \in \Sigma$ such that one of the following holds:

(1) *$\mathcal{F}$ is of type 1 (i.e. $N_n \equiv 1$) and $\phi_n = \phi$ where $\phi\in\Sigma$ is a weak limit of f_n in Σ.*
(2) *$\mathcal{F}$ is of type 2, either $t_n \equiv 0$ or $N_n^2 t_n \to \pm\infty$, and $\phi_n = e^{it_nH}G_n S_n \phi$ where $\phi \in \dot{H}^1(\mathbf{R}^d)$ is a weak limit of $G_n^{-1}e^{-it_nH}f_n$ in $\dot{H}^1$. Moreover, if $\mathcal{F}$ is of type 2a, then ϕ also belongs to $L^2(\mathbf{R}^d)$.*

The functions ϕ_n have the following properties:
$$\liminf_n \|\phi_n\|_\Sigma \gtrsim A\left(\tfrac{\varepsilon}{A}\right)^{\frac{d(d+2)}{8}}, \tag{4.4}$$
$$\lim_{n\to\infty} \|f_n\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} - \|f_n-\phi_n\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} - \|\phi_n\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} = 0, \tag{4.5}$$

$$\lim_{n\to\infty} \|f_n\|_\Sigma^2 - \|f_n - \phi_n\|_\Sigma^2 - \|\phi_n\|_\Sigma^2 = 0. \tag{4.6}$$

Proof. Our plan is as follows. First we identify the parameters t_n, x_n, N_n, which define the location of the bubble ϕ_n and its characteristic size, and dispose of the case where $N_n \equiv 1$.

The case where $N_n \to \infty$ is more involved. First we define the profile ϕ_n and verify the assertions (4.4) and (4.6). Passing to a subsequence, we may assume that the sequence $N_n^2 t_n$ converges in $[-\infty, \infty]$. If the limit is infinite, decoupling (4.5) in the $L^{\frac{2d}{d-2}}$ norm will also follow. If instead $N_n^2 t_n$ has a finite limit, we show that in fact the time parameter t_n can actually be redefined to be identically zero after making a negligible correction to the profile ϕ_n, and verify that the modified profile (with $t_n = 0$ now) satisfies property (4.5) in addition to (4.4) and (4.6). We shall see along the way that in this regime of short time scales and initial data concentrated near the origin, the potential may be essentially regarded as constant.

By Proposition 4.1, there exist frequencies N_n such that

$$\|P_{N_n} e^{-itH} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \gtrsim \varepsilon^{\frac{d+2}{4}} A^{-\frac{d-2}{4}}.$$

The comparison of Littlewood-Paley projectors (2.10) implies

$$\|\tilde{P}_{N_n} e^{-itH} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \gtrsim \varepsilon^{\frac{d+2}{4}} A^{-\frac{d-2}{4}},$$

where $\tilde{P}_N = e^{-H/N^2} - e^{-4H/N^2}$ denote the projections based on the heat kernel. By Hölder, Strichartz, and Bernstein,

$$\begin{aligned}
\varepsilon^{\frac{d+2}{4}} A^{-\frac{d-2}{4}} &\lesssim \|\tilde{P}_{N_n} e^{-itH} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \lesssim \|\tilde{P}_{N_n} e^{-itH} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{d-2}{d}} \|\tilde{P}_{N_n} e^{-itH} f_n\|_{L_{t,x}^\infty}^{\frac{2}{d}} \\
&\lesssim (N_n^{-1} A)^{\frac{d-2}{d}} \|\tilde{P}_{N_n} e^{-itH} f_n\|_{L_{t,x}^\infty}^{\frac{2}{d}}.
\end{aligned}$$

Therefore, there exist $(t_n, x_n) \in I \times \mathbf{R}^d$ such that

$$|e^{-it_n H} \tilde{P}_{N_n} f_n(x_n)| \gtrsim N_n^{\frac{d-2}{2}} A(\tfrac{\varepsilon}{A})^{\frac{d(d+2)}{8}}. \tag{4.7}$$

The parameters t_n, x_n, N_n will determine the center and width of a bubble.

We observe first that the boundedness of f_n in Σ limits how far the bubble can live from the spatial origin.

Lemma 4.4. *We have*

$$|x_n| \le C_{A,\varepsilon} N_n.$$

Proof. Put $g_n = |e^{-it_n H} f_n|$. By the kernel bound (2.9),

$$N_n^{\frac{d-2}{2}} A(\tfrac{\varepsilon}{A})^{\frac{d(d+2)}{8}} \lesssim |\tilde{P}_{N_n} e^{-it_n H} f_n(x_n)| \lesssim \tilde{P}^{\Delta}_{\le N_n} g_n(x_n) + \tilde{P}^{\Delta}_{\le N_n/2} g_n(x_n).$$

Thus one of the terms on the right side is at least half as large as the left side, and it suffices to consider the case when

$$\tilde{P}^{\Delta}_{\le N_n} g_n(x_n) \gtrsim N_n^{\frac{d-2}{2}} A(\tfrac{\varepsilon}{A})^{\frac{d(d+2)}{8}}$$

since the argument with N_n replaced by $N_n/2$ differs only cosmetically. Informally, $\tilde{P}^{\Delta}_{\le N_n} g_n$ is essentially constant over length scales of order N_n^{-1}, so if it is large at a point x_n then it is large on the ball $|x - x_n| \le N_n^{-1}$. More precisely, when $|x - x_n| \le N_n^{-1}$ we have

$$\begin{aligned}
\tilde{P}^{\Delta}_{\le N_n/2} g_n(x) &= \frac{N_n^d}{2^d (2\pi)^{\frac{d}{2}}} \int g_n(x-y) e^{-\frac{N_n^2 |y|^2}{8}} \, dy \\
&= \frac{N_n^d}{2^d (4\pi)^{\frac{d}{2}}} \int g_n(x_n - y) e^{-\frac{N_n^2 |y+x-x_n|^2}{8}} \, dy \\
&\ge e^{-1} \frac{N_n^d}{2^d (4\pi)^{\frac{d}{2}}} \int g_n(x_n - y) e^{-\frac{N_n^2 |y|^2}{2}} \, dy = e^{-1} 2^{-d} \tilde{P}^{\Delta}_{\le N_n} g_n(x_n) \\
&\gtrsim N_n^{\frac{d-2}{2}} A(\tfrac{\varepsilon}{A})^{\frac{d(d+2)}{8}}.
\end{aligned}$$

On the other hand, the mapping properties of the heat kernel imply that

$$\|\tilde{P}^{\Delta}_{\le N_n/2} g_n\|_\Sigma \lesssim (1 + N_n^{-2}) A.$$

Thus,

$$A \gtrsim \|\tilde{P}^{\Delta}_{\leq N_n/2} g_n\|_{\Sigma} \gtrsim \|x\tilde{P}^{\Delta}_{\leq N_n/2} g_n\|_{L^2(|x-x_n|\leq N_n^{-1})} \gtrsim |x_n| N_n^{-\frac{d}{2}} N_n^{\frac{d-2}{2}} A(\tfrac{\varepsilon}{A})^{\frac{d(d+2)}{8}},$$

which yields the claim. □

Case 1. Suppose the N_n have a bounded subsequence, so that (passing to a subsequence) $N_n \equiv N_\infty$. The x_n's stay bounded by Lemma 4.4, so after passing to a subsequence we may assume $x_n \to x_\infty$. We may also assume $t_n \to t_\infty$ since the interval I is compact. The functions f_n are bounded in Σ, hence (after passing to a subsequence) converge weakly in Σ to some ϕ.

To see that ϕ is nontrivial in Σ, we have

$$\begin{aligned}\langle \phi, e^{it_\infty H}\tilde{P}_{N_\infty}\delta_{x_\infty}\rangle &= \lim_n \langle f_n, e^{it_\infty H}\tilde{P}_{N_\infty}\delta_{x_\infty}\rangle \\ &= \lim_{n\to\infty} [e^{-it_n H}\tilde{P}_{N_\infty} f_n(x_n) + \langle f_n, (e^{it_\infty H} - e^{it_n H})\tilde{P}_{N_\infty}\delta_{x_n}\rangle \\ &+ \langle f_n, e^{it_\infty H}\tilde{P}_{N_n}(\delta_{x_\infty} - \delta_{x_n})\rangle].\end{aligned}$$

Using the heat kernel bounds (2.9) and the fact that, by the compactness of the embedding $\Sigma \subset L^2$, the sequence f_n converges to ϕ in L^2, one verifies easily that the second and third terms on the right side vanish. So

$$|\langle \phi, e^{it_\infty H}\tilde{P}_{N_\infty}\delta_{x_\infty}\rangle| = \lim_{n\to\infty} |e^{-it_n H}\tilde{P}_{N_\infty} f_n(x_n)| \gtrsim N_\infty^{\frac{d-2}{2}} \varepsilon^{\frac{d(d+2)}{8}} A^{-\frac{(d-2)(d+4)}{8}}.$$

On the other hand, by Hölder and (2.9),

$$\begin{aligned}|\langle \phi, e^{it_\infty H}\tilde{P}_{N_\infty}\delta_{x_\infty}\rangle| &\leq \|e^{-it_\infty H}\phi\|_{L^{\frac{2d}{d-2}}} \|\tilde{P}_{N_\infty}\delta_{x_\infty}\|_{L^{\frac{2d}{d+2}}} \\ &\lesssim \|\phi\|_\Sigma N_\infty^{\frac{d-2}{2}}.\end{aligned}$$

Therefore

$$\|\phi\|_\Sigma \gtrsim \varepsilon^{\frac{d(d+2)}{8}} A^{-\frac{(d-2)(d+4)}{8}}.$$

Set

$$\phi_n \equiv \phi,$$

and define the augmented frame $(t_n, x_n, N_n, N_n') \equiv (0,0,1,1)$. The decoupling in Σ (4.6) can be proved as in Case 2 below, and we refer the reader to the argument detailed there. It remains to establish decoupling in $L^{\frac{2d}{d-2}}$. As the embedding $\Sigma \subset L^2$ is compact, the sequence f_n, which converges weakly to $\phi \in \Sigma$, converges to ϕ strongly in L^2. After passing to a subsequence we obtain convergence pointwise a.e. The decoupling (4.5) now follows from Lemma 2.6. This completes the case where N_n have a bounded subsequence.

Case 2. Now we address the case where $N_n \to \infty$. The main nuisance is that the weak limits ϕ will usually be merely in $\dot{H}^1(\mathbf{R}^d)$, not in Σ, so defining the profiles ϕ_n will require spatial cutoffs.

As the functions $N_n^{-(d-2)/2}(e^{-it_n H} f_n)(N_n^{-1}\cdot + x_n)$ are bounded in $\dot{H}^1(\mathbf{R}^d)$, the sequence has a weak subsequential limit

$$N_n^{-\frac{d-2}{2}}(e^{-it_n H} f_n)(N_n^{-1}\cdot + x_n) \rightharpoonup \phi \text{ in } \dot{H}^1(\mathbf{R}^d). \tag{4.8}$$

By Lemma 4.4, after passing to a further subsequence we may assume

$$\lim_{n\to\infty} N_n^{-1}|x_n| = r_\infty < \infty \quad \text{and} \quad \lim_{n\to\infty} N_n^2 t_n = t_\infty \in [-\infty, \infty]. \tag{4.9}$$

It will be necessary to distinguish the cases $r_\infty > 0$ and $r_\infty = 0$, corresponding to whether the frame $\{(t_n, x_n, N_n)\}$ is type 2a or 2b, respectively.

Lemma 4.5. *If $r_\infty > 0$, the function ϕ defined in* (4.8) *also belongs to L^2.*

Proof. By (4.8) and the Rellich-Kondrashov compactness theorem, for each $R \geq 1$ we have

$$N_n^{-\frac{d-2}{2}}(e^{-it_n H} f_n)(N_n^{-1}\cdot + x_n) \to \phi \text{ in } L^2(\{|x| \leq R\}).$$

By a change of variables,

$$\begin{aligned}N_n^{-\frac{d-2}{2}}(e^{-it_n H} f_n)(N_n^{-1}\cdot + x_n)\|_{L^2(|x|\leq R)} &= N_n \|e^{-it_n H} f_n\|_{L^2(|x-x_n|\leq R N_n^{-1})} \\ &\lesssim \|x e^{-it_n H} f_n\|_{L^2}\end{aligned}$$

whenever $|x_n| \geq \frac{N_n r_\infty}{2}$ and $RN_n^{-1} \leq \frac{r_\infty}{10}$, so we have uniformly in $R \geq 1$ that

$$\limsup_n \|N_n^{-\frac{d-2}{2}}(e^{-it_nH}f_n)(N_n^{-1}\cdot+x_n)\|_{L^2(|x|\leq R)} \lesssim \sup_n \|e^{-it_nH}f_n\|_\Sigma \lesssim 1.$$

Therefore $\|\phi\|_{L^2} = \lim_{R\to\infty}\|\phi\|_{L^2(|x|\leq R)} \lesssim 1.$ □

Remark. The claim fails if $r_\infty = 0$. Indeed, if $\phi \in \dot{H}^1(\mathbf{R}^d) \setminus L^2(\mathbf{R}^d)$, then $f_n = N_n^{(d-2)/2}\phi(N_n\cdot)\chi(\cdot)$ are bounded in Σ, and $N_n^{-(d-2)/2}f_n(N_n^{-1}\cdot) = \phi(\cdot)\chi(N_n^{-1}\cdot)$ converges strongly in $\dot{H}^1$ to ϕ.

Next we prove that ϕ is nontrivial in $\dot{H}^1$.

Lemma 4.6. $\|\phi\|_{\dot{H}^1} \gtrsim A\left(\frac{\varepsilon}{A}\right)^{\frac{d(d+2)}{8}}$.

Proof. From (2.9) and (4.7),

$$N_n^{\frac{d-2}{2}} A\left(\tfrac{\varepsilon}{A}\right)^{\frac{d(d+2)}{8}} \lesssim \tilde{P}^\Delta_{\leq N_n}|e^{-it_nH}f_n|(x_n) + \tilde{P}^\Delta_{\leq N_n/2}|e^{-it_nH}f_n|(x_n),$$

so one of the terms on the right is at least half the left side. Suppose first that

$$\tilde{P}^\Delta_{\leq N_n}|e^{-it_nH}f_n|(x_n) \gtrsim N_n^{\frac{d-2}{2}} A\left(\tfrac{\varepsilon}{A}\right)^{\frac{d(d+2)}{8}}.$$

Put $\check{\psi} = \tilde{P}^\Delta_{\leq 1}\delta_0 = e^\Delta\delta_0$. Since $\check{\psi}$ is Schwartz,

$$|\langle|\phi|, \check{\psi}\rangle_{L^2}| \leq \|\phi\|_{\dot{H}^1}\|\check{\psi}\|_{\dot{H}^{-1}} \lesssim \|\phi\|_{\dot{H}^1}.$$

On the other hand, as the absolute values $N_n^{-\frac{d-2}{2}}|e^{-it_nH}f_n|(N_n^{-1}\cdot+x_n)$ converge weakly in $\dot{H}^1$ to $|\phi|$,

$$\begin{aligned}\langle|\phi|, \check{\psi}\rangle_{L^2} &= \lim_n \langle N_n^{-\frac{d-2}{2}}|e^{-it_nH}f_n|(N_n^{-1}\cdot+x_n), \check{\psi}\rangle_{L^2} \\ &= \lim_n \tilde{P}^\Delta_{\leq N_n}|e^{-it_nH}f_n|(x_n) \gtrsim A\left(\tfrac{\varepsilon}{A}\right)^{\frac{d(d+2)}{8}}.\end{aligned}$$

from which the claim follows. Similarly if

$$\tilde{P}^\Delta_{\leq N_n/2}|e^{-it_nH}f_n|(x_n) \gtrsim N_n^{\frac{d-2}{2}} A\left(\tfrac{\varepsilon}{A}\right)^{\frac{d(d+2)}{8}},$$

then we obtain $\|\phi\|_{\dot{H}^1} \sim \|\phi(2\cdot)\|_{\dot{H}^1} \gtrsim N_n^{\frac{d-2}{2}} A\left(\frac{\varepsilon}{A}\right)^{\frac{d(d+2)}{8}}$. □

Having extracted a nontrivial bubble ϕ, we are ready to define the ϕ_n. The basic idea is to undo the operations applied to f_n in the definition (4.8) of ϕ. However, we need to first apply a spatial cutoff to embed ϕ in Σ.

With the frame $\{(t_n, x_n, N_n)\}$ defined according to (4.7), form the augmented frame $\{(t_n, x_n, N_n, N_n')\}$ with the cutoff parameter N_n' chosen according to the second case in Definition 4.2. Let $G_n,\ S_n$ be the $\dot{H}^1$ isometries and spatial cutoff operators associated to $\{(t_n, x_n, N_n, N_n')\}$. Set

$$\phi_n = e^{it_nH}G_nS_n\phi = e^{it_nH}[N_n^{\frac{d-2}{2}}\phi(N_n(\cdot - x_n))\chi(N_n'(\cdot - x_n))]. \tag{4.10}$$

Let us check that ϕ_n satisfies the various properties asserted in the proposition.

Lemma 4.7. $A\left(\frac{\varepsilon}{A}\right)^{\frac{d(d+2)}{8}} \lesssim \liminf_{n\to\infty}\|\phi_n\|_\Sigma \leq \limsup_{n\to\infty}\|\phi_n\|_\Sigma \lesssim 1.$

Proof. By the definition of the Σ norm and a change of variables,

$$\|\phi_n\|_\Sigma = \|G_nS_n\|_\Sigma \geq \|S_n\phi\|_{\dot{H}^1}.$$

Lemma 4.6 and the remarks following Definition 4.2 together imply the lower bound

$$\liminf_n \|\phi_n\|_\Sigma \gtrsim A\left(\tfrac{\varepsilon}{A}\right)^{\frac{d(d+2)}{8}}.$$

The upper bound follows immediately from the case $(q, r) = (\infty, 2)$ in Lemma 4.2. □

Next we verify the decoupling assertion (4.6). By the Pythagorean theorem,

$$\begin{aligned}\|f_n\|_\Sigma^2 - \|f_n - \phi_n\|_\Sigma^2 - \|\phi_n\|_\Sigma^2 &= 2\,\mathrm{Re}\langle f_n - \phi_n, \phi_n\rangle_\Sigma) \\ &= 2\,\mathrm{Re}\langle e^{-it_nH} f_n - G_nS_n\phi, G_nS_n\phi\rangle_\Sigma \\ &= 2\,\mathrm{Re}\langle w_n, G_nS_n\phi\rangle_\Sigma.\end{aligned}$$

where $w_n = e^{-it_nH} f_n - G_nS_n\phi$. By definition,

$$\langle w_n, G_nS_n\phi\rangle_\Sigma = \langle w_n, G_nS_n\phi\rangle_{\dot H^1} + \langle xw_n, xG_nS_n\phi\rangle_{L^2}.$$

From (4.3) and the definition (4.8) of ϕ, it follows that

$$G_n^{-1} w_n \to 0 \quad \text{weakly in } \dot H^1 \text{ as } n \to \infty.$$

Hence

$$\lim_{n\to\infty} \langle w_n, G_nS_n\phi\rangle_{\dot H^1} = \lim_{n\to\infty} \langle G_n^{-1} w_n, S_n\phi\rangle_{\dot H^1} = \lim_{n\to\infty} \langle G_n^{-1} w_n, \phi\rangle_{\dot H^1} = 0.$$

We turn to the second component of the inner product. Fix $R > 0$, and estimate

$$\begin{aligned}&|\langle xw_n, xG_nS_n\phi\rangle_{L^2}| \\ &\le \int_{\{|x-x_n|\le RN_n^{-1}\}} |xw_n||xG_nS_n\phi|\,dx + \int_{\{|x-x_n|> RN_n^{-1}\}} |xw_n||xG_nS_n\phi|\,dx \\ &= (I) + (II)\end{aligned}$$

Use a change of variable and the bound $|x_n| \lesssim N_n$ to obtain

$$(I) \lesssim \int_{|x|\le R} |G_n^{-1} w_n||\phi|\,dx \to 0 \quad \text{as} \quad n\to\infty.$$

Next, apply Cauchy-Schwartz and the upper bound of Lemma 4.7 to see that

$$\begin{aligned}(II)^2 &\lesssim \int_{\{|x-x_n|>RN_n^{-1}\}} |xG_nS_n\phi|^2\,dx \\ &\lesssim N_n^{-2} \int_{R\le|x|\lesssim \frac{N_n}{N_n'}} |x_n + N_n^{-1}x|^2|\phi(x)|^2 dx \\ &\lesssim (N_n^{-2}|x_n|^2 + N_n^{-2}(N_n')^{-2}) \int_{R\le|x|\lesssim \frac{N_n}{N_n'}} |\phi(x)|^2\,dx.\end{aligned}$$

Suppose that the frame $\{(t_n, x_n, N_n)\}$ is of type 2a, so that $\lim_n N_n^{-1}|x_n| > 0$. By Lemma 4.5 and dominated convergence, the right side above is bounded by

$$\int_{R\le|x|} |\phi(x)|^2\,dx \to 0 \quad \text{as} \quad R\to\infty,$$

uniformly in n. If instead $\{(t_n, x_n, N_n)\}$ is of type 2b, use Hölder to see that the right side is bounded by

$$(N_n^{-2}|x_n|(\tfrac{N_n}{N_n'})^2 + (N_n')^{-4})\|\phi\|_{L^{\frac{2d}{d-2}}}.$$

By Sobolev embedding and the construction of the parameter N_n' in Definition 4.2, the above vanishes as $n\to\infty$. In either case, we obtain

$$\lim_{R\to\infty} \limsup_{n\to\infty} (II) = 0.$$

Combining the two estimates and choosing R arbitrarily large, we conclude as required that

$$\lim_{n\to\infty} |\langle xw_n, xG_nS_n\phi\rangle_{L^2}| = 0.$$

To close this subsection, we verify the $L^{\frac{2d}{d-2}}$ decoupling property (4.5) when $N_n^2 t_n \to \pm\infty$. Assume first that the ϕ appearing in the definition (4.10) of ϕ_n has compact support. By the dispersive estimate (2.5) and a change of variables,

$$\lim_{n\to\infty} \|\phi_n\|_{L^{\frac{2d}{d-2}}} \lesssim |t_n|^{-1}\|G_n\phi\|_{L^{\frac{2d}{d+2}}} \lesssim (N_n^2|t_n|)^{-1}\|\phi\|_{L^{\frac{2d}{d+2}}} = 0.$$

The claimed decoupling follows immediately.

For general ϕ in H^1 or $\dot{H}^1$ (depending on whether $\lim_n N_n^{-1}|x_n|$ is positive or zero), select $\psi^\varepsilon \in C_c^\infty$ converging to ϕ in the appropriate norm as $\varepsilon \to 0$. Then for all n large enough, we have

$$\|\phi_n\|_{L^{\frac{2d}{d-2}}} \leq \|e^{it_nH}G_nS_n[\phi - \psi^\varepsilon]\|_{L^{\frac{2d}{d-2}}} + \|e^{it_nH}G_nS_n\psi^\varepsilon\|_{L^{\frac{2d}{d-2}}},$$

and decoupling follows from Lemmas 2.3 and 4.2 and the special case just proved. □

4.2. **Convergence of linear propagators.** To complete the proof of Proposition 4.3, we need a more detailed understanding of how the linear propagator e^{-itH} interacts with the $\dot{H}^1$-symmetries G_n associated to a frame in certain limits. This section is inspired by the discussion surrounding [20, Lemma 5.2], which proves analogous results relating the linear propagators of the 2D Schrödinger equation and the complexified Klein-Gordon equation $-iv_t + \langle\nabla\rangle v = 0$.

Definition 4.3. We say two frames $\mathcal{F}^1 = \{(t_n^1, x_n^1, N_n^1)\}$ and $\mathcal{F}^2 = \{(t_n^2, x_n^2, N_n^2)\}$ (where the superscripts are indices, not exponents) are *equivalent* if

$$\tfrac{N_n^1}{N_n^2} \to R_\infty \in (0,\infty),\ N_n^1(x_n^2 - x_n^1) \to x_\infty \in \mathbf{R}^d,\ (N_n^1)^2(t_n^1 - t_n^2) \to t_\infty \in \mathbf{R}.$$

The frames are *orthogonal* should any of the above statements fail. Note that replacing the N_n^1 in the second and third expressions above by N_n^2 yields an equivalent definition of orthogonality.

Remark. If $\mathcal{F}^1$ and $\mathcal{F}^2$ are equivalent, it follows from the above definition that they must be of the same type in Definition 4.1, and that $\lim_n (N_n^1)^{-1}|x_n^1|$ and $\lim_n (N_n^2)^{-1}|x_n^2|$ are either both zero or both positive.

The following lemma and its corollary make precise the heuristic that when acting on functions concentrated at a point, e^{-itH} can be approximated for small t by regarding the $|x|^2/2$ potential as essentially constant on the support of the initial data; thus one obtains a modulated free particle propagator $e^{-it|x_0|^2/2}e^{it\Delta/2}$, where x_0 is the spatial center of the data.

Lemma 4.8 (Strong convergence)**.** *Suppose*

$$\mathcal{F}^M = (t_n^M, x_n, M_n), \quad \mathcal{F}^N = (t_n^N, y_n, N_n)$$

are equivalent frames. Define

$$\begin{aligned} &R_\infty = \lim_{n\to\infty} \tfrac{M_n}{N_n},\ t_\infty = \lim_{n\to\infty} M_n^2(t_n^M - t_n^N),\ x_\infty = \lim_{n\to\infty} M_n(y_n - x_n), \\ &r_\infty = \lim_n M_n^{-1}|x_n| = \lim_n M_n^{-1}|y_n|. \end{aligned}$$

Let G_n^M, G_n^N be the scaling and translation operators attached to the frames $\mathcal{F}^M$ and $\mathcal{F}^N$ respectively. Then $(e^{-it_n^N H}G_n^N)^{-1}e^{-it_n^M H}G_n^M$ converges strongly as operators on Σ to the operator U_∞ defined by

$$U_\infty\phi = e^{-\frac{it_\infty(r_\infty)^2}{2}} R_\infty^{\frac{d-2}{2}} [e^{\frac{it_\infty\Delta}{2}}\phi](R_\infty \cdot + x_\infty).$$

Proof. If $M_n \equiv 1$, then by the definition of a frame we must have $\mathcal{F}^M = \mathcal{F}^N = \{(1,0,0)\}$, so the claim is trivial. Thus we may assume that $M_n \to \infty$. Put $t_n = t_n^M - t_n^N$. Using Mehler's formula (2.4), we write

$$\begin{aligned} &(e^{-it_n^N H}G_n^N)^{-1}e^{-it_n^M H}G_n^M\phi(x) = (G_n^N)^{-1}e^{-it_nH}G_n^M\phi(x) \\ &= (\tfrac{M_n}{N_n})^{\frac{d-2}{2}} e^{i\gamma(t_n)|y_n + N_n^{-1}x|^2} e^{\frac{iM_n^2\sin(t_n)\Delta}{2}} [e^{i\gamma(t_n)|x_n + M_n^{-1}\cdot|^2}\phi](\tfrac{M_n}{N_n}x + M_n(y_n - x_n)). \end{aligned}$$

where

$$\gamma(t) = \tfrac{\cos t - 1}{2\sin t} = -\tfrac{t}{4} + O(t^3).$$

Observe that

$$e^{i\gamma(t_n)|x_n + M_n^{-1}\cdot|^2}\phi \to e^{-\frac{it_\infty(r_\infty)^2}{4}}\phi \quad \text{in } \Sigma.$$

Indeed,

$$\begin{aligned} &\|\nabla[e^{i\gamma(t_n)|x_n+M_n^{-1}\cdot|^2}\phi - e^{i\gamma(t_n)|x_n|^2}\phi]\|_{L^2} = \|\nabla[(e^{i\gamma(t_n)[M_n^{-2}|x|^2 + 2M_n^{-1}x_n\cdot x]} - 1)\phi]\|_{L^2} \\ &\lesssim \|t_n(M_n^{-2}x + 2M_n^{-1}x_n)\phi\|_{L^2} + \|(e^{i\gamma(t_n)[M_n^{-2}|x|^2 + 2M_n^{-1}x_n\cdot x]} - 1)\nabla\phi\|_{L^2} \\ &\lesssim |t_n|M_n^{-2}\|x\phi\|_{L^2} + |t_n||x_n|M_n^{-1}\|\phi\|_{L^2} + \|(e^{i\gamma(t_n)[M_n^{-2}|x|^2 2M_n^{-1}x_n\cdot x]} - 1)\nabla\phi\|_{L^2}. \end{aligned}$$

As $n \to \infty$, the first two terms vanish because $\|x\phi\|_2 + \|\phi\|_2 \lesssim \|\phi\|_\Sigma$, while the third term vanishes by dominated convergence. Dominated convergence also implies that

$$\|x[e^{i\gamma(t_n)|x_n + M_n^{-1}x|^2}\phi - e^{i\gamma(t_n)|x_n|^2}\phi]\|_{L^2} \to 0 \text{ as } n \to \infty.$$

On the other hand, since

$$\gamma(t_n)|x_n|^2 = -\frac{M_n^2 t_n M_n^{-2}|x_n|^2}{4} + O(M_n^{-4}) \to -\frac{t_\infty (r_\infty)^2}{4},$$

it follows that

$$\|e^{i\gamma(t_n)|x_n + M_n^{-1}\cdot|^2}\phi - e^{-\frac{it_\infty (r_\infty)^2}{4}}\phi\|_\Sigma \to 0$$

as claimed. As $e^{\frac{iM_n^2 \sin(t_n)\Delta}{2}} \to e^{\frac{it_\infty \Delta}{2}}$ strongly, we obtain

$$e^{\frac{iM_n^2 \sin(t_n)\Delta}{2}}[e^{i\gamma(t_n)|x_n + M_n^{-1}\cdot|^2}\phi] \to e^{-\frac{it_\infty (r_\infty)^2}{4}} e^{\frac{it_\infty \Delta}{2}}\phi \text{ in } \Sigma,$$

and the conclusion quickly follows. □

Corollary 4.9. *Let $\{(t_n^M, x_n, M_n, M_n')\}$ and $\{(t_n^N, y_n, N_n, N_n')\}$ be equivalent frames, and S_n^M, S_n^N be the associated spatial cutoff operators. Then*

$$\lim_{n\to\infty} \|e^{-it_n^M H} G_n^M S_n^M \phi - e^{-it_n^N H} G_n^N S_n^N U_\infty \phi\|_\Sigma = 0 \tag{4.11}$$

and

$$\lim_{n\to\infty} \|e^{-it_n^M H} G_n^M S_n^M \phi - e^{-it_n^N H} G_n^N U_\infty S_n^N \phi\|_\Sigma = 0 \tag{4.12}$$

whenever $\phi \in H^1$ if the frames conform to case 2a and $\phi \in \dot{H}^1$ if they conform to case 2b in Definition 4.2.

Proof. As before, the result is immediate if $M_n \equiv 1$ since all operators in sight are trivial. Thus we may assume $M_n \to \infty$. Suppose first that $\phi \in C_c^\infty$. Using the unitarity of e^{-itH} on Σ, the operator bounds (4.3), and the fact that $S_n^M \phi = \phi$ for all n sufficiently large, we write the left side of (4.11) as

$$\begin{aligned}
&\|G_n^N[(G_n^N)^{-1} e^{-i(t_n^M - t_n^N)H} G_n^M \phi - S_n^N U_\infty \phi]\|_\Sigma \\
&\lesssim \|(G_n^N)^{-1} e^{-i(t_n^M - t_n^N)H} G_n^M \phi - S_n^N U_\infty \phi\|_\Sigma \\
&\lesssim \|(G_n^N)^{-1} e^{-i(t_n^M - t_n^N)H} G_n^M \phi - U_\infty \phi\|_\Sigma + \|(1 - S_n^N) U_\infty \phi\|_\Sigma
\end{aligned}$$

which goes to zero by Lemma 4.8 and dominated convergence. This proves (4.11) under the additional hypothesis that $\phi \in C_c^\infty$.

We now remove this crutch and take $\phi \in H^1$ or $\dot{H}^1$ depending on whether the frames are of type 2a or 2b in Definition 4.2, respectively. For each $\varepsilon > 0$, choose $\phi^\varepsilon \in C_c^\infty$ such that $\|\phi - \phi^\varepsilon\|_{H^1} < \varepsilon$ or $\|\phi - \phi^\varepsilon\|_{\dot{H}^1} < \varepsilon$, respectively. Then

$$\begin{aligned}
&\|e^{-it_n^M H} G_n^M S_n^M \phi - e^{-it_n^N H} G_n^N S_n^N U_\infty \phi\|_\Sigma \le \|e^{-it_n^M H} G_n^M S_n^M (\phi - \phi^\varepsilon)\|_\Sigma \\
&+ \|e^{-it_n H} G_n^M S_n^M \phi^\varepsilon - e^{-it_n^N H} G_n^N S_n^N U_\infty \phi^\varepsilon\|_\Sigma + \|e^{-it_n^N H} G_n^N S_n^N U_\infty (\phi - \phi^\varepsilon)\|_\Sigma
\end{aligned}$$

In the limit as $n \to \infty$, the middle term vanishes and we are left with a quantity at most a constant times

$$\limsup_{n\to\infty} \|G_n^M S_n^M (\phi - \phi^\varepsilon)\|_\Sigma + \limsup_{n\to\infty} \|G_n^N S_n^N U_\infty (\phi - \phi^\varepsilon)\|_\Sigma.$$

Applying Lemma 4.2 and using the mapping properties of U_∞ on $\dot{H}^1$ and H^1, we see that

$$\limsup_{n\to\infty} \|e^{-it_n H} G_n^M S_n^M \phi - e^{it_n^N H} G_n^N S_n^N U_\infty \phi\|_\Sigma \lesssim \varepsilon$$

for every $\varepsilon > 0$. This proves the claim (4.11). Similar considerations deal with the second claim (4.12). □

Lemma 4.10. *Suppose the frames $\{(t_n^M, x_n, M_n)\}$ and $\{(t_n^N, y_n, N_n)\}$ are equivalent. Put $t_n = t_n^M - t_n^N$. Then for $f, g \in \Sigma$ we have*

$$\langle (G_n^N)^{-1} e^{-it_n H} G_n^M f, g\rangle_{\dot{H}^1} = \langle f, (G_n^M)^{-1} e^{it_n H} G_n^N g\rangle_{\dot{H}^1} + R_n(f, g),$$

where $|R_n(f, g)| \le C|t_n| \|G_n^M f\|_\Sigma \|G_n^N g\|_\Sigma$.

Remark. It follows from Lemma 4.8 that

$$\lim_{n\to\infty} \langle (G_n^N)^{-1} e^{-it_nH} G_n^M f, g\rangle_{\dot H^1} = \lim_{n\to\infty} \langle f, (G_n^M)^{-1} e^{it_nH} G_n^N g\rangle_{\dot H^1}$$

for fixed $f, g \in \Sigma$. The content of this lemma lies in the quantitative error bound.

Proof. We have

$$\langle (G_n^N)^{-1} e^{-it_nH} G_N^M f, g\rangle_{\dot H^1} = \langle f, (G_n^M)^{-1} e^{it_nH} G_n^N g\rangle_{\dot H^1} + R_n(f,g)$$

where $R_n(f,g) = \langle [\nabla, e^{-it_nH}] G_n^M f, \nabla G_n^N g\rangle_{L^2} - \langle \nabla G_n^M f, [\nabla, e^{it_nH}] G_n^N g\rangle_{L^2}$. The claim follows from Cauchy-Schwartz and the commutator estimate

$$\|[\nabla, e^{-itH}]\|_{\Sigma\to L^2} = O(t),$$

which is a consequence of the standard identities

$$\begin{aligned} e^{itH} i\nabla e^{-itH} &= i\nabla \cos t - x\sin t \\ e^{itH} x e^{-itH} &= i\nabla \sin t + x\cos t. \end{aligned}$$

□

Next we prove a converse to Lemma 4.8.

Lemma 4.11 (Weak convergence)**.** *Assume the frames $\mathcal{F}^M = \{(t_n^M, x_n, M_n)\}$ and $\mathcal{F}^N = \{(t_n^N, y_n, N_n)\}$ are orthogonal. Then for any $f \in \Sigma$,*

$$(e^{-it_n^N H} G_n^N)^{-1} e^{-it_n^M H} G_n^M f \to 0 \quad \textit{weakly in } \dot H^1.$$

Proof. Put $t_n = t_n^M - t_n^N$, and suppose that $|M_n^2 t_n| \to \infty$. Then

$$\|(G_n^N)^{-1} e^{-it_nH} G_n^M f\|_{L^{\frac{2d}{d-2}}} \to 0$$

for $f \in C_c^\infty$ by a change of variables and the dispersive estimate, thus for general $f \in \Sigma$ by a density argument. Therefore $(G_n^N)^{-1} e^{-it_nH} G_n^M f$ converges weakly in $\dot H^1$ to 0. Now consider the case where $M_n^2 t_n \to t_\infty \in \mathbf{R}$. The orthogonality of $\mathcal{F}^M$ and $\mathcal{F}^N$ implies that either $N_n^{-1} M_n$ converges to 0 or ∞, or $M_n|x_n - y_n|$ diverges as $n \to \infty$. In either case, one verifies easily that $(G_n^N)^{-1} G_n^M$ converge to zero weakly as operators on $\dot H^1$. By Lemma 4.8, $(G_n^N)^{-1} e^{-it_nH} G_n^M f = (G_n^N)^{-1} G_n^M (G_n^M)^{-1} e^{-it_nH} G_n^M f$ converges to zero weakly in $\dot H^1$. □

Corollary 4.12. *Let $\{(t_n^M, x_n, M_n, M_n')\}$ and $\{(t_n^N, y_n, N_n, N_n')\}$ be orthogonal with corresponding operators G_n^M, S_n^M and G_n^N, S_n^N. Then*

$$(e^{-it_n^N H} G_n^N)^{-1} e^{-it_n^M H} G_n^M S_n^M \phi \rightharpoonup 0 \quad \textit{in} \quad \dot H^1$$

whenever $\phi \in H^1$ if $\mathcal{F}^M$ is of type 2a and $\phi \in \dot H^1$ if $\mathcal{F}^M$ is of type 2b.

Proof. If $\phi \in C_c^\infty$, then $S_n^M \phi = \phi$ for all large n, and the claim follows from Lemma 4.11. The case of general ϕ in H^1 or $\dot H^1$ then follows from an approximation argument similar to the one used to prove Corollary 4.9. □

4.3. **End of proof of inverse Strichartz.** We return to the proof of Proposition 4.3. Thus far, we have identified a frame $\{(t_n, x_n, N_n, N_n')\}$ and an associated profile ϕ_n such that the sequence $N_n^2 t_n$ has a limit in $[-\infty, \infty]$ as $n \to \infty$. The ϕ_n were shown to satisfy properties (4.4), (4.5), and (4.6) if either $(t_n, x_n, N_n) = (0, 0, 1)$ or $N_n \to \infty$ and $N_n^2 t_n \to \pm\infty$. Thus, it remains to prove that if $N_n \to \infty$ and $N_n^2 t_n$ remains bounded, then we may modify the frame so that t_n is identically zero and find a profile ϕ_n corresponding to this new frame which satisfies all the properties asserted in the proposition. The following lemma will therefore complete the proof of the proposition.

Lemma 4.13. *Let $f_n \in \Sigma$ satisfy the hypotheses of Proposition 4.3. Suppose $\{(t_n, x_n, N_n, N_n')\}$ is an augmented frame with $N_n \to \infty$ and $N_n^2 t_n \to t_\infty \in \mathbf{R}$ as $n \to \infty$. Then there is a profile $\phi_n' = G_n S_n \phi'$ associated to the frame $\{(0, x_n, N_n, N_n')\}$ such that properties* (4.4), (4.5)*, and* (4.6) *hold with ϕ_n' in place of ϕ_n.*

Proof. Let $\phi_n = e^{it_nH}G_nS_n\phi$ be the profile defined by (4.10). We have already seen that ϕ_n satisfies properties (4.4) and (4.6), and that

$$\phi = \dot{\mathrm{H}}^1\text{-w-lim}_{n\to\infty} G_n^{-1}e^{-it_nH}f_n.$$

As the sequence $G_n^{-1}f_n$ is bounded in $\dot{H}^1$, it has a weak subsequential limit

$$\phi' = \dot{\mathrm{H}}^1\text{-w-lim}_{n\to\infty} G_n^{-1}f_n.$$

For any $\psi \in C_c^\infty$, apply Lemma 4.10 with $f = G_n^{-1}e^{-it_nH}f_n$ to see that

$$\begin{aligned}\langle \phi', \psi\rangle_{\dot{H}^1} &= \lim_{n\to\infty}\langle G_n^{-1}f_n, \psi\rangle_{\dot{H}^1} = \lim_{n\to\infty}\langle G_n^{-1}e^{it_nH}G_nG_n^{-1}e^{-it_nH}f_n, \psi\rangle_{\dot{H}^1}\\ &= \lim_{n\to\infty}\langle G_n^{-1}e^{-it_nH}f_n, G_n^{-1}e^{-it_nH}G_n\psi\rangle_{\dot{H}^1} = \langle \phi, U_\infty\psi\rangle_{\dot{H}^1},\end{aligned}$$

where $U_\infty = \text{s-lim}_{n\to\infty} G_n^{-1}e^{-it_nH}G_n$ is the strong operator limit guaranteed by Lemma 4.8. As U_∞ is unitary on $\dot{H}^1$, we have the relation $\phi = U_\infty\phi'$.

Put $\phi_n' = G_nS_n\phi'$. By Corollary 4.9,

$$\|\phi_n - \phi_n'\|_\Sigma = \|e^{it_nH}G_nS_n\phi - G_nS_nU_\infty^{-1}\phi\|_\Sigma \to 0 \quad \text{as} \quad n\to\infty.$$

Hence ϕ_n' inherits property (4.4) from ϕ_n. The same proof as for ϕ_n shows that Σ decoupling (4.6) holds as well. It remains to verify the last decoupling property (4.5). As $G_n^{-1}f_n$ converges weakly in $\dot{H}^1$ to ϕ', by Rellich-Kondrashov and a diagonalization argument we may assume after passing to a subsequence that $G_n^{-1}f_n$ converges to ϕ' almost everywhere on $\mathbf{R}^d$. By the Lemma 2.6, the fact that $\lim_{n\to\infty}\|G_nS_n\phi' - G_n\phi'\|_{\frac{2d}{d-2}} = 0$, and a change of variables,

$$\begin{aligned}&\lim_{n\to\infty}\left[\|f_n\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} - \|f_n - \phi_n'\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} - \|\phi_n'\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}}\right]\\ &= \lim_{n\to\infty}\left[\|G_n^{-1}f_n\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} - \|G_n^{-1}f_n - \phi'\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} - \|\phi'\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}}\right]\\ &= 0.\end{aligned}$$

□

Remark. As $\lim_{n\to\infty}\|\phi_n - \phi_n'\|_\Sigma = 0$, we see by Sobolev embedding that the decoupling (4.5) also holds for the original profile $\phi_n = e^{it_nH}G_nS_n\phi$ with nonzero time parameter t_n.

4.4. **Linear profile decomposition.** As before, I will denote a fixed interval containing 0 of length at most 1, and all spacetime norms are taken over $I \times \mathbf{R}^d$ unless indicated otherwise.

Proposition 4.14. *Let f_n be a bounded sequence in Σ. After passing to a subsequence, there exists $J^* \in \{0, 1, \dots\} \cup \{\infty\}$ such that for each finite $1 \le j \le J^*$, there exist an augmented frame $\mathcal{F}^j = \{(t_n^j, x_n^j, N_n^j, (N_n^j)')\}$ and a function ϕ^j with the following properties.*

- *Either $t_n^j \equiv 0$ or $(N_n^j)^2(t_n^j) \to \pm\infty$ as $n \to \infty$.*
- *ϕ^j belongs to Σ, H^1, or $\dot{H}^1$ depending on whether $\mathcal{F}^j$ is of type 1, 2a, or 2b, respectively.*

For each finite $J \le J^*$, we have a decomposition

$$f_n = \sum_{j=1}^J e^{it_n^jH}G_n^jS_n^j\phi^j + r_n^J = \sum_{j=1}^J \phi_n^j + r_n^J, \tag{4.13}$$

where G_n^j, S_n^j are the $\dot{H}^1$-isometry and spatial cutoff operators associated to $\mathcal{F}^j$. This decomposition has the following properties:

$$(G_n^J)^{-1}e^{-it_n^JH}r_n^J \overset{\dot{H}^1}{\rightharpoonup} 0 \quad \text{for all } J \le J^*, \tag{4.14}$$

$$\sup_J \lim_{n\to\infty}\left|\|f_n\|_\Sigma^2 - \sum_{j=1}^J\|\phi_n^j\|_\Sigma^2 - \|r_n^J\|_\Sigma^2\right| = 0, \tag{4.15}$$

$$\sup_J \lim_{n\to\infty}\left|\|f_n\|_{L_x^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} - \sum_{j=1}^J\|\phi_n^j\|_{L_x^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} - \|r_n^J\|_{L_x^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}}\right| = 0. \tag{4.16}$$

Whenever $j \neq k$, the frames $\{(t_n^j, x_n^j, N_n^j)\}$ and $\{(t_n^k, x_n^k, N_n^k)\}$ are orthogonal:

$$\lim_{n\to\infty} \tfrac{N_n^j}{N_n^k} + \tfrac{N_n^k}{N_n^j} + N_n^j N_n^k |t_n^j - t_n^k| + \sqrt{N_n^j N_n^k}|x_n^j - x_n^k| = \infty. \tag{4.17}$$

Finally, we have

$$\lim_{J\to J^*} \limsup_{n\to\infty} \|e^{-it_n H} r_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} = 0, \tag{4.18}$$

Remark. One can also show a posteriori using (4.17) and (4.18) the fact, which we will neither prove nor use, that

$$\sup_J \lim_{n\to\infty} \Big| \|e^{-itH} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{2(d+2)}{d-2}} - \sum_{j=1}^J \|e^{-itH}\phi_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{2(d+2)}{d-2}} - \|e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{2(d+2)}{d-2}} \Big| = 0.$$

The argument uses similar ideas as in the proofs of [17][Lemma 2.7] or Lemma 6.3; we omit the details.

Proof. Proceed inductively using Proposition 4.3. Let $r_n^0 = f_n$. Assume that we have a decomposition up to level $J \geq 0$ obeying properties (4.14) through (4.16). After passing to a subsequence, define

$$A_J = \lim_n \|r_n^J\|_\Sigma \quad \text{and} \quad \varepsilon_J = \lim_n \|e^{-it_n H} r_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}.$$

If $\varepsilon_J = 0$, stop and set $J^* = J$. Otherwise apply Proposition 4.3 to the sequence r_n^J to obtain a frame $(t_n^{J+1}, x_n^{J+1}, N_n^{J+1}, (N_n^{J+1})')$ and functions

$$\phi^{J+1} \in \dot{H}^1, \quad \phi_n^{J+1} = e^{it_n^{J+1}H} G_n^{J+1} S_n^{J+1} \phi^{J+1} \in \Sigma$$

which satisfy the conclusions of Proposition 4.3. In particular ϕ^{J+1} is the $\dot{H}^1$ weak limit of the sequence $(G_n^{J+1})^{-1} e^{-it_n^{J+1}H} r_n^J$. Let $r_n^{J+1} = r_n^J - \phi_n^{J+1}$. By the induction hypothesis, (4.15) and (4.16) are satisfied with J replaced by $J+1$. Also,

$$(G_n^{J+1})^{-1} e^{-it_n^{J+1}H} r_n^{J+1} = [(G_n^{J+1})^{-1} e^{-it_n^{J+1}H} r_n^J - \phi^{J+1}] + (1 - S_n^{J+1})\phi^{J+1}.$$

As $n \to \infty$, the first term goes to zero weakly in $\dot{H}^1$ while the second term goes to zero strongly. Thus (4.14) holds at level $J+1$ as well. After passing to a subsequence, we may define

$$A_{J+1} = \lim_n \|r_n^{J+1}\|_\Sigma \quad \text{and} \quad \varepsilon_{J+1} = \lim_n \|e^{-itH} r_n^{J+1}\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}.$$

If $\varepsilon_{J+1} = 0$, stop and set $J^* = J+1$. Otherwise continue the induction. If the algorithm never terminates, set $J^* = \infty$. From (4.15) and (4.16), the parameters A_J and ε_J satisfy the inequality

$$A_{J+1}^2 \leq A_J^2 [1 - C(\tfrac{\varepsilon_J}{A_J})^{\frac{d(d+2)}{4}}].$$

If $\limsup_{J\to J^*} \varepsilon_J = \varepsilon_\infty > 0$, then as A_J are decreasing there would exist infinitely many J's so that

$$A_{J+1}^2 \leq A_J^2 [1 - C(\tfrac{\varepsilon_\infty}{A_0})^{\frac{d(d+2)}{4}}],$$

which implies that $\lim_{J\to J^*} A_J = 0$. But this contradicts the Strichartz inequality which dictates that $\limsup_{J\to J^*} A_J \gtrsim \limsup_{J\to J^*} \varepsilon_J = \varepsilon_0$. We conclude that

$$\lim_{J\to J^*} \varepsilon_J = 0.$$

Thus (4.18) holds.

It remains to prove the assertion (4.17). Suppose otherwise, and let $j < k$ be the first two indices for which $\mathcal{F}^j$ and $\mathcal{F}^k$ are equivalent. Thus $\mathcal{F}^\ell$ and $\mathcal{F}^k$ are orthogonal for all $j < \ell < k$. By the construction of the profiles, we have

$$r_n^{j-1} = e^{it_n^j H} G_n^j S_n^j \phi^j + e^{it_n^k H} G_n^k S_n^k \phi^k + \sum_{j<\ell<k} e^{it_n^\ell H} G_n^\ell S_n^\ell \phi^\ell + r_n^k,$$

therefore

$$\begin{aligned}(e^{it_n^j H} G_n^j)^{-1} r_n^{j-1} &= (e^{it_n^j H} G_n^j)^{-1} e^{it_n^j H} G_n^j S_n^j \phi^j + (e^{it_n^j H} G_n^j)^{-1} e^{it_n^k H} G_n^k S_n^k \phi^k \\ &+ \sum_{j<\ell<k} (e^{it_n^j H} G_n^j)^{-1} e^{it_n^\ell H} G_n^\ell S_n^\ell \phi^\ell + (e^{it_n^j H} G_n^j)^{-1} r_n^k.\end{aligned}$$

As $n \to \infty$, the left side converges to ϕ^j weakly in $\dot{H}^1$. On the right side, we apply Corollary 4.9 to see that the first and second terms converge in $\dot{H}^1$ to ϕ^j and $U_\infty^{jk}\phi^k$, respectively, for some isomorphism U_∞^{jk} of $\dot{H}^1$. By Corollary 4.12, each of the terms in the summation converges to zero weakly in $\dot{H}^1$. Taking for granted the claim that

$$(e^{it_n^j H}G_n^j)^{-1} r_n^k \to 0 \quad \text{weakly in } \dot{H}^1, \tag{4.19}$$

it follows that

$$\phi^j = \phi^j + U_\infty^{jk}\phi^k,$$

so $\phi^k = 0$, which contradicts the nontriviality of ϕ^k. Therefore, the proof of the proposition will be complete upon verifying the weak limit (4.19). As that sequence is bounded in $\dot{H}^1$, it suffices to check that

$$\langle (e^{it_n^j H}G_n^j)^{-1} r_n^k, \psi\rangle_{\dot{H}^1} \to 0 \quad \text{for any } \psi \in C_c^\infty(\mathbf{R}^d).$$

Write $(e^{it_n^j H}G_n^j)^{-1} r_n^k = (e^{it_n^j H}G_n^j)^{-1}(e^{it_n^k H}G_n^k)(e^{it_n^k H}G_n^k)^{-1} r_n^k$, and use Lemma 4.10 and the weak limit (4.14) to see that

$$\begin{aligned}\lim_{n\to\infty} \langle (e^{it_n^j H}G_n^j)^{-1} r_n^k, \psi\rangle_{\dot{H}^1} &= \lim_{n\to\infty} \langle (e^{it_n^k H}G_n^k)^{-1} r_n^k, (e^{it_n^k H}G_n^k)^{-1}(e^{it_n^j H}G_n^j)\psi\rangle_{\dot{H}^1} \\ &= \lim_{n\to\infty} \langle (G_n^k)^{-1} e^{-it_n^k H} r_n^k, (U_\infty^{jk})^{-1}\psi\rangle_{\dot{H}^1} \\ &= 0.\end{aligned}$$

□

5. The case of concentrated initial data

The next step in the proof of Theorem 1.2 is to establish wellposedness when the initial data consists of a highly concentrated "bubble". The picture to keep in mind is that of a single profile ϕ_n^j in Proposition 4.14 as $n \to \infty$. In the next section we combine this special case with the profile decomposition to treat general initial data. Although we state the following result as a conditional one to permit a unified exposition, by Theorem 1.1 the result is unconditionally true in most cases.

Proposition 5.1. *Let $I = [-1, 1]$. Assume that Conjecture 1.1 holds. Suppose*

$$\mathcal{F} = \{(t_n, x_n, N_n, N_n')\}$$

is an augmented frame with $t_n \in I$ and $N_n \to \infty$, such that either $t_n \equiv 0$ or $N_n^2 t_n \to \pm\infty$; that is, $\mathcal{F}$ is type 2a or 2b in Definition 4.2. Let $G_n, \tilde{G}_n$, and S_n be the associated operators defined in (4.1) *and* (4.2)*. Suppose ϕ belongs to H^1 or $\dot{H}^1$ depending on whether $\mathcal{F}$ is type 2a or 2b respectively. Then, for n sufficiently large, there is a unique solution $u_n : I \times \mathbf{R}^d \to \mathbf{C}$ to the defocusing equation* (1.1)*, $\mu = 1$, with initial data*

$$u_n(0) = e^{it_n H} G_n S_n \phi.$$

This solution satisfies a spacetime bound

$$\limsup_{n\to\infty} S_I(u_n) \le C(E(u_n)).$$

Suppose in addition that $\{(q_k, r_k)\}$ is any finite collection of admissible pairs with $2 < r_k < d$. Then for each $\varepsilon > 0$ there exists $\psi^\varepsilon \in C_c^\infty(\mathbf{R} \times \mathbf{R}^d)$ such that

$$\limsup_{n\to\infty} \sum_k \| u_n - \tilde{G}_n [e^{-\frac{itN_n^{-2}|x_n|^2}{2}} \psi^\varepsilon] \|_{L_t^{q_k} \Sigma_x^{r_k}(I\times\mathbf{R}^d)} < \varepsilon. \tag{5.1}$$

Assuming also that $\|\nabla\phi\|_{L^2} < \|\nabla W\|_{L^2}$ and $E_\Delta(\phi) < E_\Delta(W)$, we have the same conclusion as above for the focusing equation (1.1)*, $\mu = -1$.*

The proof proceeds in several steps. First we construct an approximate solution on I in the sense of Proposition 3.3. Roughly speaking, when N_n is large and $t = O(N_n^{-2})$, solutions to (1.1) are well-approximated up to a phase factor by solutions to the energy-critical NLS with no potential, which by Conjecture 1.1 exist globally and scatter. In the long-time regime $N_n^{-2} << |t| \le 1$, the solution to (1.1) has dispersed and resembles a linear evolution $e^{-itH}\phi$. By patching these approximations together, we obtain an approximate solution over the entire time interval I with arbitrarily small error as N_n becomes large. It then follows by Proposition 3.3 that for n large, (1.1) admits a solution on I with controlled spacetime bound. The last claim about approximating the solution by functions in $C_c^\infty(\mathbf{R}\times\mathbf{R}^d)$ will follow essentially from our construction of the approximate solutions.

We shall need a commutator estimate. In the sequel, $P_{\le N}, P_N$ will denote the standard Littlewood-Paley projectors based on $-\Delta$.

Lemma 5.2. *Let v be a global solution to*

$$(i\partial_t + \tfrac{1}{2}\Delta)v = F(v),\ \ v(0)\in \dot H^1(\mathbf{R}^d)$$

where $F(z) = \pm|z|^{\frac{4}{d-2}}z$. Then on any compact time interval I,

$$\lim_{N\to\infty} \|P_{\le N}F(v) - F(P_{\le N}v)\|_{L_t^2 H_x^{1,\frac{2d}{d+2}}(I\times\mathbf{R}^d)} = 0$$

Proof. Recall [29, Lemma 3.11] that as a consequence of the spacetime bound (1.7), ∇v is finite in all Strichartz norms:

$$\|\nabla v\|_{S(\mathbf{R})} < C(\|v(0)\|_{\dot H^1}) < \infty. \tag{5.2}$$

It suffices to show separately that

$$\lim_{n\to\infty} \|P_{\le N}F(v) - F(P_{\le N}v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} = 0, \tag{5.3}$$

$$\lim_{n\to\infty} \|\nabla[P_{\le N}F(v) - F(P_{\le N}v)]\|_{L_t^2 L_x^{\frac{2d}{d+2}}} = 0. \tag{5.4}$$

Write

$$\begin{aligned}\|\nabla[P_{\le N}F(v) - F(P_{\le N}v)]\|_{L_t^2 L_x^{\frac{2d}{d+2}}} &\le \|\nabla P_{>N}F(v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\ &\quad + \|\nabla[F(v) - F(P_{\le N}v)]\|_{L_t^2 L_x^{\frac{2d}{d+2}}}.\end{aligned} \tag{5.5}$$

As $P_{>N} = 1 - P_{\le N}$ and

$$\|\nabla F(v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim \|v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \|\nabla v\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}} \le C(\|v(0)\|_{\dot H^1}),$$

dominated convergence implies that

$$\lim_{N\to\infty} \|\nabla P_{>N}F(v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} = 0.$$

To treat the second term on the right side of (5.5), observe first that with $F(z) = |z|^{\frac{4}{d-2}}z$,

$$|F_z(z) - F_z(w)| + |F_{\bar z}(z) - F_{\bar z}(w)| \lesssim \begin{cases} |z-w|(|z|^{\frac{6-d}{d-2}} + |w|^{\frac{6-d}{d-2}}), & 3\le d\le 5 \\ |z-w|^{\frac{4}{d-2}}, & d\ge 6.\end{cases}$$

Combining this with the pointwise bound

$$\begin{aligned}|\nabla[F(v) - F(P_{\le N}v)]| &\le (|F_z(v) - F_z(P_{\le N}v)| + |F_{\bar z}(v) - F_{\bar z}(P_{\le N}v)|)|\nabla v| \\ &\quad + (|F_z(P_{\le N}v)| + |F_{\bar z}(P_{\le N}v)|)|\nabla P_{>N}v|,\end{aligned}$$

Hölder, and dominated convergence, when $d\ge 6$ we have

$$\begin{aligned}&\|\nabla[F(v) - F(P_{\le N}v)]\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\ &\lesssim \||P_{>N}v|^{\frac{4}{d-2}}|\nabla v|\|_{L_t^2 L_x^{\frac{2d}{d+2}}} + \||P_{\le N}v|^{\frac{4}{d-2}}|\nabla P_{>N}v|\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\ &\lesssim \|P_{>N}v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \|\nabla v\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}} + \|v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \|P_{>N}\nabla v\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}} \\ &\to 0 \text{ as } N\to\infty.\end{aligned} \tag{5.6}$$

If $3\le d\le 5$, the first term in the second line of (5.6) is replaced by

$$\begin{aligned}&\||P_{>N}v|(|v|^{\frac{6-d}{d-2}} + |P_{\le N}v|^{\frac{6-d}{d-2}})|\nabla v|\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\ &\le \|P_{>N}v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \|v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{6-d}{d-2}} \|\nabla v\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}}\end{aligned}$$

which goes to 0 by dominated convergence. This establishes (5.4). The proof of (5.3)is similar. Write

$$\|P_{\le N}F(v) - F(P_{\le N}v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \le \|P_{>N}F(v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} + \|F(v) - F(P_{\le N}v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}}.$$

By Hölder, Bernstein, and the chain rule,

$$\|P_{>N}F(v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim N^{-1}\|v\|^{\frac{4}{d-2}}_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \|\nabla v\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}} = O(N^{-1}).$$

Using Bernstein, Hölder, and Sobolev embedding, and the pointwise bound

$$|F(v) - F(P_{\le N}v)| \lesssim |P_{>N}v|(|v|^{\frac{4}{d-2}} + |P_{\le N}v|^{\frac{4}{d-2}}),$$

we obtain

$$\begin{aligned}\|F(v) - F(P_{\le N}v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} &\le \|(|v|^{\frac{4}{d-2}} + |P_{\le N}v|^{\frac{4}{d-2}})P_{>N}v\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\ &\lesssim_{|I|} (\|\nabla v\|^{\frac{4}{d-2}}_{L_t^\infty L_x^2} + +\|\nabla v\|^{\frac{4}{d-2}}_{L_t^\infty L_x^2})\|\nabla P_{>N}v\|_{L_t^\infty L_x^2}.\end{aligned}$$

As $v \in C_t^0 \dot{H}_x^1(I \times \mathbf{R}^d)$, the orbit $\{v(t)\}_{t\in I}$ is compact in $\dot{H}^1(\mathbf{R}^d)$. The Riesz characterization of L^2 compactness therefore implies that the right side goes to 0 as $N \to \infty$. □

Now suppose that $\phi_n = e^{it_n H} G_n S_n \phi$ as in the statement of Proposition 5.1. If $\mu = -1$, assume also that $\|\phi\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, $E(\phi) < E_\Delta(W)$. We first construct functions $\tilde{v}_n$ which obey all of the conditions of the Proposition 3.3 except possibly the hypothesis in (3.2) about matching initial data. A slight modification of the $\tilde{v}_n$ will then yield genuine approximate solutions.

If $t_n \equiv 0$, let v be the solution to the potential-free problem (1.6) provided by Conjecture 1.1 with $v(0) = \phi$. If $N_n^2 t_n \to \pm\infty$, let v be the solution to (1.6) which scatters in $\dot{H}^1$ to $e^{\frac{it\Delta}{2}}\phi$ as $t \to \mp\infty$. Note the reversal of signs.

Put

$$\tilde{N}_n' = (\tfrac{N_n}{N_n'})^{\frac{1}{2}}, \tag{5.7}$$

let $T > 0$ denote a large constant to be chosen later, and define

$$\tilde{v}_n^T(t) = \begin{cases} e^{-\frac{it|x_n|^2}{2}} \tilde{G}_n[S_n P_{\le \tilde{N}_n'} v](t + t_n) & |t| \le TN_n^{-2} \\ e^{-i(t - TN_n^{-2})H}\tilde{v}_n^T(TN_n^{-2}), & TN_n^{-2} \le t \le 2 \\ e^{-i(t + TN_n^{-2})H}\tilde{v}_n^T(-TN_n^{-2}) & -2 \le t \le -TN_n^{-2}. \end{cases} \tag{5.8}$$

The awkward time translation by t_n is needed to undo the time translation built into the operator $\tilde{G}_n$; see (4.1). We shall suppress the superscript T unless the role of that parameter needs to be emphasized. Introducing the notation

$$\begin{aligned} v_n(t,x) &= [\tilde{G}_n v](t + t_n, x) = N_n^{\frac{d-2}{2}} v(N_n^2 t, N_n(x - x_n)), \\ \chi_n(x) &= \chi(N_n'(x - x_n)), \end{aligned}$$

where χ is the function used to define the spatial cutoff operator S_n in (4.2), and using the identity $\tilde{G}_n \chi = \chi_n \tilde{G}_n$, we can also write the top expression in (5.8) as

$$\tilde{v}_n(t) = e^{-\frac{it|x_n|^2}{2}} \chi_n P_{\le \tilde{N}_n' N_n} v_n, \quad |t| \le TN_n^{-2}.$$

As discussed previously, during the initial time window $\tilde{v}_n$ is essentially a modulated solution to (1.6) with cutoffs applied in both space, to place the solution in $C_t\Sigma_x$, and frequency, to enable taking an extra derivative in the error analysis below.

If $\phi \in \dot{H}^1$, use Lemma 4.2 and the fact that $\|v\|_{L_t^\infty \dot{H}_x^1} \le C(\|\phi\|_{\dot{H}^1})$ (energy conservation) to deduce

$$\limsup_n \|\tilde{v}_n\|_{L_t^\infty \Sigma_x(|t| \le TN_n^{-2})} \le C(\|\phi\|_{\dot{H}^1}),$$

therefore

$$\limsup_n \|\tilde{v}_n\|_{L_t^\infty \Sigma_x([-2,2])} \le C(\|\phi\|_{\dot{H}^1}). \tag{5.9}$$

From (1.7), (5.9), and Strichartz, we obtain

$$\|\tilde{v}_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}([-2,2]\times\mathbf{R}^d)} \le C(\|\phi\|_{\dot{H}^1}) \quad \text{for } n \text{ large.} \tag{5.10}$$

Due to mass conservation, a similar bound holds when $\phi \in H^1$. Now let

$$e_n = (i\partial_t - H)\tilde{v}_n - F(\tilde{v}_n).$$

We show that

$$\lim_{T\to\infty} \limsup_{n\to\infty} \|H^{\frac{1}{2}} e_n\|_{N([-2,2])} = 0, \tag{5.11}$$

so that by taking T large enough the $\tilde{v}_n$ will satisfy the second error condition in (3.2) for all n sufficiently large.

First we deal with the time interval $|t| \le TN_n^{-2}$.

Lemma 5.3. $\lim_{T\to\infty} \limsup_{n\to\infty} \|H^{\frac{1}{2}} e_n\|_{N(|t|\le TN_n^{-2})} = 0.$

Proof. When $-TN_n^{-2} \le t \le TN_n^{-2}$, compute

$$\begin{aligned} e_n &= e^{-\frac{it|x_n|^2}{2}} [\chi_n P_{\le \tilde{N}_n' N_n} F(v_n) - \chi_n^{\frac{d+2}{d-2}} F(P_{\le \tilde{N}_n' N_n} v_n) \\ &+ \frac{|x_n|^2 - |x|^2}{2} (P_{\le \tilde{N}_n' N_n} v_n)\chi_n + \frac{1}{2}(P_{\le \tilde{N}_n' N_n} v_n)\Delta\chi_n + (\nabla P_{\le \tilde{N}_n' N_n} v_n)\cdot\nabla\chi_n] \\ &= e^{-\frac{it|x_n|^2}{2}} [(a) + (b) + (c) + (d)], \end{aligned}$$

and estimate each term separately in the dual Strichartz space $N(\{|t| \le TN_n^{-2}\})$. Write

$$\begin{aligned} (a) &= \chi_n P_{\le \tilde{N}_n' N_n} F(v_n) - \chi_n^{\frac{d+2}{d-2}} F(P_{\le \tilde{N}_n' N_n} v_n) \\ &= \chi_n [P_{\le \tilde{N}_n' N_n} F(v_n) - F(P_{\le_{\tilde{N}_n' N_n}} v_n)] + \chi_n (1 - \chi_n^{\frac{4}{d-2}}) F(P_{\le \tilde{N}_n' N_n} v_n) \\ &= (a') + (a''). \end{aligned}$$

By the Leibniz rule and a change of variables,

$$\begin{aligned} &\|\nabla(a')\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le TN_n^{-2})} \\ &\le \|\nabla[P_{\le \tilde{N}_n'} F(v) - F(P_{\le \tilde{N}_n'} v)]\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le T)} \\ &+ \|[P_{\le \tilde{N}_n' N_n} F(v_n) - F(P_{\le \tilde{N}_n' N_n} v_n)]\nabla\chi_n\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le TN_n^{-2})}. \end{aligned} \tag{5.12}$$

By Lemma 5.2, the first term disappears in the limit as $n \to \infty$. That lemma also applies to the second term after a change of variables to give

$$\begin{aligned} &\|[P_{\le \tilde{N}_n' N_n} F(v_n) - F(P_{\le \tilde{N}_n' N_n} v_n)]\nabla\chi_n\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le TN_n^{-2})} \\ &\lesssim N_n' \|P_{\le \tilde{N}_n' N_n} F(v_n) - F(P_{\le \tilde{N}_n' N_n} v_n)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le TN_n^{-2})} \\ &\lesssim \tfrac{N_n'}{N_n} \|P_{\le \tilde{N}_n'} F(v) - F(P_{\le \tilde{N}_n'} v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le T)} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} \|\nabla(a')\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le TN_n^{-2})} = 0.$$

By changing variables, using the bound $|x_n| \lesssim N_n$, and referring to Lemma 5.2 once more,

$$\begin{aligned} &\||x|(a')\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim N_n \|P_{\le \tilde{N}_n' N_n} F(v_n) - F(P_{\le \tilde{N}_n' N_n} v_n)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le TN_n^{-2})} \\ &\lesssim \|P_{\le \tilde{N}_n'} F(v) - F(P_{\le \tilde{N}_n'} v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le T)} \to 0 \text{ as } n \to \infty. \end{aligned}$$

It follows from Lemma 2.2 that

$$\lim_{n\to\infty} \|H^{\frac{1}{2}}(a')\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le TN_n^{-2})} = 0.$$

To estimate (a''), we use the Leibniz rule, a change of variables, Hölder, Sobolev embedding, the bound (5.2), and dominated convergence to obtain

$$\begin{aligned}
&\|\nabla(a'')\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim \||P_{\le \tilde N_n N_n} v_n|^{\frac{4}{d-2}} \nabla P_{\le \tilde N_n' N_n} v_n\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le T N_n^{-2},\ |x-x_n|\sim (N_n')^{-1})} \\
&+ \tfrac{N_n'}{N_n} \|P_{\le \tilde N_n' N_n} v_n\|_{L_t^\infty L_x^{\frac{2d}{d-2}}}^{\frac{d+2}{d-2}} \\
&\lesssim \|\nabla v\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}} \|P_{\le \tilde N_n'} v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(|t|\le T,\ |x|\sim \frac{N_n}{N_n'})}^{\frac{4}{d-2}} + O(\tfrac{N_n'}{N_n}) \\
&\lesssim C(E(v))(\|P_{>\tilde N_n'} v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} + \|v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(|t|\le T, |x|\gtrsim \frac{N_n}{N_n'})})^{\frac{4}{d-2}} + O(\tfrac{N_n'}{N_n}) \\
&= o(1) + O(\tfrac{N_n'}{N_n}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\||x|(a'')\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \sim \|F(P_{\le \tilde N_n'} v)\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d}{d-2}}(|t|\le T, |x|\sim \frac{N_n}{N_n'})} \\
&\lesssim (\|P_{>\tilde N_n'} v\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d}{d-2}}(|t|\le T)} + \|v\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d}{d-2}}(|t|\le T, |x|\sim \frac{N_n}{N_n'})})^{\frac{d+2}{d-2}} \\
&= o(1).
\end{aligned}$$

Therefore

$$\lim_{N\to\infty} \|H^{\frac12}(a'')\|_{L_t^2 L_{t,x}^{\frac{2d}{d+2}}(|t|\le T N_n^{-2})} = 0$$

as well. This completes the analysis for (a).

For (b), note that on the support of the function we have $\big||x_n|^2 - |x|^2\big| = |x_n - x||x_n + x| \sim N_n (N_n')^{-1}$. Thus by Hölder and Sobolev embedding,

$$\begin{aligned}
\|\nabla(b)\|_{L_t^1 L_x^2(|t|\le T N_n^{-2})} &\lesssim \tfrac{N_n}{N_n'} \|\nabla P_{\le \tilde N_n' N_n} v_n\|_{L_t^1 L_x^2(|t|\le T N_n^{-2})} \\
&+ N_n \|P_{\le \tilde N_n' N_n} v_n\|_{L_t^1 L_x^2(|t|\le T N_n^{-2},\ |x-x_n|\sim (N_n')^{-1})} \\
&\lesssim (N_n' N_n)^{-1} \|\nabla v_n\|_{L_t^\infty L_x^2} \to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Using Hölder and Sobolev embedding, we have

$$\begin{aligned}
\||x|(b)\|_{L_t^1 L_x^2(|t|\le T N_n^{-2})} &\sim \tfrac{N_n^2}{N_n'} \|P_{\le \tilde N_n' N_n} v_n\|_{L_t^1 L_x^2(|t|\le T N_n^{-2}, |x-x_n|\lesssim (N_n')^{-1})} \\
&\lesssim \begin{cases} (N_n')^{-2} \|\nabla v_n\|_{L_t^\infty L_x^2}, & \lim_{n\to\infty} N_n^{-1}|x_n| = 0 \\ \|v_n\|_{L_t^\infty L_x^2} = O(N_n^{-1}), & \lim_{n\to\infty} N_n^{-1}|x_n| > 0, \end{cases}
\end{aligned}$$

which vanishes as $n\to\infty$ in either case. Thus $\|H^{1/2}(b)\|_{L_t^1 L_x^2} \to 0$. The term (c) is dealt with similarly. Finally, to estimate (d), apply Hölder, Bernstein, and the definition (5.7) of the frequency cutoffs $\tilde N_n'$ to obtain

$$\begin{aligned}
\|\nabla(d)\|_{L_t^1 L_x^2(|t|\le T N_n^{-2})} &\lesssim N_n' \||\nabla|^2 P_{\le \tilde N_n' N_n} v_n\|_{L_t^1 L_x^2} + \||\nabla P_{\le \tilde N_n' N_n} v_n|(|\nabla|^2 \chi_n)\|_{L_t^1 L_x^2} \\
&\lesssim \left[\left(\tfrac{N_n'}{N_n}\right)^{\frac12} + \left(\tfrac{N_n'}{N_n}\right)^2 \right] \|\nabla v_n\|_{L_t^\infty L_x^2} \to 0.
\end{aligned}$$

Using Hölder in time, we get

$$\||x|(d)\|_{L_t^1 L_x^2(|t|\le T N_n^{-2})} \lesssim \tfrac{N_n'}{N_n} \|\nabla v_n\|_{L_t^\infty L_x^2} \to 0.$$

This completes the proof of the lemma. □

Next, we estimate the error over the time intervals $[-2, T N_n^{-2}]$ and $[T N_n^{-2}, 2]$.

Lemma 5.4. $\lim_{T\to\infty} \limsup_{n\to\infty} \|H^{\frac12} e_n\|_{N([-2, T N_n^{-2}] \cup [T N_n^{-2}, 2])} = 0.$

Proof. We consider just the forward time interval as the other interval is treated similarly. Since $\tilde{v}_n^T$ solves the linear equation, the error e_n is just the nonlinear term:

$$e_n = (i\partial_t - H)\tilde{v}_n^T - F(\tilde{v}_n^T) = -F(\tilde{v}_n^T).$$

By the chain rule (Corollary 2.4) and Strichartz,

$$\|H^{\frac{1}{2}} e_n\|_{N([TN_n^{-2},2])} \lesssim \|\tilde{v}_n^T\|^{\frac{4}{d-2}}_{L_{t,x}^{\frac{2(d+2)}{d-2}}([TN_n^{-2},2])} \|\tilde{v}_n^T(TN_n^{-2})\|_\Sigma.$$

By definition $\tilde{v}_n^T(TN_n^{-2}) = e^{-\frac{iTN_n^{-2}|x_n|^2}{2}} \tilde{G}_n S_n P_{\le \tilde{N}_n'} v(TN_n^{-2} - t_n)$, so Lemma 4.2 implies that

$$\limsup_{n\to\infty} \|\tilde{v}_n^T(TN_n^{-2})\|_\Sigma \lesssim \begin{cases} \|v|_{L_t^\infty \dot{H}_x^1}, & \lim_{n\to\infty} N_n^{-1}|x_n| = 0, \\ \|v\|_{L_t^\infty H_x^1}, & \lim_{n\to\infty} N_n^{-1}|x_n| > 0 \end{cases}$$

is bounded in either case. Using Strichartz and interpolation, it suffices to show

$$\lim_{T\to\infty} \limsup_{n\to\infty} \|\tilde{v}_n^T\|_{L_T^\infty L_x^{\frac{2d}{d-2}}([TN_n^{-2},2])} = 0.$$

As we are assuming Conjecture 1.1, there exists $v_\infty \in \dot{H}^1$ so that

$$\lim_{t\to\infty} \|v(t) - e^{\frac{it\Delta}{2}} v_\infty\|_{\dot{H}_x^1} = 0;$$

if $v(0) \in H^1$ the same limit holds with respect to the H^1 norm. Then one also has

$$\lim_{t\to\infty} \limsup_{n\to\infty} \|P_{\le \tilde{N}_n'} v(t) - e^{\frac{it\Delta}{2}} v_\infty\|_{\dot{H}_x^1} = 0,$$

(with the obvious modification if $v(0) \in H^1$) and Lemma 4.2 implies that

$$\lim_{T\to\infty} \limsup_{n\to\infty} \|\tilde{v}_n(TN_n^{-2}) - e^{-\frac{iTN_n^{-2}|x_n|^2}{2}} G_n S_n (e^{\frac{iT\Delta}{2}} v_\infty)\|_\Sigma = 0.$$

An application of Strichartz and Corollary 4.9 yields

$$\begin{aligned} \tilde{v}_n(t) &= e^{-i(t-TN_n^{-2})H}[\tilde{v}_n(TN_n^{-2})] \\ &= e^{-i(t-TN_n^{-2})H}[e^{-\frac{iTN_n^{-2}|x_n|^2}{2}} G_n S_n e^{\frac{iT\Delta}{2}} v_\infty] + \text{error} \\ &= e^{-itH}[G_n S_n v_\infty] + \text{error} \end{aligned}$$

where $\lim_{T\to\infty} \limsup_{n\to\infty} \|\text{error}\|_\Sigma = 0$ uniformly in t. By Sobolev embedding,

$$\begin{aligned} &\lim_{T\to\infty} \limsup_{n\to\infty} \|\tilde{v}_n\|_{L_t^\infty L_x^{\frac{2d}{d-2}}([TN_n^{-2},2])} \\ &= \lim_{T\to\infty} \limsup_{n\to\infty} \|e^{-itH}[G_n S_n v_\infty]\|_{L_t^\infty L_x^{\frac{2d}{d-2}}([TN_n^{-2},2])}. \end{aligned}$$

A standard density argument using the dispersive estimate for e^{-itH} shows that the last limit is zero. □

Lemmas 5.3 and 5.4 together establish (5.11).

Lemma 5.5 (Matching initial data)**.** *Let $u_n(0) = e^{it_n H} G_n S_n \phi$ as in Proposition 5.1. Then*

$$\lim_{T\to\infty} \limsup_{n\to\infty} \|\tilde{v}_n^T(-t_n) - u_n(0)\|_\Sigma = 0.$$

Proof. If $t_n \equiv 0$, then by definition $\tilde{v}_n^T(0) = G_n S_n P_{\le N_n'} \phi$, so Lemma 4.2 and the definition (5.7) of the frequency parameter N_n' imply

$$\lim_{n\to\infty} \|\tilde{v}_n^T(0) - u_n(0)\|_\Sigma \lesssim \lim_{n\to\infty} \left\{ \begin{array}{ll} \|P_{>N_n'} \phi\|_{H^1}, & \lim_{n\to\infty} N_n^{-1}|x_n| > 0 \\ \|P_{>N_n'} \phi\|_{\dot{H}^1}, & \lim_{n\to\infty} N_n^{-1}|x_n| = 0 \end{array} \right\} = 0.$$

Next we consider the case $N_n^2 t_n \to \infty$; the case $N_n^2 t_n \to -\infty$ works similarly. Arguing as in the previous lemma and recalling that in this case, the solution v was chosen to scatter *backward* in time to $e^{\frac{it\Delta}{2}}\phi$, for n large we have

$$\tilde{v}_n^T(-t_n) = e^{it_n H}[G_n S_n \phi] + \text{error}$$

where $\lim_{T\to\infty} \limsup_{n\to\infty} \|\text{error}\|_\Sigma \to 0$. The claim follows. □

For each fixed $T > 0$, set

$$\tilde{u}_n^T(t) = \tilde{v}_n^T(t - t_n), \tag{5.13}$$

which is defined for $t \in [-1, 1]$. Then for a fixed large value of T, this is an approximate solution for all n sufficiently large in the sense of Proposition 3.3. Indeed, by (5.9) and (5.10), $\tilde{u}_n^T$ satisfy the hypotheses (3.1) with $E, L = C(\|\phi\|_{\dot{H}^1})$. Lemmas 5.3, 5.4, 5.5, Sobolev embedding, and Strichartz show that for any $\varepsilon > 0$, there exists $T > 0$ so that $\tilde{u}_n^T$ satisfies the hypotheses (3.2) for all large n. Invoking Proposition 3.3, we obtain the first claim of Proposition 5.1 concerning the existence of solutions.

The remaining assertion of Proposition 5.1 regarding approximation by smooth functions will follow from the next lemma. Recall the notation

$$\|f\|_{L_t^q \Sigma_x^r} = \|H^{\frac{1}{2}} f\|_{L_t^q L_x^r}.$$

Lemma 5.6. *Fix finitely many admissible (q_k, r_k) with $2 \le r_k < d$. For every $\varepsilon > 0$, there exists a smooth function $\psi^\varepsilon \in C_c^\infty(\mathbf{R} \times \mathbf{R}^d)$ such that for all k*

$$\limsup_{T\to\infty} \limsup_{n\to\infty} \|\tilde{v}_n^T - \tilde{G}_n[e^{-\frac{itN_n^{-2}|x_n|^2}{2}} \psi^\varepsilon](t + t_n)\|_{L_T^{q_k} \Sigma_x^{r_k}([-2,2])} < \varepsilon.$$

Proof. We continue using the notation defined at the beginning. Let

$$\tilde{w}_n^T = \begin{cases} e^{-\frac{it|x_n|^2}{2}} \tilde{G}_n[S_n v](t + t_n), & |t| \le TN_n^{-2} \\ e^{-i(t - TN_n^{-2})H}[\tilde{w}_n^T(TN_n^{-2})], & t \ge TN_n^{-2} \\ e^{-i(t + TN_n^{-2})H}[\tilde{w}_n^T(-TN_n^{-2})], & t \le -TN_n^{-2} \end{cases}$$

This is essentially $\tilde{v}_n^T$ in (5.8) without the frequency cutoffs. We see first that $\tilde{v}_n^T$ can be well-approximated by $\tilde{w}_n^T$ in spacetime:

$$\begin{aligned} &\limsup_{n\to\infty} \|\tilde{v}_n^T - \tilde{w}_n^T\|_{L_t^{q_k} \Sigma_x^{r_k}([-2,2])} = 0, \\ &\sup_{T>0} \limsup_{n\to\infty} \|\tilde{w}_n^T\|_{L_t^{q_k} \Sigma_x^{r_k}([-2,2])} < \infty. \end{aligned} \tag{5.14}$$

Indeed by dominated convergence,

$$\|\nabla(v - P_{\le \tilde{N}_n'} v)\|_{L_t^{q_k} L_x^{r_k}(\mathbf{R} \times \mathbf{R}^d)} \to 0 \text{ as } n \to \infty,$$

thus (5.14) follows from Lemma 4.2 and the Strichartz inequality for e^{-itH}.

A consequence of the dispersive estimate is that most of the spacetime norm of $\tilde{w}_n^T$ is concentrated in the time interval $|t| \le TN_n^{-2}$:

$$\lim_{T\to\infty} \limsup_{n\to\infty} \|\tilde{w}_n^T\|_{L_t^{q_k} \Sigma_x^{r_k}([-2, -TN_n^{-2}] \cup [TN_n^{-2}, 2])} = 0. \tag{5.15}$$

To see this, it suffices by symmetry to consider the forward interval. Recall that v scatters forward in $\dot{H}^1$ (and in H^1 if $v(0) \in H^1$) to some $e^{\frac{it\Delta}{2}} v_\infty$. By Lemma 4.2,

$$\lim_{T\to\infty} \limsup_{n\to\infty} \|(\tilde{G}_n S_n v(TN_n^{-2} + t_n) - G_n S_n(e^{\frac{iT\Delta}{2}} v_\infty)\|_\Sigma = 0.$$

By Strichartz,

$$\lim_{T\to\infty} \limsup_{n\to\infty} \|e^{\frac{iTN_n^{-2}|x_n|^2}{2}} \tilde{w}_n^T - e^{-i(t - TN_n^{-2})H}[G_n S_n(e^{\frac{iT\Delta}{2}} v_\infty)]\|_{L_t^{q_k} \Sigma_x^{r_k}([TN_n^{-2}, 2])} = 0$$

By Corollary 4.9 and Strichartz, for each $T > 0$ we have

$$\lim_{n\to\infty} \|e^{-i(t - TN_n^{-2})H}[G_n S_n(e^{\frac{iT\Delta}{2}} v_\infty)] - e^{\frac{iT(r_\infty)^2}{2}} e^{-itH}[G_n S_n v_\infty]\|_{L_t^{q_k} \Sigma_x^{r_k}} = 0.$$

For each $\varepsilon > 0$, choose $v_\infty^\varepsilon \in C_c^\infty$ such that $\|v_\infty - v_\infty^\varepsilon\|_{\dot{H}^1} < \varepsilon$. By the dispersive estimate,

$$\|e^{-itH}[G_n v_\infty^\varepsilon]\|_{L_t^{q_k} L_x^{r_k}([TN_n^{-2}, 2])} \lesssim T^{-\frac{1}{q_k}} \|v_\infty^\varepsilon\|_{L_x^{r_k'}}$$

Combining the above with Strichartz and Lemma 4.2, we get

$$\limsup_{n\to\infty} \|\tilde{w}_n^T\|_{L_t^{q_k} \Sigma_x^{r_k}([TN_n^{-2}, 2])} \lesssim o(1) + \varepsilon + O_{\varepsilon, q_k}(T^{-\frac{1}{q_k}}) \text{ as } T \to \infty.$$

Taking $T \to \infty$, we find

$$\limsup_{T\to\infty} \limsup_{n\to\infty} \|\tilde{w}_n^T\|_{L_t^{q_k}\Sigma_x^{r_k}([TN_n^{-2},2])} \lesssim \varepsilon$$

for any $\varepsilon > 0$, thereby establishing (5.15).

Choose $\psi^\varepsilon \in C_c^\infty(\mathbf{R}\times\mathbf{R}^d)$ such that $\sum_{k=1}^N \|v-\psi^\varepsilon\|_{L_t^{q_k}\dot{H}_x^{1,r_k}} < \varepsilon$. By combining Lemma 4.2 with (5.14) and (5.15), we get

$$\lim_{T\to\infty} \limsup_{n\to\infty} \|\tilde{v}_n(t,x) - e^{-\frac{it|x_n|^2}{2}} \tilde{G}_n \psi^\varepsilon(t+t_n)\|_{L_t^{q_k}\Sigma_x^{r_k}([-2,2])} \lesssim \varepsilon.$$

This completes the proof of the lemma, hence Proposition 5.1. □

Remark. From the proof it is clear that that the proposition also holds if the interval $I=[-1,1]$ is replaced by any smaller interval.

6. Palais-Smale and the proof of Theorem 1.2

In this section we prove a Palais-Smale-type compactness property for sequences of blowing up solutions to (1.1). This will quickly lead to Theorem 1.2.

For a maximal solution u to (1.1), define

$$S_*(u) = \sup\{S_I(u) : I \text{ is an open interval with } \ \leq 1\},$$

where we set $S_I(u)=\infty$ if u is not defined on I. All solutions in this section are assumed to be maximal. Set

$$\begin{aligned} \Lambda_d(E) &= \sup\{S_*(u) : u \text{ solves (1.1)}, \ \mu=+1, \ E(u)=E\} \\ \Lambda_f(E) &= \sup\{S_*(u) : u \text{ solves (1.1)}, \ \mu=-1, \ E(u)=E, \\ &\qquad \|\nabla u(0)\|_{L^2} < \|\nabla W\|_{L^2}\}. \end{aligned}$$

Finally, define

$$\mathcal{E}_d = \{E : \Lambda_d(E) < \infty\}, \quad \mathcal{E}_f = \{E : \Lambda_f(E) < \infty\}.$$

By the local theory, Theorem 1.2 is equivalent to the assertions

$$\mathcal{E}_d = [0,\infty), \quad \mathcal{E}_f = [0, E_\Delta(W)).$$

Suppose Theorem 1.2 failed. By the small data theory, $\mathcal{E}_d, \ \mathcal{E}_f$ are nonempty and open, and the failure of Theorem 1.2 implies the existence of a critical energy $E_c > 0$, with $E_c < E_\Delta(W)$ in the focusing case, such that $\Lambda_d(E), \ \Lambda_f(E) = \infty$ for $E > E_c$ and $\Lambda_d(E), \ \Lambda_f(E) < \infty$ for all $E < E_c$.

Define the spaces

$$\dot{X}^1 = \begin{cases} L_{t,x}^{10} \cap L_t^5 \Sigma_x^{\frac{30}{11}}([-\frac12,\frac12]\times\mathbf{R}^3), & d=3 \\ L_{t,x}^{\frac{2(d+2)}{d-2}} \cap L_t^{\frac{2(d+2)}{d}} \Sigma_x^{\frac{2(d+2)}{d}}([-\frac12,\frac12]\times\mathbf{R}^d), & d\geq 4. \end{cases}$$

When $d=3$, also define

$$\dot{Y}^1 = \dot{X}^1 \cap L_t^{\frac{10}{3}} \Sigma_x^{\frac{10}{3}}([-\tfrac12,\tfrac12]\times\mathbf{R}^3).$$

Remark. The case $d=3$ is singled out for technical reasons. Our choice of Strichartz norm $L_t^5\Sigma_x^{30/11}$ is guided by the fact that $\frac{30}{11} < 3$, which is needed for Sobolev embedding. In higher dimensions the symmetric Strichartz norm suffices since $\frac{2(d+2)}{d} < d$ for all $d \geq 4$. This distinction necessitates a separate but essentially parallel treatment of various estimates when $d=3$.

Proposition 6.1 (Palais-Smale)**.** *Assume Conjecture 1.1 holds. Suppose that $u_n : (t_n - \frac12, t_n + \frac12)\times\mathbf{R}^d \to \mathbf{C}$ is a sequence of solutions with*

$$\lim_{n\to\infty} E(u_n) = E_c, \ \lim_{n\to\infty} S_{(t_n-\frac12,t_n]}(u_n) = \lim_{n\to\infty} S_{[t_n,t_n+\frac12)}(u_n) = \infty.$$

In the focusing case, assume also that $E_c < E_\Delta(W)$ and $\|\nabla u_n(t_n)\|_{L^2} < \|\nabla W\|_{L^2}$. Then there exists a subsequence such that $u_n(t_n)$ converges in Σ.

Let us first see how this would imply the main theorem.

Proof of Theorem 1.2. Suppose the theorem failed. In the defocusing case, there exist $E_c \in (0, \infty)$ and a sequence of solutions u_n with $E(u_n) \to E_c$, $S_{(-\frac{1}{4},0]}(u_n) \to \infty$, and $S_{[0,\frac{1}{4})}(u_n) \to \infty$. The same is true in the focusing case except E_c is restricted to the interval $(0, E_\Delta(W))$ and $\limsup_n \|u_n(0)\|_{\dot H^1} < \|W\|_{\dot H^1}$. By Proposition 6.1, after passing to a subsequence $u_n(0)$ converges in Σ to some ϕ. Then $E(\phi) = \lim_n E(u_n(0)) = E_c$.

Let $u_\infty : (-T_{min}, T_{max}) \to \mathbf{C}$ be the maximal lifespan solution to (1.1) with $u_\infty(0) = \phi$. By comparing u with the u_n and applying Proposition 3.3, we see that $S_{([0,\frac{1}{2})}(u_\infty) = S_{(-\frac{1}{2},0]}(u_\infty) = \infty$. So $-1/2 \le -T_{min} < T_{max} \le 1/2$. But Proposition 6.1 implies that the orbit $\{u_\infty(t) : t \in (-T_{min}, T_{max})\}$ is precompact in Σ, thus there is a sequence of times t_n increasing to T_{max} such that $u_\infty(t_n)$ converges in Σ to some ψ. Taking a local solution with initial data equal to ψ, we can then invoke Proposition 3.3 to extend u_∞ to some larger interval $(-T_{min}, T_{max} + \eta)$, contradicting the maximality of u_∞. □

Proof of Proposition 6.1. By replacing $u_n(t)$ with $u_n(t + t_n)$, we may assume $t_n \equiv 0$. Note that by energy conservation and Corollary 7.2, this time translation does not change the hypotheses of the focusing case.

Observe (referring to the discussion in Section 7 for the focusing case) that the sequence $u_n(0)$ is bounded in Σ. Applying Proposition 4.14, after passing to a subsequence we have a decomposition

$$u_n(0) = \sum_{j=1}^J e^{it_n^j H} G_n S_n \phi^j + w_n^J = \sum_{j=1}^J \phi_n^j + w_n^J$$

with the properties stated in that proposition. In particular, the remainder has asymptotically trivial linear evolution:

$$\lim_{J \to J^*} \limsup_{n \to \infty} \|e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}, \tag{6.1}$$

and the energies asymptotically decouple:

$$\sup_J \lim_{n \to \infty} |E(u_n) - \sum_{j=1}^J E(\phi_n^j) - E(w_n^J)| = 0. \tag{6.2}$$

Observe that $\liminf_n E(\phi_n^j) \ge 0$. This is obvious in the defocusing case. In the focusing case, (4.15) and the discussion in Section 7 imply that

$$\sup_j \limsup_n \|\phi_n^j\|_\Sigma \le \|u_n\|_\Sigma < \|\nabla W\|_{L^2},$$

so the claim follows from Lemma 7.1. Therefore, there are two possibilities.

Case 1: $\sup_j \limsup_{n\to\infty} E(\phi_n^j) = E_c$.

By combining (6.2) with the fact that the profiles ϕ_n^j are nontrivial in Σ, it follows that $J^* = 1$ and

$$u_n(0) = e^{it_n H} G_n S_n \phi + w_n, \ \lim_{n \to \infty} \|w_n\|_\Sigma = 0.$$

We argue that $N_n \equiv 1$ (thus $x_n = 0$ and $t_n = 0$). Suppose $N_n \to \infty$.

Proposition 5.1 implies that for all large n, there exists a unique solution u_n on $[-\frac{1}{2}, \frac{1}{2}]$ with $u_n(0) = e^{it_n H} G_n S_n \phi$ and $\limsup_{n\to\infty} S_{(-\frac{1}{2},\frac{1}{2})}(u_n) \le C(E_c)$. By perturbation theory (Proposition 3.3),

$$\limsup_{n\to\infty} S_{[-\frac{1}{2},\frac{1}{2}]}(u_n) \le C(E_c),$$

which is a contradiction. Therefore, $N_n \equiv 1$, $t_n^j \equiv 0$, $x_n^j \equiv 0$, and

$$u_n(0) = \phi + w_n$$

for some $\phi \in \Sigma$. This is the desired conclusion.

Case 2: $\sup_j \limsup_{n\to\infty} E(\phi_n^j) \le E_c - 2\delta$ for some $\delta > 0$.

By the definition of E_c, there exist solutions $v_n^j : (-\frac{1}{2}, \frac{1}{2}) \times \mathbf{R}^d \to \mathbf{C}$ with

$$\|v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}([-\frac{1}{2},\frac{1}{2}])} \lesssim_{E_c,\delta} E(\phi_n^j)^{\frac{1}{2}}.$$

By standard arguments (c.f. [29, Lemma 3.11]), this implies the seemingly stronger bound

$$\|v_n^j\|_{\dot X^1} \lesssim_{E_c,\delta} E(\phi_n^j)^{\frac{1}{2}}. \tag{6.3}$$

In the case $d = 3$, we also have $\|v_n^j\|_{\dot{Y}^1} \lesssim E(\phi_n^j)^{\frac{1}{2}}$. Put

$$u_n^J = \sum_{j=1}^J v_n^j + e^{-itH} w_n^J. \tag{6.4}$$

We claim that for sufficiently large J and n, u_n^J is an approximate solution in the sense of Proposition 3.3. To prove this claim, we check that u_n^J has the following three properties:

(i) $\lim_{J\to J^*} \limsup_{n\to\infty} \|u_n^J(0) - u_n(0)\|_\Sigma = 0$.
(ii) $\limsup_{n\to\infty} \|u_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}([-T,T])} \lesssim_{E_c,\delta} 1$ uniformly in J.
(iii) $\lim_{J\to J^*} \limsup_{n\to\infty} \|H^{\frac{1}{2}} e_n^J\|_{N([-\frac{1}{2},\frac{1}{2}])} = 0$, where

$$e_n = (i\partial_t - H)u_n^J - F(u_n^J).$$

There is nothing to check for part (i) as $u_n^J(0) = u_n(0)$ by construction. The verification of (ii) relies on the asymptotic decoupling of the nonlinear profiles v_n^j, which we record in the following two lemmas.

Lemma 6.2 (Orthogonality). *Suppose that two frames $\mathcal{F}^j = (t_n^j, x_n^j, N_n^j)$, $\mathcal{F}^k = (t^k, x_n^k, N_n^k)$ are orthogonal, and let $\tilde{G}_n^j$, $\tilde{G}_n^k$ be the associated spacetime scaling and translation operators as defined in* (4.1). *Then for all ψ^j, ψ^k in $C_c^\infty(\mathbf{R}\times\mathbf{R}^d)$,*

$$\|(\tilde{G}_n^j\psi^j)(\tilde{G}_n^k\psi^k)\|_{L_{t,x}^{\frac{d+2}{d-2}}} + \|(\tilde{G}_n^j\psi^j)\nabla(\tilde{G}_n^k\psi^k)\|_{L_{t,x}^{\frac{d+2}{d-1}}} + \||x|(\tilde{G}_n^j\psi^j)(\tilde{G}_n^k\psi^k)\|_{L_{t,x}^{\frac{d+2}{d-1}}}$$
$$+ \||x|^2(\tilde{G}_n^j\psi^j)(\tilde{G}_n^k\psi^k)\|_{L_{t,x}^{\frac{d+2}{d}}} + \|(\nabla\tilde{G}_n^j\psi^j)(\nabla\tilde{G}_n^k\psi^k)\|_{L_{t,x}^{\frac{d+2}{d}}} \to 0$$

as $n\to\infty$. When $d = 3$, we also have

$$\||x|^2(\tilde{G}_n^j\psi^j)(\tilde{G}_n^k\psi^k)\|_{L_t^5 L_x^{\frac{15}{11}}} + \|(\nabla\tilde{G}_n^j\psi^j)(\nabla\tilde{G}_n^k\psi^k)\|_{L_t^5 L_x^{\frac{15}{11}}} \to 0.$$

Proof. The arguments for each term are similar, and we only supply the details for the second term. Suppose $N_n^k(N_n^j)^{-1} \to \infty$. By the chain rule, a change of variables, and Hölder,

$$\|(\tilde{G}_n^j\psi^j)\nabla(\tilde{G}_n^k\psi^k)\|_{L_{t,x}^{\frac{d+2}{d-1}}} = \|\psi^j\nabla(\tilde{G}_n^j)^{-1}\tilde{G}_n^k\psi^k\|_{L_{t,x}^{\frac{d+2}{d-1}}}$$
$$\le \|\psi^j\chi_n\|_{L_{t,x}^{\frac{2(d+2}{d-2}}} \|\nabla\psi^k\|_{L_{t,x}^{\frac{2(d+2)}{d}}},$$

where χ_n is the characteristic function of the support of $\nabla(\tilde{G}_n^j)^{-1}\tilde{G}_n^k\psi^k$. As the support of χ_n has measure shrinking to zero, we have

$$\lim_{n\to\infty} \|\psi^j\chi_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} = 0.$$

A similar argument deals with the case where $N_n^j(N_n^k)^{-1} \to \infty$. Therefore, we may suppose that

$$\frac{N_n^k}{N_n^j} \to N_\infty \in (0,\infty).$$

Make the same change of variables as before, and compute

$$\nabla(\tilde{G}_n^j)^{-1}\tilde{G}_n^k\psi^k(t,x) = (\tfrac{N_n^k}{N_n^j})^{\frac{d}{2}}(\nabla\psi^k)[\tfrac{N_n^k}{N_n^j}t + (N_n^k)^2(t_n^j - t_n^k), \tfrac{N_n^k}{N_n^j}x + N_n^k(x_n^j - x_n^k)].$$

The decoupling statement (4.17) implies that

$$(N_n^k)^2(t_n^j - t_n^k) + N_n^k|x_n^j - x_n^k| \to \infty.$$

Therefore, the supports of ψ^j and $\nabla(\tilde{G}_n^j)^{-1}\tilde{G}_n^k\psi^k$ are disjoint for large n. □

Lemma 6.3 (Decoupling of nonlinear profiles). *Let v_n^j be the nonlinear solutions defined above. Then when $d \ge 4$,*

$$\|v_n^jv_n^k\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} + \|v_n^j\nabla v_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}} + \||x|v_n^jv_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}}$$
$$+ \|(\nabla v_n^j)(\nabla v_n^k)\|_{L_{t,x}^{\frac{2(d+2)}{d}}} + \||x|^2v_n^jv_n^k\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \to 0$$

as $n \to \infty$. When $d = 3$, the same statement holds with the last two expressions replaced by

$$\|(\nabla v_n^j)(\nabla v_n^k)\|_{L_t^5 L_x^{\frac{15}{11}}} + \||x|^2 v_n^j v_n^k\|_{L_t^5 L_x^{\frac{30}{11}}} \to 0.$$

Proof. We spell out the details for the $\|v_n^j |x| v_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}}$ term. Consider first the case $d \geq 4$. As $2 < \frac{2(d+2)}{d} < d$, by Proposition 5.1 we can approximate v_n^j in $\dot{X}^1$ by test functions

$$c_n^j \tilde{G}_n \psi^j, \quad \psi^j \in C_c^\infty(\mathbf{R} \times \mathbf{R}^d), \quad c_n^j(t) = e^{-\frac{i(t-t_n^j)|x_n^j|^2}{2}}.$$

By Hölder and a change of variables,

$$\begin{aligned}
&\|v_n^j |x| v_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}} \leq \|(v_n^j - c_n^j \tilde{G}_n^j \psi^j)|x| v_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}} \\
&+ \||x| \tilde{G}_n^j \psi^j (v_n^k - c_n^k \tilde{G}_n^k \psi^k)\|_{L_{t,x}^{\frac{d+2}{d-1}}} + \||x| \tilde{G}_n^j \psi^j \tilde{G}_n^k \psi^k\|_{L_{t,x}^{\frac{d+2}{d-1}}} \\
&\leq \|(v_n^j - c_n^j \tilde{G}_n^j \psi^j)\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \|v_n^k\|_{\dot{X}^1} \\
&+ \|\psi^j\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \|(v_n^k - c_n^k \tilde{G}_n^k \psi^k)\|_{\dot{X}^1} + \|(\tilde{G}_n^j \psi^j)|x|(\tilde{G}_n^k \psi^k)\|_{L_{t,x}^{\frac{d+2}{d-1}}}
\end{aligned}$$

By first choosing ψ^j, then ψ^k, then invoking the previous lemma, we obtain for any $\varepsilon > 0$ that

$$\limsup_{n \to \infty} \|v_n^j |x| v_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}} \leq \varepsilon.$$

When $d = 3$, we also approximate v_n^j in $\dot{X}^1$ (which is possible because the exponent $\frac{30}{11}$ in the definition of $\dot{X}^1$ is less than 3), and estimate

$$\begin{aligned}
&\|v_n^j |x| v_n^k\|_{L_{t,x}^{\frac{5}{2}}} \\
&\leq \|(v_n^j - c_n^j \tilde{G}_n^j \psi^j)|x| v_n^k\|_{L_{t,x}^{\frac{5}{2}}} + \||x| \tilde{G}_n^j \psi^j (v_n^k - c_n^k \tilde{G}_n^k \psi^k)\|_{L_{t,x}^{\frac{5}{2}}} + \||x| \tilde{G}_n^j \psi^j \tilde{G}_n^k \psi^k\|_{L_{t,x}^{\frac{5}{2}}} \\
&\leq \|(v_n^j - c_n^j \tilde{G}_n^j \psi^j)\|_{L_{t,x}^{10}} \|v_n^k\|_{\dot{Y}^1} \\
&+ \|\psi^j\|_{L_t^5 L_x^{30}} \|v_n^k - c_n^k \tilde{G}_n^k \psi^k\|_{\dot{X}^1} + \|(\tilde{G}_n^j \psi^j)|x|(\tilde{G}_n^k \psi^k)\|_{L_{t,x}^{\frac{5}{2}}}
\end{aligned}$$

which, just as above, can be made arbitrarily small as $n \to \infty$. Similar approximation arguments deal with the other terms. □

Let us verify Claim (ii) above. In fact we shall show that

$$\limsup_{n \to \infty} \|u_n^J\|_{\dot{X}^1([-\frac{1}{2}, \frac{1}{2}])} \lesssim_{E_c, \delta} 1 \text{ uniformly in } J. \tag{6.5}$$

First, observe that

$$S(u_n^J) = \iint |\sum_{j=1}^J v_n^j + e^{-itH} w_n^J|^{\frac{2(d+2)}{d-2}} \, dx dt \lesssim S(\sum_{j=1}^J v_n^j) + S(e^{-itH} w_n^J).$$

By the properties of the LPD, $\lim_{J \to J^*} \limsup_{n \to \infty} S(e^{-itH} w_n^J) = 0$. Recalling (6.3), write

$$\begin{aligned}
S(\sum_{j=1}^J v_n^j) &= \Big\|(\sum_{j=1}^J v_n^j)^2\Big\|_{L_{t,x}^{\frac{d+2}{d-2}}}^{\frac{d+2}{d-2}} \leq (\sum_{j=1}^J \|v_n^j\|^2_{L_{t,x}^{\frac{2(d+2)}{d-2}}} + \sum_{j \neq k} \|v_n^j v_n^k\|_{L_{t,x}^{\frac{d+2}{d-2}}})^{\frac{d+2}{d-2}} \\
&\lesssim (\sum_{j=1}^J E(\phi_n^j) + o_J(1))^{\frac{d+2}{d-2}}
\end{aligned}$$

where the last line used Lemma 6.3. As energy decoupling implies $\limsup_{n \to \infty} \sum_{j=1}^J E(\phi_n^j) \leq E_c$, we obtain $\lim_{J \to J^*} \limsup_{n \to \infty} S(u_n^J) \lesssim_{E_c, \delta} 1$.

By mimicking this argument one also obtains

$$\limsup_{n \to \infty} (\|\nabla u_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}} + \||x| u_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}}) \lesssim_{E_c, \delta} 1 \text{ uniformly in } J.$$

Property (ii) is therefore verified in the case $d \geq 4$. The case $d = 3$ is dealt with similarly.

Remark. The above argument shows that for each J and each $\eta > 0$, there exists $J' \leq J$ such that

$$\limsup_{n\to\infty} \| \sum_{j=J'}^{J} v_n^j \|_{\dot{X}^1([-\frac12,\frac12])} \leq \eta.$$

It remains to check property (iii) above, namely, that

$$\lim_{J\to J^*} \limsup_{n\to\infty} \|H^{1/2} e_n^J\|_{N([-\frac12,\frac12])} = 0. \tag{6.6}$$

Let $F(z) = |z|^{\frac{4}{d-2}} z$ and decompose

$$e_n^J = [\sum_{j=1}^{J} F(v_n^j) - F(\sum_{j=1}^{J} v_n^j)] + [F(u_n^J - e^{-itH} w_n^J) - F(u_n^J)] = (a) + (b). \tag{6.7}$$

Consider (a) first. Suppose $d \geq 6$. Using the chain rule $\nabla F(u) = F_z(u)\nabla u + F_{\overline{z}}(u)\overline{\nabla u}$ and the estimates

$$|F_z(z)| + |F_{\overline{z}}(z)| = O(|z|^{\frac{4}{d-2}}),\ |F_z(z) - F_z(w)| + |F_{\overline{z}}(z) - F_{\overline{z}}(w)| = O(|z-w|^{\frac{4}{d-2}}),$$

we compute

$$|\nabla (a)| \lesssim \sum_{j=1}^{J} \sum_{k\neq j} |v_n^k|^{\frac{4}{d-2}} |\nabla v_n^j|.$$

By Hölder, Lemma 6.3, and the induction hypothesis (6.3),

$$\|\nabla(a)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \lesssim \sum_{j=1}^{J}\sum_{k\neq j} \||v_n^k||\nabla v_n^j|\|_{L_{t,x}^{\frac{d+2}{d-1}}}^{\frac{4}{d-2}} \|\nabla v_n^k\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{d-6}{d-2}} = o_J(1)$$

as $n \to \infty$. When $3 \leq d \leq 5$, we have instead

$$|\nabla(a)| \lesssim \sum_{j=1}^{J}\sum_{k\neq j} |v_n^k||\nabla v_n^j| O(\Big|\sum_{k=1}^{J} v_n^k\Big|^{\frac{6-d}{d-2}} + |v_n^j|^{\frac{6-d}{d-2}}),$$

thus

$$\|\nabla(a)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \lesssim_J \left(\sum_{j=1}^{J} \|v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{6-d}{d-2}} \right) \sum_{j=1}^{J}\sum_{k\neq j} \||v_n^k||\nabla v_n^j|\|_{L_{t,x}^{\frac{d+2}{d-1}}} = o_J(1).$$

Similarly, writing

$$|(a)| \leq \sum_{j=1}^{J} \Big| |v_n^j|^{\frac{4}{d-2}} - |\sum_{k=1}^{J} v_n^k|^{\frac{4}{d-2}} \Big| |v_n^j| \lesssim \sum_{j=1}^{J}\sum_{k\neq j} |v_n^j||v_n^k|^{\frac{4}{d-2}},$$

we have

$$\|x(a)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \sum_{j=1}^{J}\sum_{k\neq j} \||x|v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{d-6}{d-2}} \||x|v_n^j v_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}}^{\frac{4}{d-2}} = o_J(1).$$

When $3 \leq d \leq 5$,

$$|(a)| \lesssim \sum_{j=1}^{J}\sum_{k\neq j} |v_n^j| v_n^k| O(\Big|\sum_{k=1}^{J} v_n^k\Big|^{\frac{6-d}{d-2}} + |v_n^j|^{\frac{6-d}{d-2}}),$$

hence also

$$\||x|(a)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} = o_J(1).$$

Summing up,

$$\|H^{1/2}(a)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \lesssim \|\nabla(a)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} + \|x(a)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} = o_J(1).$$

Next we estimate (b), restricting temporarily to dimensions $d \geq 4$. When $d \geq 6$, write

$$\begin{aligned}(b) &= F(u_n^J - e^{-itH} w_n^J) - F(u_n^J) \\ &= (|u_n^J - e^{-itH} w_n^J|^{\frac{4}{d-2}} - |u_n^J|^{\frac{4}{d-2}}) \sum_{j=1}^J v_n^j - (e^{-itH} w_n^J)|u_n^J|^{\frac{4}{d-2}} \\ &= O(|e^{-itH} w_n^J|^{\frac{4}{d-2}}) \sum_{j=1}^J v_n^j - (e^{-itH} w_n^J)|u_n^J|^{\frac{4}{d-2}},\end{aligned}$$

and apply Hölder's inequality:

$$\begin{aligned}\||x|(b)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} &\lesssim \|e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \|\sum_{j=1}^J |x| v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \\ &\quad + \||x| u_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{4}{d-2}} \||x| e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{d-6}{d-2}} \|e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}}\end{aligned} \tag{6.8}$$

When $d = 4, 5$,

$$(b) = (e^{-itH} w_n^J) O(|u_n^J|^{\frac{6-d}{d-2}} + |u_n^J - e^{-itH} w_n^J|^{\frac{6-d}{d-2}}) \sum_{j=1}^J v_n^j - (e^{-itH} w_n^J)|u_n^J|^{\frac{4}{d-2}},$$

thus

$$\begin{aligned}&\||x|(b)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \\ &\lesssim \|e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \||x| \sum_{j=1}^J v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d}}} (\|u_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{6-d}{d-2}} + \|e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{6-d}{d-2}}) \\ &+ \|e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \|x u_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \|u_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{6-d}{d-2}}.\end{aligned}$$

Using (6.5), Strichartz, and the decay property (6.1), we get

$$\lim_{J \to J*} \limsup_{n \to \infty} \||x|(b)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} = 0.$$

It remains to bound $\nabla(b)$. By the chain rule,

$$\begin{aligned}\nabla(b) &\lesssim |e^{-itH} w_n^J|^{\frac{4}{d-2}} \Big|\sum_{j=1}^J \nabla v_n^j\Big| + |u_n^J|^{\frac{4}{d-2}} |\nabla e^{-itH} w_n^J| \\ &= (b') + (b'').\end{aligned}$$

The first term (b') can be handled in the manner of (6.8) above. To deal with (b''), fix a small parameter $\eta > 0$, and use the above remark to obtain $J' = J'(\eta) \leq J$ such that

$$\|\sum_{j=J'}^J v_n^j\|_{\dot{X}^1} \leq \eta.$$

By the subadditivity of $z \mapsto |z|^{\frac{4}{d-2}}$ (which is true up to a constant when $d = 4, 5$) and Hölder,

$$\begin{aligned}&\|(b'')\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} = \||\sum_{j=1}^J v_n^j + e^{-itH} w_n^J|^{\frac{4}{d-2}} |\nabla e^{-itH} w_n^J|\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \\ &\lesssim \|e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \|H^{1/2} e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}} + \|\sum_{j=J'}^J v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \|H^{1/2} e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \\ &+ C_{J'} \sum_{j=1}^{J'-1} \|\nabla e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{d-6}{d-2}} \||v_n^j| |\nabla e^{-itH} w_n^J|\|_{L_{t,x}^{\frac{d+2}{d-1}}}^{\frac{4}{d-2}}.\end{aligned}$$

By Strichartz and the decay of $e^{-itH}w_n^J$ in $L_{t,x}^{\frac{2(d+2)}{d-2}}$, the first term goes to 0 as $J \to \infty,\ n \to \infty$. By Strichartz and the definition of J', the second term is bounded by

$$\eta^{\frac{4}{d-2}} \|w_n^J\|_\Sigma$$

which can be made arbitrarily small since $\limsup_{n\to\infty} \|w_n^J\|_\Sigma$ is bounded uniformly in J. To finish, we check that for each fixed j

$$\lim_{J\to J^*} \limsup_{n\to\infty} \| |v_n^j| \nabla e^{-itH} w_n^J\|_{L_{t,x}^{\frac{d+2}{d-1}}} = 0. \tag{6.9}$$

For any $\varepsilon > 0$, there exist $\psi^j \in C_c^\infty(\mathbf{R} \times \mathbf{R}^d)$ such that if

$$c_n^j = e^{-\frac{i(t-t_n^j)|x_n^j|^2}{2}}$$

then

$$\limsup_{n\to\infty} \|v_n^j - c_n^j \tilde{G}_n^j \psi^j\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}([-\frac12,\frac12])} < \varepsilon,$$

Note that $\tilde{G}_n^j \psi^j$ is supported on the set

$$\{|t - t_n^j| \lesssim (N_n^j)^{-2},\ |x - x_n^j| \lesssim (N_n^j)^{-1}\}.$$

Thus for all n sufficiently large,

$$\begin{aligned}
&\|v_n^j \nabla e^{-itH} w_n^J\|_{L_{t,x}^{\frac{d+2}{d-1}}} \\
&\le \|v_n^j - c_n^j \tilde{G}_n^j \psi^j\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \|\nabla e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}} + \|\tilde{G}_n^j \psi^j \nabla e^{-itH} w_n^J\|_{L_{t,x}^{\frac{d+2}{d-1}}} \\
&\lesssim_{E_c} \varepsilon + \|(\tilde{G}_n^j \psi^j) \nabla e^{-itH} w_n^J\|_{L_{t,x}^{\frac{d+2}{d-1}}}.
\end{aligned}$$

By Hölder, noting that $\frac{d+2}{d-1} \le 2$ whenever $d \ge 4$,

$$\begin{aligned}
\|(\tilde{G}_n^j \psi^j) \nabla e^{-itH} w_n^J\|_{L_{t,x}^{\frac{d+2}{d-1}}} &\lesssim_\varepsilon (N_n^j)^{\frac{d-2}{2}} \|\nabla e^{-itH} w_n^J\|_{L_{t,x}^{\frac{d+2}{d-1}}(|t-t_n^j|\lesssim (N_n^j)^{-2}, |x-x_n^j|\lesssim (N_n^j)^{-1})} \\
&\lesssim N_n^j \|\nabla e^{-itH} w_n^J\|_{L_{t,x}^2(|t-t_n^j|\lesssim (N_n^j)^{-2}, |x-x_n^j|\lesssim (N_n^j)^{-1})};
\end{aligned}$$

Corollary 2.10 implies

$$\|v_n^j \nabla e^{-itH} w_n^J\|_{L_{t,x}^{\frac{d+2}{d-1}}} \lesssim \varepsilon + C_{\varepsilon,E_c} \|e^{-itH} w_n^J\|^{\frac13}_{L_{t,x}^{\frac{2(d+2)}{d-2}}}.$$

Sending $n \to \infty$, then $J \to J^*$, then $\varepsilon \to 0$ establishes (6.9), and with it, Property (iii).

When $d = 3$, we estimate (b) in (6.7) instead in the $L_t^{\frac53} L_x^{\frac{30}{23}}$ dual Strichartz norm. Write

$$(b) = (e^{-itH} w_n^J) v_n^j O(|u_n^J|^3 + |u_n^J - e^{-itH} w_n^J|^3) \sum_{j=1}^J v_n^j - (e^{-itH} w_n^J)|u_n^J|^4,$$

and apply Hölder's inequality:

$$\begin{aligned}
\| |x| (b)\|_{L_t^{\frac53} L_x^{\frac{30}{23}}} &\lesssim \|e^{-itH} w_n^J\|_{L_{t,x}^{10}} \|u_n^J\|^3_{L_{t,x}^{10}} \|H^{1/2} u_n^J\|_{L_t^5 L_x^{\frac{30}{11}}} \\
&\quad + \|e^{-itH} w_n^J\|_{L_{t,x}^{10}} (\|u_n^J\|^3_{L_{t,x}^{10}} + \|e^{-itH} w_n^J\|^3_{L_{t,x}^{10}}) \|H^{\frac12} \sum_{j=1}^J v_n^j\|_{L_t^5 L_x^{\frac{30}{11}}}.
\end{aligned} \tag{6.10}$$

Using (6.1) and (6.5), we have

$$\lim_{J\to J*} \limsup_{n\to\infty} \| |x| (b)\|_{L_t^{\frac53} L_x^{\frac{30}{11}}} = 0.$$

It remains to bound $\nabla(b)$. By the chain rule,

$$\nabla(b) = O\left((|u_n^J - e^{-itH} w_n^J|^4 - |u_n^J|^4) \nabla \sum_{j=1}^J v_n^j \right) + |u_n^J|^4 |\nabla e^{-itH} w_n^J|$$

$$= (b') + (b'').$$

The first term (b') can be treated in the manner of $\| |x| (b) \|_{L_t^{\frac{5}{3}} L_x^{\frac{30}{23}}}$ above. We now concern ourselves with (b''). Fix a small parameter $\eta > 0$, and use the above remark to obtain $J' = J'(\eta) \le J$ such that

$$\| \sum_{j=J'}^{J} v_n^j \|_{\dot{X}^1} \le \eta.$$

Thus by the triangle inequality and Hölder,

$$\begin{aligned}
\|(b'')\|_{L_t^{\frac{5}{3}} L_x^{\frac{30}{23}}} &= \| |\sum_{j=1}^{J} v_n^j + e^{-itH} w_n^J|^4 (e^{-itH} w_n^J) \|_{L_t^{\frac{5}{3}} L_x^{\frac{30}{23}}} \\
&\lesssim \|e^{-itH} w_n^J\|_{L_{t,x}^{10}}^4 \|H^{\frac{1}{2}} e^{-itH} w_n^J\|_{L_t^5 L_x^{\frac{30}{11}}} \\
&+ \| | \sum_{j=J'}^{J} v_n^j |^4 |\nabla e^{-itH} w_n^J| \|_{L_t^{\frac{5}{3}} L_x^{\frac{30}{23}}} + C_{J'} \sum_{j=1}^{J'} \| |v_n^j|^4 \nabla e^{-itH} w_n^J \|_{L_t^{\frac{5}{3}} L_x^{\frac{30}{23}}} \\
&\lesssim \|e^{-itH} w_n^J\|_{L_{t,x}^{10}}^4 \|H^{\frac{1}{2}} e^{-itH} w_n^J\|_{L_t^5 L_x^{\frac{30}{11}}} \\
&+ \| \sum_{j=J'}^{J} v_n^j \|_{\dot{X}^1}^4 \| |\nabla e^{-itH} w_n^J| \|_{L_t^5 L_x^{\frac{30}{11}}} + C_{J'} \sum_{j=1}^{J'} \| |v_n^j|^4 \nabla e^{-itH} w_n^J \|_{L_t^{\frac{5}{3}} L_x^{\frac{30}{23}}}
\end{aligned}$$

By Strichartz and the decay of $e^{-itH} w_n^J$ in $L_{t,x}^{10}$, the first term goes to 0 as $J \to \infty,\ n \to \infty$. By Strichartz and the definition of J', the second term is bounded by

$$\eta^4 \|w_n^J\|_{\Sigma}$$

which can be made arbitrarily small since $\limsup_{n\to\infty} \|w_n^J\|_{\Sigma}$ is bounded uniformly in J. To finish, we show that for each fixed j

$$\lim_{J\to J^*} \limsup_{n\to\infty} \| |v_n^j|^4 \nabla e^{-itH} w_n^J \|_{L_t^{\frac{5}{3}} L_x^{\frac{30}{11}}} = 0.$$

By Hölder,

$$\| |v_n^j|^4 \nabla e^{-itH} w_n^J \|_{L_t^{\frac{5}{3}} L_x^{\frac{30}{23}}} \le \|v_n^j\|_{L_{t,x}^{10}}^3 \|v_n^j \nabla e^{-itH} w_n^J\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}}},$$

so by (6.3) it suffices to show

$$\lim_{J\to J^*} \limsup_{n\to\infty} \|v_n^j \nabla e^{-itH} w_n^J\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}}} = 0. \tag{6.11}$$

For any $\varepsilon > 0$, there exists $\psi^j \in C_c^\infty(\mathbf{R} \times \mathbf{R}^3)$ and functions $c_n^j(t),\ |c_n^j| \equiv 1$ such that

$$\limsup_{n\to\infty} \|v_n^j - c_n^j \tilde{G}_n^j \psi^j\|_{L_{t,x}^{10}([-\frac{1}{2}, \frac{1}{2}])} < \varepsilon,$$

Note that $\tilde{G}_n^j \psi^j$ is supported on the set

$$\{|t - t_n^j| \lesssim (N_n^j)^{-2},\ |x - x_n^j| \lesssim (N_n^j)^{-1}\}.$$

Thus for all n sufficiently large,

$$\begin{aligned}
&\|v_n^j \nabla e^{-itH} w_n^J\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}}} \\
&\le \|v_n^j - c_n^j \tilde{G}_n^j \psi^j\|_{L_{t,x}^{10}} \|\nabla e^{-itH} w_n^J\|_{L_t^5 L_x^{\frac{30}{11}}} + \|\tilde{G}_n^j \psi^j \nabla e^{-itH} w_n^J\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}}} \\
&\lesssim_{E_c} \varepsilon + \|(\tilde{G}_n^j \psi^j) \nabla e^{-itH} w_n^J\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}}}.
\end{aligned}$$

From the definition of the operators $\tilde{G}_n^j$, we have

$$\|(\tilde{G}_n^j \psi^j) \nabla e^{-itH} w_n^J\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}}} \lesssim_\varepsilon N_n^{\frac{1}{2}} \|\nabla e^{-itH} w_n^J\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}} (|t - t_n^j| \lesssim (N_n^j)^{-2}, |x - x_n^j| \lesssim (N_n^j)^{-1})}.$$

Corollary 2.10 implies

$$\|v_n^j \nabla e^{-itH} w_n^J\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}}} \lesssim \varepsilon + C_\varepsilon \|e^{-itH} w_n^J\|_{L_{t,x}^{10}}^{\frac{1}{9}} \|w_n^J\|_\Sigma^{\frac{8}{9}}.$$

Sending $n \to \infty$, then $J \to J^*$, then $\varepsilon \to 0$ establishes (6.11), and with it, Property (iii). This completes the treatment of the case $d = 3$.

By perturbation theory, $\limsup_{n\to\infty} S_{(-T,T)} \le C(E_c) < \infty$, contrary to the Palais-Smale hypothesis. This rules out Case 2 and completes the proof of Proposition 6.1. □

7. Proof of Theorem 1.3

We begin by recalling some facts about the *ground state*

$$W(x) = (1 + \tfrac{|x|^2}{d(d-2)})^{-\frac{d-2}{2}} \in \dot H^1(\mathbf{R}^d)$$

This function satisfies the elliptic equation

$$\tfrac{1}{2}\Delta W + W^{\frac{4}{d-2}} W = 0.$$

It is well-known (c.f. Aubin [1] and Talenti [27]) that the functions witnessing the sharp constant in the Sobolev inequality

$$\|f\|_{L^{\frac{2d}{d-2}}(\mathbf{R}^d)} \le C_d \|\nabla f\|_{L^2(\mathbf{R}^d)},$$

are precisely those of the form $f(x) = \alpha W(\beta(x - x_0)),\ \alpha \in \mathbf{C},\ \beta > 0,\ x_0 \in \mathbf{R}^d$.

For the reader's convenience, we reiterate the definitions of the energy associated to the focusing energy-critical NLS with and without potential:

$$E_\Delta(u) = \int_{\mathbf{R}^d} \tfrac{1}{2}|\nabla u|^2 - (1 - \tfrac{2}{d})|u|^{\frac{2d}{d-2}}\, dx,$$
$$E(u) = E_\Delta(u) + \tfrac{1}{2}\|xu\|_{L^2}^2.$$

Lemma 7.1 (Energy trapping [16])**.** *Suppose* $E_\Delta(u) \le (1-\delta_0)E_\Delta(W)$.

- *Either* $\|\nabla u\|_{L^2} < \|\nabla W\|_{L^2}$ *or* $\|\nabla u\|_{L^2} > \|\nabla W\|_{L^2}$.
- *If* $\|\nabla u\|_{L^2} < \|\nabla W\|_{L^2}$, *then there exists* $\delta_1 > 0$ *depending on* δ_0 *such that*
$$\|\nabla u\|_{L^2} \le (1-\delta_1)\|\nabla W\|_{L^2},$$
and $E_\Delta(u) \ge 0$.
- *If* $\|\nabla u\|_{L^2} > \|\nabla W\|_{L^2}$ *then there exists* $\delta_2 > 0$ *depending on* δ_0 *such that*
$$\|\nabla u\|_{L^2} \ge (1+\delta_2)\|\nabla W\|_{L^2},$$
and $\frac{1}{2}\|\nabla u\|_{L^2}^2 - \|u\|_{L^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} \le -\delta_0 E_\Delta(W)$.

Now suppose $E(u) < E_\Delta(W)$ and $\|\nabla u\|_{L^2} \le \|\nabla W\|_{L^2}$. The energy inequality can be written as

$$\|u\|_\Sigma^2 + (1-\tfrac{2}{d})(\|W\|_{L^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} - \|u\|_{L^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}}) \le \|\nabla W\|_{L^2}^2.$$

By the variational characterization of W, the difference of norms on the left side is nonnegative; therefore

$$\|u\|_\Sigma \le \|\nabla W\|_{L^2}.$$

Combining the above with conservation of energy and a continuity argument, we obtain

Corollary 7.2. *Suppose* $u : I \times \mathbf{R}^d \to \mathbf{C}$ *is a solution to the focusing equation* (1.1) *with* $E(u) \le (1 - \delta_0)E_\Delta(W)$. *Then there exist* $\delta_1, \delta_2 > 0$, *depending on* δ_0, *such that*

- *If* $\|u(0)\|_{\dot H^1} \le \|W\|_{\dot H^1}$, *then*
$$\sup_{t\in I} \|u(t)\|_\Sigma \le (1-\delta_1)\|W\|_{\dot H^1} \quad \text{and} \quad E(u) \ge 0.$$
- *If* $\|u(0)\|_{\dot H^1} \ge \|W\|_{\dot H^1}$, *then*
$$\inf_{t\in I} \|u(t)\|_\Sigma \ge (1+\delta_2)\|W\|_{\dot H^1} \quad \text{and} \quad \tfrac{1}{2}\|\nabla u\|_2^2 - \|u\|_{L^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} \le -\delta_0 E_\Delta(W).$$

Proof of Theorem 1.3. Let u be the maximal solution to (1.1) with

$$u(0) = u_0, \quad E(u_0) < E_\Delta(W), \quad \|\nabla u_0\|_2 \geq \|\nabla W\|_2.$$

Let $f(t) = \int_{\mathbf{R}^d} |x|^2 |u(t,x)|^2 \, dx$. It can be shown [8] that f is C^2 on the interval of existence and

$$f''(t) = \int |\nabla u(t,x)|^2 - 2|u(t,x)|^{\frac{2d}{d-2}} - \tfrac{1}{2}|x|^2|u(t,x)|^2 \, dx.$$

By the corollary, f'' is bounded above by some fixed $C < 0$. Therefore

$$f(t) \leq A + Bt + \tfrac{C}{2}t^2$$

for some constants A and B. It follows that u has a finite lifespan in both time directions. □

8. Bounded linear potentials

In this section we show using a perturbative argument that

$$i\partial_t u = (-\tfrac{1}{2}\Delta + V)u + |u|^{\frac{4}{d-2}}u, \quad u(0) = u_0 \in H^1(\mathbf{R}^d) \tag{8.1}$$

is globally wellposed whenever V is a real-valued function with

$$V_{max} := \|V\|_{L^\infty} + \|\nabla V\|_{L^\infty} < \infty.$$

This equation defines the Hamiltonian flow of the energy functional

$$E(u(t)) = \int_{\mathbf{R}^d} \tfrac{1}{2}|\nabla u(t,x)|^2 + V|u(t,x)|^2 + \tfrac{d-2}{d}|u|^{\frac{2d}{d-2}} \, dx = E(u(0)). \tag{8.2}$$

Solutions to (8.1) also conserve *mass*:

$$M(u(t)) = \int_{\mathbf{R}^d} |u(t,x)|^2 \, dx = M(u(0)).$$

It will be convenient to assume V is positive and bounded away from 0. This hypothesis allows us to bound the H^1 norm of u purely in terms of E instead of both E and M, and causes no loss of generality because for sign-indefinite V we could simply consider the conserved quantity $E + CM$ for some positive constant C.

Theorem 8.1. *For any $u_0 \in H^1(\mathbf{R}^d)$, (8.1) has a unique global solution $u \in C^0_{t,loc}H^1_x(\mathbf{R} \times \mathbf{R}^d)$. Further, u obeys the spacetime bounds*

$$S_I(u) \leq C(\|u_0\|_{H^1}, |I|)$$

for any compact interval $I \subset \mathbf{R}$.

The proof follows the strategy pioneered by [29] and treats the term Vu as a perturbation to (1.6), which is globally wellposed. Thus Duhamel's formula reads

$$u(t) = e^{\frac{it\Delta}{2}}u(t_0) - i\int_0^t e^{\frac{i(t-s)\Delta}{2}}[|u(s)|^{\frac{4}{d-2}}u(s) + Vu(s)]ds. \tag{8.3}$$

We record mostly without proof some standard results in the local theory of (8.1). Introduce the notation

$$\|u\|_{X(I)} = \|\nabla u\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}} (I \times \mathbf{R}^d)}.$$

Lemma 8.2 (Local wellposedness)**.** *Fix $u_0 \in H^1(\mathbf{R}^d)$, and suppose $T_0 > 0$ is such that*

$$\|e^{\frac{it\Delta}{2}}u_0\|_{X([-T_0,T_0])} \leq \eta \leq \eta_0$$

where $\eta_0 = \eta_0(d)$ is a fixed parameter. Then there exists a positive

$$T_1 = T_1(\|u_0\|_{H^1}, \eta, V_{max})$$

such that (8.1) has a unique (strong) solution $u \in C^0_t H^1_x([-T_1, T_1] \times \mathbf{R}^d)$. Further, if $(-T_{min}, T_{max})$ is the maximal lifespan of u, then $\|\nabla u\|_{S(I)} < \infty$ for every compact interval $I \subset (-T_{min}, T_{max})$, where $\|\cdot\|_{S(I)}$ is the Strichartz norm defined in Section 2.1.

Proof sketch. Run the usual contraction mapping argument using the Strichartz estimates to show that

$$\mathcal{I}(u)(t) = e^{\frac{it\Delta}{2}} u_0 - i \int_0^t e^{\frac{i(t-s)\Delta}{2}} [|u(s)|^{\frac{4}{d-2}} u(s) + V u(s)] dx$$

has a fixed point in a suitable function space. Estimate the terms involving V in the $L_t^1 L_x^2$ dual Strichartz norm and choose the parameter T_1 to make those terms sufficiently small after using Hölder in time. □

Lemma 8.3 (Blowup criterion). *Let $u : (T_0, T_1) \times \mathbf{R}^d \to \mathbf{C}$ be a solution to* (8.1) *with*

$$\|u\|_{X((T_0,T_1))} < \infty.$$

If $T_0 > -\infty$ or $T_1 < \infty$, then u can be extended to a solution on a larger time interval.

Our argument uses the stability theory for the energy-critical NLS (1.6).

Lemma 8.4 (Stability [28]). *Let $\tilde{u} : I \times \mathbf{R}^d \to \mathbf{C}$ be an approximate solution to equation* (1.6) *in the sense that*

$$i\partial_t \tilde{u} = -\tfrac{1}{2}\Delta u \pm |\tilde{u}|^{\frac{4}{d-2}} \tilde{u} + e$$

for some function e. Assume that

$$\|\tilde{u}\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \le L, \quad \|\nabla u\|_{L_t^\infty L_x^2} \le E, \tag{8.4}$$

and that for some $0 < \varepsilon < \varepsilon_0(E, L)$,

$$\|\tilde{u}(t_0) - u_0\|_{\dot{H}^1} + \|\nabla e\|_{N(I)} \le \varepsilon, \tag{8.5}$$

where $\|\cdot\|_{N(I)}$ was defined in Section 2.1. Then there exists a unique solution $u : I \times \mathbf{R}^d \to \mathbf{C}$ to (1.6) *with $u(t_0) = u_0$ which further satisfies the estimates*

$$\|\tilde{u} - u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} + \|\nabla(\tilde{u} - u)\|_{S(I)} \le C(E, L)\varepsilon^c \tag{8.6}$$

where $0 < c = c(d) < 1$ and $C(E, L)$ is a function which is nondecreasing in each variable.

Proof of Theorem 8.1. It suffices to show that for T sufficiently small depending only on $E = E(u_0)$, the solution u to (8.1) on $[0, T]$ satisfies an *a priori* estimate

$$\|u\|_{X([0,T])} \le C(E). \tag{8.7}$$

From Lemma 8.3 and energy conservation, it will follow that u is a global solution with the desired spacetime bound.

By Theorem 1.1, the equation

$$(i\partial_t + \tfrac{1}{2}\Delta) w = |w|^{\frac{4}{d-2}} w, \quad w(0) = u(0).$$

has a unique global solution $w \in C_{t,loc}^0 \dot{H}_x^1(\mathbf{R} \times \mathbf{R}^d)$ with the spacetime bound (1.7). Fix a small parameter $\eta > 0$ to be determined shortly, and partition $[0, \infty)$ into $J(E, \eta)$ intervals $I_j = [t_j, t_{j+1})$ so that

$$\|w\|_{X(I_j)} \le \eta. \tag{8.8}$$

For some $J' < J$, we then have

$$[0, T] = \bigcup_{j=0}^{J'-1} ([0, T] \cap I_j).$$

We make two preliminary estimates. By Hölder in time,

$$\|Vu\|_{N(I_j)} + \|\nabla(Vu)\|_{N(I_j)} \lesssim C_V T \|u\|_{L_t^\infty H_x^1(I_j)} \le \varepsilon \tag{8.9}$$

for any ε provided that $T = T(E, V, \varepsilon)$ is sufficiently small. Further, observe that

$$\|e^{\frac{i(t-t_j)\Delta}{2}} w(t_j)\|_{X(I_j)} \le 2\eta \tag{8.10}$$

for η sufficiently small depending only on d. Indeed, from the Duhamel formula

$$w(t) = e^{\frac{i(t-t_j)\Delta}{2}} w(t_j) - i \int_{t_j}^t e^{\frac{i(t-s)\Delta}{2}} (|w|^{\frac{4}{d-2}} w)(s) ds,$$

Strichartz, and the chain rule, it follows that

$$\begin{aligned}\|e^{\frac{i(t-t_j)\Delta}{2}} w(t_j)\|_{X(I_j)} &\le \|w\|_{X(I_j)} + c_d \|\nabla(|w|^{\frac{4}{d-2}} w)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I_j)} \\ &\le \eta + c_d \|w\|_{X(I_j)}^{\frac{d+2}{d-2}} \\ &\le \eta + c_d \eta^{\frac{d+2}{d-2}}.\end{aligned}$$

Choosing η sufficiently small relative to c_d yields (8.10).

Take $\varepsilon < \eta$ in (8.9) (taking T small) and apply the Duhamel formula (8.3), Strichartz, Hölder, and (8.10) to obtain

$$\begin{aligned}\|u\|_{X(I_0)} &\le \|e^{\frac{it\Delta}{2}} u(0)\|_{X(I_0)} + c_d \|u\|_{X(I_0)}^{\frac{d+2}{d-2}} + C\|Vu\|_{L_t^1 H_x^1(I_0)} \\ &\le 2\eta + c_d \|u\|_{X(I_0)}^{\frac{d+2}{d-2}} + C_V T \|u\|_{L_t^\infty H_x^1(I_0)} \\ &\le 3\eta + c_d \|u\|_{X(I_0)}^{\frac{d+2}{d-2}}.\end{aligned}$$

By a continuity argument,

$$\|u\|_{X(I_0)} \le 4\eta. \tag{8.11}$$

Choose ε sufficiently small in (8.9) so that the smallness condition (8.5) is satisfied, and invoke Lemma 8.4 with $\|u(0) - w(0)\|_{\dot{H}^1} = 0$ to find that

$$\|\nabla(u - w)\|_{S(I_0)} \le C(E)\varepsilon^c. \tag{8.12}$$

On the interval I_1, use (8.10), (8.12), and the usual estimates to obtain

$$\begin{aligned}\|u\|_{X(I_1)} &\le \|e^{\frac{i(t-t_1)\Delta}{2}} u(t_1)\|_{X(I_1)} + c_d \|u\|_{X(I_1)}^{\frac{d+2}{d-2}} + C_V T \|u\|_{L_t^\infty H_x^1(I_1)} \\ &\le C(E)\varepsilon^c + 2\eta + c\|u\|_{X(I_1)}^{\frac{d+2}{d-2}} + \eta,\end{aligned}$$

where the $C(E)$ in the last line has absorbed the Strichartz constant c; this redefinition of $C(E)$ will cause no trouble because the number of times it will occur depends only on E, d, and V. By taking ε sufficiently small relative to η and using continuity, we see that

$$\|u\|_{X(I_1)} \le 4\eta.$$

As before, taking T sufficiently small yields

$$\begin{aligned}\|\nabla(Vu)\|_{\dot{N}^0(I_1)} &\le \varepsilon \\ \|e^{\frac{i(t-t_1)\Delta}{2}} [u(t_1) - w(t_1)]\|_{X(I_1)} &\le C(E)\varepsilon^c\end{aligned}$$

for any $\varepsilon \le \varepsilon_0(E, L)$. Therefore by Lemma 8.4,

$$\|\nabla(u - w)\|_{S(I_1)} \le C(E)\varepsilon^c.$$

The parameters η, ε, T are chosen so that each depends only on the preceding parameters and on the fixed quantities d, E, V. After iterating at most J' times and summing the bounds over $0 \le j \le J' - 1$, we conclude that for T sufficiently small depending on E and V,

$$\|u\|_{X([0,T])} \le 4J'\eta \le C(E).$$

This establishes the bound (8.7). □

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