THE QUINTIC NLS ON PERTURBATIONS OF $\mathbf{R}^3$

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Abstract. Consider the defocusing quintic nonlinear Schrödinger equation on $\mathbf{R}^3$ with initial data in the energy space. This problem is "energy-critical" in view of a certain scale-invariance, which is a main source of difficulty in the analysis of this equation. It is a nontrivial fact that all finite-energy solutions scatter to linear solutions. We show that this remains true under small compact deformations of the Euclidean metric. Our main new ingredient is a long-time microlocal weak dispersive estimate that accounts for the refocusing of geodesics.

1. Introduction

Let g be a smooth Riemannian metric on $\mathbf{R}^3$. We consider the large-data Cauchy problem for the defocusing nonlinear Schrödinger equation

$$i\partial_t u + \Delta_g u = |u|^4 u,\ u(0,x) = u_0(x) \in \dot{H}^1, \tag{1.1}$$

where Δ_g is the Laplace-Beltrami operator. More precise assumptions on g shall be prescribed shortly.

This equation admits a conserved energy

$$E(u) = \int_{\mathbf{R}^3} \frac{1}{2} g^{jk} \partial_j u \overline{\partial_k u} + \frac{1}{6}|u|^6 \, dg, \tag{1.2}$$

where $dg = \sqrt{|g|}dx$ is the Riemannian measure.

If $g = \delta$ is the standard Euclidean metric, one recovers the well-known energy-critical NLS. A key feature of that equation—and a major source of analytical headaches—is the scaling symmetry $u_\lambda(t,x) = \lambda^{-1/2} u(\lambda^{-2}t, \lambda^{-1}x)$. As the energy is also invariant under this rescaling, conservation of energy alone does not rule out the possibility that some solutions may concentrate at a point and blow up in finite time. It is a difficult theorem that all finite-energy solutions to that equation scatter [Bou99, CKS+08].

Although the exact scaling symmetry no longer holds for general g, it reemerges at small length scales in the sense that solutions highly concentrated near a point x_0 will evolve, for short times, approximately as though g were the constant metric $g(x_0)$. Despite this parallel, however, equation (1.1) is not a simple perturbation of the Euclidean energy-critical equation. Indeed, disturbing the highest order terms may destroy fundamental linear smoothing and decay estimates. This breakdown is linked to the geometry of the geodesic flow.

For a general metric g, the linear local smoothing estimate $L^2 \to L^2 H^{1/2}_{loc}$ is known to fail in the presence of trapping [Doi96]. Also, on a curved background, multiple geodesics emanating from a point may converge at another point. Linear solutions exhibit weaker decay with such refocusing; in particular, by the parametrix construction of Hassell-Wunsch [HW05], the Euclidean dispersive estimate

$$\|e^{it\Delta}\|_{L^1(\mathbf{R}^d) \to L^\infty(\mathbf{R}^d)} \lesssim |t|^{-d/2} \tag{1.3}$$

necessarily fails whenever the metric admits conjugate points. In general one can only recover a frequency-localized version which holds at most for times inversely proportional to frequency; see [BGT04]. The time window stops the flow well before refocusing of geodesics can occur.

While trapping does not occur if g is sufficiently close to flat, arbitrarily small perturbations of the flat metric may cause rays to refocus. Thus (1.3) typically fails on curved backgrounds. This has substantial implications for both the linear and nonlinear analysis.

The standard abstract approach to linear Strichartz estimates combines the dispersive estimate with a TT^* argument [KT98]. This method is not directly applicable where the dispersive estimate is not available. Nonetheless, lossless Strichartz inequalities have been obtained for curved backgrounds, starting with the influential work of Staffilani and Tataru [ST02] and generalized substantially since [RT07, HTW06, Tat08, BT08, MMT08]. The basic strategy in these papers is to exploit microlocal versions of the dispersive estimate through suitable parametrices and to control the errors using local smoothing, which holds in greater generality compared to the dispersive estimate.

Linear dispersion also plays a key role in the study of nonlinear solutions, in particular, when trying to control highly concentrated nonlinear profiles that arise as potential obstructions to global existence, and for proving the decoupling of nonlinear profiles. There are by now several examples of such an analysis; see for example [IPS12], [KVZb], or [Jao16]. We briefly recall the main idea from the latter reference, which discusses the energy-critical NLS with the Schrödinger operator for a quantum harmonic oscillator. By the Mehler formula, the linear propagator obeys the dispersive estimate locally in time.

Suppose u_n is a sequence of solutions to the defocusing quintic harmonic oscillator on $\mathbf{R}^3$ with initial data $u_n(0) = \lambda_n^{-1/2}\phi(\lambda_n^{-1}\cdot)$ for some $\lambda_n \to 0$ and some compactly supported ϕ. For short times (more precisely, when $|t| \le T\lambda_n^2$ for any $T > 0$), the harmonic oscillator solution u_n perceives the potential as essentially constant and is well-approximated by the solution $\tilde{u}_n$ to the Euclidean energy-critical equation with the same initial data. Using as a black box the theorem that Euclidean solutions exist globally and scatter, one deduces via stability theory that u_n is well-behaved for $t \le O(\lambda_n^2)$.

For $t \ge T\lambda_n^2$, the (local in time) dispersive estimate for the harmonic oscillator and the scattering of Euclidean solutions ensure that for large T and small λ_n, the nonlinearity $|u_n|^4 u_n$ is a negligible perturbation of the linear harmonic oscillator. That is, for such t, u_n evolves essentially according to the linear flow applied to $u(T\lambda_n^2)$, which is perfectly well behaved. Thus, linear decay allows one to control concentrated nonlinear solutions for times when the Euclidean approximation no longer holds.

We investigate the situation where g coincides with the flat metric outside the unit ball and all geodesics escape to infinity. This is the simplest nontrivial generalization of the Euclidean metric and is a natural counterpart to the scenario considered recently by Killip, Visan, and Zhang [KVZb], who proved scattering for the analogue of equation (1.1) in the exterior of a hard convex obstacle. We prove

Theorem 1.1. *Let g be a smooth metric on $\mathbf{R}^3$ which coincides with the Euclidean metric outside the unit ball. Also assume that g is nontrapping, i.e. all geodesics eventually leave every compact set. For any $u_0 \in \dot{H}^1$, there is a unique global solution to* (1.1)*. Moreover, there exists $\varepsilon > 0$ such that if $\|g - \delta\|_{C^3} \le \varepsilon$ then the solutions obey global spacetime bounds*

$$\|u\|_{L^{10}_{t,x}(\mathbf{R}\times\mathbf{R}^3)} \le C(E(u_0)).$$

We use the Kenig-Merle concentration compactness and rigidity framework [Ker01, KM06], an evolution of the earlier induction on energy method of [Bou99, CKS$^+$08]. In particular, we follow quite closely the mold of [KVZb]. Assuming that the scattering fails, we show that there must exist a global-in-time blowup solution u_c with minimal energy among all counterexamples to the theorem. In view of this minimality, u_c is also shown to be almost-periodic in the sense that $u(t)$ is trapped in some compact subset of $\dot{H}^1$. In these arguments, the asymptotic behavior of the Euclidean NLS (which we use as a black box) plays a key role. However, under the smallness assumption on the metric, a Morawetz inequality will imply that solutions to equation (1.1) can never be almost-periodic. The smallness condition for scattering is probably artificial, but we do not see at this time how to dispense with it.

The heart of the matter is how to overcome the reduced linear dispersion, which is the main obstacle to the linear and nonlinear profile decompositions. In Section 4, we prove a weak analogue of the usual dispersive estimate which nonetheless suffices for our purposes. This is a long-time variant of the Burq-Gerard-Tzvetkov dispersion estimate [BGT04] where we track the microlocalized Schrödinger flow on timescales that permit refocusing.

Several recent works have exploited analogous weak dispersive estimates to study energy-critical NLS in non-Euclidean geometries. The dispersion results from different mechanisms in each case. En route to proving global wellposedness for the quintic NLS on $\mathbf{T}^3$, Ionescu-Pausader introduce an "extinction lemma" [IP12, Lemma 4.2] to control concentrated nonlinear profiles at times beyond the "Euclidean window". Afterwards, Pausader-Tzvetkov-Wang [PTW14] obtained the analogous result on $\mathbf{S}^3$, also relying crucially on an extinction lemma. The arguments there lean on the special structure of the underlying manifold, using for instance Fourier analysis on the torus (which, when combined with number theoretic arguments, yield good bounds on the Schrödinger propagator) or the concentration properties of spherical harmonics.

In a different vein, Killip-Visan-Zhang [KVZb] also obtained an extinction lemma in the exterior of a convex obstacle. The geodesics in that domain are broken straight lines. To study the linear evolution of a profile concentrating near the obstacle, they construct a gaussian wavepacket parametrix and carefully study how the wavepackets reflect off the obstacle. The essential geometric fact in their favor is that due to the convexity assumption, any two rays diverge after reflecting off the obstacle.

When the hard obstacle is replaced by a lens, refracted rays can certainly refocus. However, some decay still occurs for a different reason. By the uncertainty principle, a solution which is initially highly concentrated in space must be broadly distributed in momentum (frequency). Thus, it will spread out along geodesics as the slower parts lag behind. This is an observation of D. Tataru communicated to the author by R. Killip and M. Visan. We make this heuristic precise in Section 4 by building a wavepacket parametrix and studying the geodesic flow.

Outline of paper. Section 2 collects technical points concerning Sobolev spaces and some linear theory. From the linear estimates it is a standard matter to obtain the perturbative theory, and we merely state the main results.

Sections 3 and 4 contain the main technical contribution of this paper. In Section 3, we study linear solutions in various situations where the variation in the metric is intuitively negligible (for instance, when considering initial data supported far from the origin). The most interesting case is when the solution is initially concentrated near the origin, where it experiences nontrivial effects from the geometry. Analyzing this case relies on an extinction lemma which is the subject of Section 4.

With those considerations out of the way, we construct the linear profile decomposition in Sections 5. We also show in Section 6 that highly concentrated nonlinear profiles are well-behaved; here the extinction lemma and the existing scattering result for the Euclidean quintic equation both play a critical role.

In Section 7, we use a nonlinear profile decomposition and induction on energy to reduce Theorem 1.1 to considering almost-periodic minimal-energy counterexamples. This will already imply global wellposedness. Some care is needed to control the interaction between linear and nonlinear profiles; see the discussion preceding Lemma 7.6.

Finally, in Section 8 we prove scattering under the smallness assumption via a Bourgain-Morawetz inequality.

In the appendix, we use the ideas from Section 4 to give a small refinement to the Burq-Gerard-Tzvetkov semiclassical dispersive estimate which may be of independent interest.

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2. Preliminaries

2.1. **Sobolev spaces.** The energy space $\dot H^1 = \dot H^1(g)$ is defined as the completion of test functions $C_0^\infty(\mathbf{R}^3)$ with respect to the quadratic form

$$\|u\|_{\dot H^1}^2 = \int_{\mathbf{R}^3} |du|_g^2\, dg(x) = \int_{\mathbf{R}^3} g^{jk}\partial_j u \overline{\partial_k u}\, dg(x).$$

As $\|u\|_{\dot H^1(g)} \sim \|(-\Delta_\delta)^{1/2} u\|_{L^2(dx)} = \|u\|_{\dot H^1(\delta)}$, where $\dot H^1(\delta)$ is the Euclidean homogeneous Sobolev space, the spaces $\dot H^1(g)$ and $\dot H^1(\delta)$ are equal as sets and have equivalent inner products. We make this distinction because we shall occasionally use the fact that Δ_g is self-adjoint with respect to the inner product for $\dot H^1(g)$; when the difference is irrelevant we just write $\dot H^1(\mathbf{R}^3)$ or just $\dot H^1$.

For $1 < p < \infty$, define the homogeneous Sobolev spaces $\dot H^{1,p}(\delta)$ and $\dot H^{1,p}(g)$ as the completion of C_0^∞ under the norms

$$\|u\|_{\dot H^{1,p}(\delta)} := \|(-\Delta_\delta)^{1/2} u\|_{L^p}, \quad \|u\|_{\dot H^{1,p}(g)} := \|(-\Delta_g)^{1/2} u\|_{L^p}. \tag{2.1}$$

As noted above, these two definitions coincide when $p = 2$. Less trivially, these norms are equivalent for all $1 < p < \infty$. This is a consequence of the following boundedness result for the Riesz transform $d(-\Delta_g)^{-1/2}$ on asymptotically Euclidean manifolds.

Proposition 2.1 ([CCH06, Remark 5.2])**.** *Let (M, g) be a Riemannian manifold such that for some $R > 0$, $M \setminus B(0,R)$ is Euclidean. Then the Riesz transform $d(-\Delta_g)^{-1/2}$ is bounded from $L^p(M)$ to $L^p(M; T^*M)$ for all $1 < p < \infty$.*

By a well-known duality argument (see for example [CD03, Section 2.1]), this implies the reverse inequality whose proof we give for completeness:

Corollary 2.2.

$$\|(-\Delta_g)^{1/2} u\|_{L^p} \lesssim_p \|du\|_{L^p},\ \forall u \in C_0^\infty,\ 1 < p < \infty.$$

Proof. By duality, it suffices to show

$$|\langle (-\Delta_g)^{1/2} u, v\rangle| \lesssim \|du\|_{L^p} \|\|v\|_{L^{p'}}.$$

Then

$$\begin{aligned}\langle (-\Delta_g)^{1/2} u, v\rangle &= \langle u, (-\Delta_g)^{1/2} v\rangle = \langle u, (-\Delta_g)(-\Delta_g)^{-1/2} v\rangle \\ &= \langle du, d(-\Delta_g)^{-1/2} v\rangle \lesssim \|du\|_{L^p}\|v\|_{L^{p'}}.\end{aligned}$$

Note that while the intermediate manipulations are justified for v spectrally localized away from 0 and ∞, we may then pass to general $v \in L^{p'}$ using (2.4) below. □

Noting also that

$$\|df\|_{L^p} = \|d(-\Delta_g)^{-1/2}(-\Delta_g)^{1/2} f\|_{L^p} \lesssim \|(-\Delta_g)^{1/2} f\|_{L^p},$$

we summarize the previous two estimates in the following

Corollary 2.3 (Equivalence of Sobolev norms)**.** *For all $1 < p < \infty$ and $f \in C_0^\infty$,*

$$\|(-\Delta_\delta)^{1/2} u\|_{L^p} \sim_p \|df\|_{L^p} \sim_p \|(-\Delta_g)^{1/2} u\|_{L^p}.$$

This corollary lets one transfer the Euclidean Leibniz and chain rule estimates to the Sobolev norms defined by $(-\Delta_g)^{1/2}$, which are better adapted to the equation as $(-\Delta_g)^{1/2}$ commutes with the linear propagator. We shall frequently employ the following

Corollary 2.4.

$$\|(-\Delta_g)^{1/2}F(u)\|_{L^p} \lesssim \|F'(u)\|_{L^q}\|(-\Delta_g)^{1/2}u\|_{L^r}$$

for all $1 < p, r < \infty$ *and* $1 < q \le \infty$ *such that* $p^{-1} = q^{-1} + r^{-1}$. *In particular, we have*

$$\|(-\Delta_g)^{1/2}(|u|^4u)\|_{L^2L^{\frac{6}{5}}} \lesssim \|u\|^4_{L^{10}L^{10}}\|(-\Delta_g)^{1/2}\|_{L^{10}L^{\frac{30}{13}}}.$$

2.2. **Strichartz estimates.** Local-in-time Strichartz estimates without loss for compact nontrapping metric perturbations were first established by Staffilani and Tataru [ST02]. As later observed, their argument can be combined with the global local smoothing estimate of Rodnianski and Tao to deduce global-in-time Strichartz estimates [RT07]. As mentioned in the introduction, these results have since been extended to long-range metrics.

Proposition 2.5. *[*[ST02, RT07]*] For any function* $u : I \times \mathbf{R}^3 \to \mathbf{C}$,

$$\|u\|_{L^\infty L^2 \cap L^2L^6} \lesssim \|u(0)\|_{L^2} + \|(i\partial_t + \Delta_g)u\|_{L^1L^2 + L^2L^{6/5}}$$

In particular, by Sobolev embedding and Corollary 2.3,

$$\|u\|_{L^{10}L^{10}} \lesssim \|(-\Delta_g)^{1/2}u\|_{L^{10}L^{\frac{30}{13}}} \lesssim \|u(0)\|_{\dot{H}^1} + \|\nabla(i\partial_t + \Delta_g)u\|_{L^1L^2+L^2L^{6/5}}.$$

In the sequel we adopt the notation

$$Z(I) = L^{10}_t L^{10}_x(I \times \mathbf{R}^3), \quad N(I) = (L^1_tL^2_x + L^2_tL^{6/5}_x)(I \times \mathbf{R}^3).$$

2.3. **Some harmonic analysis.** In this section we set up a Littlewood-Paley theory, which will underlie the linear profile decomposition. We use the heat semigroup and follow essentially standard arguments that combine a spectral multiplier theorem with heat kernel bounds.

Gaussian heat kernel bounds for Δ_g are classical. We quote a result of Aronson, who in fact considered uniformly elliptic operators on Euclidean space; see the book [Gri09] for a comprehensive survey.

Theorem 2.6 ([Aro67])**.** *There exist constants* $c_1, c_2 > 0$ *such that*

$$e^{t\Delta_g}(x,y) \le c_1 t^{-\frac{3}{2}} e^{-\frac{d_g(x,y)^2}{ct}},$$

where $d_g(x,y)$ *is the Riemannian distance between* x *and* y.

In view of this bound, we have access to a very general spectral multiplier theorem. For simplicity we state just the special case that we need.

Theorem 2.7 ([TDOS02, Theorem 3.1])**.** *For any* F *satisfying the homogeneous symbol estimates*

$$|\lambda^k\partial^k F(\lambda)| \le C_k \text{ for all } 0 \le k \le \left\lceil \frac{3}{2} \right\rceil + 1,$$

the operator $F(-\Delta_g)$ *maps* $L^1(\mathbf{R}^3) \to L^{1,\infty}(\mathbf{R}^3)$ *and* $L^p(\mathbf{R}^3 \to L^p(\mathbf{R}^3)$ *for all* $1 < p < \infty$.

For a dyadic number $N \in 2^{\mathbf{Z}}$, define Littlewood-Paley projections in terms of the heat kernel

$$\tilde{P}_{\le N} = e^{\Delta_g/N^2}, \quad \tilde{P}_N = e^{\Delta_g/N^2} - e^{4\Delta_g/N^2}.$$

Later (see Lemma 7.6) we also introduce Littlewood-Paley projections $P_{\le N}$ and P_N using compactly supported spectral multipliers instead of the heat kernel.

The following Bernstein estimates are standard:

Proposition 2.8.

$$\|\tilde{P}_{\le N}\|_{L^p \to L^p} \le 2, \ 1 < p < \infty. \tag{2.2}$$

$$\|\tilde{P}_{\le N}\|_{L^p \to L^q} \le cN^{\frac{d}{p} - \frac{d}{q}}, \ 1 \le p \le q \le \infty. \tag{2.3}$$

$$f = \sum_N \tilde{P}_N f \quad in\ L^p, \ 1 < p < \infty.. \tag{2.4}$$

Also, for all $1 < p < \infty$*, the following square function estimate holds*

$$\|(-\Delta_g)^{\frac{s}{2}} f\|_{L^p} \sim_p \Big\| (\sum_N |N^s (\tilde{P}_N)^k f|^2)^{1/2} f \Big\|_{L^p}, \tag{2.5}$$

whenever $2k > s$*.*

Proof. By the pointwise bound (2.6) on the heat kernel,

$$\|e^{t\Delta_g}\|_{L^1 \to L^\infty} \le ct^{-3/2}. \tag{2.6}$$

Duality then yields

$$\|e^{t\Delta_g}\|_{L^1 \to L^2} = \|e^{t\Delta_g}\|_{L^2 \to L^\infty} = \|e^{2t\Delta_g}\|^{1/2}_{L^1 \to L^\infty} \le ct^{-\frac{3}{4}}.$$

Since $\int e^{t\Delta_g}(x,y)\,dg(y) = \int e^{t\Delta_g}(x,y)\,dg(x) \equiv 1$, Schur's test implies

$$\|e^{t\Delta_g}\|_{L^p \to L^p} \le 1, \ 1 \le p \le \infty.$$

The claims (2.2) and (2.3) follow from interpolating these estimates.

The convergence in (2.4) follows from the functional calculus when $p = 2$. On the other hand, Theorem 2.7 ensures boundedness in L^p for all $1 < p < \infty$. By interpolation, one gets convergence for all such p.

Finally, the square function estimate (2.5) follows the standard argument using independent random signs and the multiplier theorem 2.7. The lower bound on k ensures that the symbol for $(\tilde{P}_N)^k$ (which is not quite compactly supported) vanishes at the origin to higher order than the symbol for $(-\Delta_g)^{s/2}$; see [KVZa] for details. □

2.4. **Tools from microlocal analysis.** We recall some standard constructions from semiclassical analysis that shall be needed later. For a more comprehensive account, the reader may consult for instance [Zwo12].

For a symbol $a(x,\xi)$ and a parameter $0 < h \le 1$, the semiclassical Weyl quantization of a, denoted by $a^w(X, hD)$, is the pseudo-differential operator with Schwartz kernel

$$a^w(X,hD)(x,y) := \int_{\mathbf{R}^d} e^{\frac{i(x-y)\cdot\xi}{h}} a\Big(\frac{x+y}{2}, \xi\Big)\,d\xi.$$

In particular, if $a(x,\xi) = g^{jk}(x)\xi_j\xi_k$ is the principal symbol of the Laplacian $-\Delta_g$, one computes

$$a^w(x,hD) = -h^2 \partial_j g^{jk} \partial_k - \frac{h^2}{4}(\partial_j\partial_k g^{jk}).$$

If P is a self-adjoint pseudo-differential operator, the semiclassical functional calculus yields an asymptotic expansion of functions of P, defined via the spectral theorem, in terms of the principal symbol of P. We quote the following result from [BGT04] (see also the references therein):

Proposition 2.9. *Let* P *be an elliptic self-adjoint operator of order* $m > 0$ *on* $\mathbf{R}^d$*. Assume that the symbol* p *and the principal symbol* p_m *satisfy*

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \lesssim_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}, \quad |p_m(x,\xi)| \gtrsim |\xi|^m,$$

for all $(x,\xi) \in \mathbf{R}^d$ and all multiindices α, β. Given $\varphi \in C_0^\infty(\mathbf{R})$, there is a sequence of functions $\Psi_j \in C_b^\infty(\mathbf{R}^d \times \mathbf{R}^d)$ (smooth with all derivatives bounded) such that for any $N \in \mathbf{N}, \sigma \in [0,\infty)$, we have

$$\|\varphi(h^m P) - \sum_{j=0}^{N-1} h^j \Psi_j^w(x, hD_x)\|_{L^2 \to H^\sigma} \lesssim h^{N-\sigma} \quad \text{for all} \quad 0 < h \le 1.$$

Moreover, $\Psi_0(x,\xi) = \varphi(p_m(x,\xi))$, and Ψ_j are linear combinations of derivatives of $\varphi(p_m)$ with coefficients in $C_b^\infty(\mathbf{R}^d \times \mathbf{R}^d)$.

Another device we shall use is the wavepacket decomposition. For each $h > 0$ and (x_0, ξ_0), define

$$\psi^h_{(x_0,\xi_0)}(y) = 2^{-\frac{d}{2}} \pi^{-\frac{3d}{4}} h^{-\frac{3d}{4}} e^{\frac{i\xi_0(y-x_0)}{h}} e^{-\frac{(y-x_0)^2}{2h}},$$

which is a Gaussian wavepacket localized in phase space to the box

$$\{(x,\xi) : |x - x_0| \le h^{1/2},\ |\xi - h^{-1}\xi_0| \le h^{-1/2}\}.$$

The *FBI transform* at scale h is an isometry $T_h : L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d \times \mathbf{R}^d)$ defined by

$$T_h f(x,\xi) = \langle \psi^h_{(x,\xi)}, f\rangle = c_d h^{-\frac{3d}{4}} \int e^{\frac{i\xi(x-y)}{h}} e^{-\frac{(x-y)^2}{2h}} f(y)\, dy = c_d h^{-\frac{5d}{4}} \int e^{\frac{ix\eta}{h}} e^{-\frac{(\xi-\eta)^2}{2h}} \hat{f}(\tfrac{\eta}{h})\, d\eta.$$

For more details, consult for instance [KT05] and the references therein. From the adjoint formula

$$T_h^* F(y) = \int \psi^h_{(x,\xi)}(y) F(x,\xi)\, dx d\xi,$$

one obtains for each $f \in L^2(\mathbf{R}^d)$ a representation

$$f = T_h^* T_h f = \int \langle \psi^h_{(x,\xi)}, f\rangle \psi^h_{(x,\xi)}\, dx d\xi.$$

as a continuous superposition of width $h^{1/2}$ wavepackets. Such a decomposition is well-adapted to semiclassical Schrödinger dynamics as the Schrödinger evolution of each wavepacket $\psi^h_{(x_0,\xi_0)}$ will remain coherent and behave essentially like a classical particle on time scales of order h.

2.5. **Local wellposedness.** We summarize some standard results concerning the local existence, uniqueness, and stability of solutions. These are proved by the usual contraction mapping and bootstrap arguments for the Euclidean NLS (see [KV13] and the references therein). These arguments apply equally well in dimensions $3 \le d \le 6$. When $d > 6$, however, the stability theorem is proved in the Euclidean setting using exotic Strichartz estimates [TV05, KV13] stemming from the Euclidean dispersive estimate, which is unavailable in the present context.

Proposition 2.10. *There exists $\varepsilon_0 > 0$ such that for any $u_0 \in \dot{H}^1$, and for any interval $I \ni 0$ such that*

$$\|(-\Delta_g)^{1/2} e^{it\Delta_g} u_0\|_{L^{10} L^{\frac{30}{13}}(I \times \mathbf{R}^3)} \le \varepsilon \le \varepsilon_0,$$

there is a unique solution to (1.1) on I with $u(0,x) = u_0$, which also satisfies

$$\|(-\Delta_g)^{1/2} u\|_{L^{10} L^{\frac{30}{13}}} \le 2\varepsilon. \tag{2.7}$$

In particular, solutions with sufficiently small energy are global and scatter.

Proof. Run contraction mapping on the space X defined by the conditions

$$\|(-\Delta_g)^{1/2} u\|_{L^{10} L^{\frac{30}{13}}} \le 2\varepsilon, \quad \|(-\Delta_g)^{1/2} u\|_{L^\infty L^2} \le \|u_0\|_{\dot{H}^1} + \varepsilon$$

equipped with the metric $\rho(u,v) = \|(-\Delta_g)^{1/2}(u-v)\|_{L^{10}L^{\frac{30}{13}}}$. For each $u \in X$, let $\mathcal{I}(u)$ be the solution to the linear equation

$$(i\partial_t + \Delta_g)\mathcal{I}(u) = |u|^4 u$$

We check that for ε sufficiently small, the map $u \mapsto \mathcal{I}(u)$ is a contraction on X. By the Duhamel formula

$$u(t) = e^{it\Delta_g}u(0) - i\int_0^t e^{i(t-s)\Delta_g}(i\partial_t + \Delta_g)u(s)\,ds,$$

Strichartz, the Leibniz rule, and Sobolev embedding,

$$\begin{aligned}\|(-\Delta_g)^{1/2}\mathcal{I}(u)\|_{L^{10}L^{\frac{30}{13}}} &\le \|(-\Delta_g)^{1/2}e^{it\Delta_g}u_0\|_{L^{10}L^{\frac{30}{13}}} + c\|(-\Delta_g)^{1/2}(|u|^4u)\|_{L^2L^{\frac{6}{5}}} \\ &\le \varepsilon + c\|(-\Delta_g)^{1/2}u\|^5_{L^{10}L^{\frac{30}{13}}} \le \varepsilon + c(2\varepsilon)^5,\end{aligned}$$

$$\|(-\Delta_g)^{1/2}\mathcal{I}(u)\|_{L^\infty L^2} \le \|(-\Delta_g)^{1/2}u_0\|_{L^2} + c(2\varepsilon)^5$$

Thus $\mathcal{I}$ maps X into itself.

For $u, v \in X$, the difference $\mathcal{I}(u) - \mathcal{I}(v)$ solves the equation with right hand side

$$|u|^4u - |v|^4v = \big(|u|^4 + \overline{u}v(|u|^2 + |v|^2)\big)(u-v) + v^2(|u|^2+|v|^2)(\overline{u}-\overline{v}).$$

Hence, applying the Leibniz rule and Sobolev embedding repeatedly,

$$\begin{aligned}&\|(-\Delta_g)^{1/2}[\mathcal{I}(u) - \mathcal{I}(v)]\|_{L^{10}L^{\frac{30}{13}}} \\ &\lesssim \|(-\Delta_g)^{1/2}(u-v)\|_{L^{10}L^{\frac{30}{13}}}(\|(-\Delta_g)^{1/2}u\|^4_{L^{10}L^{\frac{30}{13}}} + \|(-\Delta_g)^{1/2}v\|^4_{L^{10}L^{\frac{30}{13}}}) \\ &\lesssim (2\varepsilon)^4\|(-\Delta_g)^{1/2}(u-v)\|_{L^{10}L^{\frac{30}{13}}}.\end{aligned}$$

□

Proposition 2.11. *Let $\tilde{u}$ solve the perturbed equation*

$$(i\partial_t + \Delta_g)\tilde{u} = \tilde{u}^4\tilde{u} + e, \tag{2.8}$$

and let $0 \in I$ be an interval such that

$$\|\tilde{u}\|_{Z(I)} \le L, \quad \|\nabla\tilde{u}\|_{L^\infty L^2} \le E.$$

Then there exists $\varepsilon_0(E, L)$ such that if $\varepsilon \le \varepsilon_0$ and

$$\|\tilde{u}(0) - u_0\|_{\dot{H}^1} + \|\nabla e\|_{N(I)} \le \varepsilon,$$

there is a unique solution u to (1.1) on I with $u(0) = u_0$, with

$$\begin{aligned}\|u - \tilde{u}\|_{Z(I)} + \|\nabla(u - \tilde{u})\|_{L^2L^6\cap L^\infty L^2} &\le C(E,L)\varepsilon \\ \|\nabla u\|_{L^2L^6\cap L^\infty L^2(I\times\mathbf{R}^3)} &\le C(E,L).\end{aligned}$$

3. Convergence of linear propagators

In this section and the next, we relate the linear Schrödinger propagators for the curved and flat metrics under several limits. This will play an essential technical part when we construct the profile decompositions, and also provides a stepping stone for an analogous analysis of the nonlinear equations in Section 6.

Theorem 3.1. *Let $(\lambda_n, x_n) \in (0,\infty) \times \mathbf{R}^3$ be a sequence of length scales and spatial centers conforming to one of the following scenarios:*

(a) $\lambda_n \to \infty$.
(b) $|x_n| \to \infty$.
(c) $x_n \to x_\infty$, $\lambda_n \to 0$.

Let $\Delta := \delta^{jk}\partial_j\partial_k$ *in the first two cases and* $\Delta := g^{jk}(x_\infty)\partial_j\partial_k$ *in the third. Then for any* $\phi \in \dot{H}^1$*, writing* $\phi_n = \lambda_n^{-\frac{d-2}{2}}\phi(\frac{\cdot - x_n}{\lambda_n})$*, we have*

$$\lim_{n\to\infty} \|e^{it\Delta_g}\phi_n - e^{it\Delta}\phi_n\|_{L^\infty L^6(\mathbf{R}\times\mathbf{R}^3)} = 0.$$

In cases (a), (b), the convergence actually occurs in $L^\infty \dot{H}^1$*.*

Proof. By approximation in $\dot{H}^1$, we may assume that ϕ is Schwartz.

Suppose first that $\lambda_n \to \infty$. By the Strichartz inequality, the equivalence of Sobolev norms, and the Leibniz rule,

$$\begin{aligned}\|e^{it\Delta_g}\phi_n - e^{it\Delta}\phi_n\|_{L^\infty L^6} &\lesssim \|(-\Delta_g)^{1/2}(\Delta_g-\Delta)e^{it\Delta}\phi_n\|_{L^2L^{6/5}}\\ &\lesssim \|\chi\nabla e^{it\Delta}\phi_n\|_{L^2L^{6/5}} + \|\chi\nabla^2 e^{it\Delta}\phi_n\|_{L^2L^{6/5}} + \|\chi\nabla^3 e^{it\Delta}\phi_n\|_{L^2L^{6/5}}\end{aligned}$$

where $\chi(x)$ is the characteristic function of the unit ball. By Hölder and the Euclidean dispersive estimate,

$$\|\chi\nabla e^{it\Delta}\phi_n\|_{L^2L^{6/5}} = \lambda_n^2\|\chi(\lambda_n\cdot + x_n)e^{it\Delta}\nabla\phi\|_{L^2L^{6/5}} \lesssim \lambda_n^{-\frac{1}{2}}\|e^{it\Delta}\nabla\phi\|_{L^2L^\infty} \lesssim \lambda_n^{-1/2}\|\nabla phi\|_{L^1};$$

to be entirely accurate, the last inequality is valid for $|t| \geq 1$, while on the time interval $|t| \leq 1$ one simply uses Sobolev embedding $\|e^{it\Delta}\nabla\phi\|_{L^\infty_x} \lesssim \|e^{it\Delta}\nabla\phi\|_{H^s_x}$ for $s > 3/2$.

The terms involving two or more derivatives enjoy even better decay since $\lambda_n \to \infty$.

Assume now that $|x_n| \to \infty$, $\lambda_n \equiv \lambda_0 \in (0,\infty)$. By the Duhamel formula and Sobolev embedding,

$$\|e^{it\Delta_g}\phi_n - e^{it\Delta}\phi_n\|_{L^\infty L^2} \lesssim \|(\Delta_g-\Delta)e^{it\Delta}\phi_n\|_{L^1L^2} \lesssim \|\chi e^{it\Delta}\nabla\phi_n\|_{L^1L^2} + \|\chi e^{it\Delta}\nabla^2\phi_n\|_{L^1L^2},$$

where χ is a bump function supported on the unit ball. For any fixed $T > 0$, decompose

$$\|\chi e^{it\Delta}\nabla\phi_n\|_{L^1L^2} \leq \|\chi_n e^{it\Delta}\nabla\phi\|_{L^1L^2(\{|t|\leq T\})} + \|\chi_n e^{it\Delta}\nabla\phi\|_{L^1L^2(\{|t|>T\})},$$

where $\chi_n = \chi(\lambda_0\cdot + x_n)$. The first term vanishes as $n \to \infty$ because the orbit $\{e^{it\Delta}\nabla\phi\}_{|t|\leq T}$ is compact in L^2. We use Hölder's inequality and the dispersive estimate to bound the second term by

$$\|e^{it\Delta}\phi\|_{L^1L^\infty(\{|t|>T\})} \lesssim T^{-\frac{1}{2}}\|\phi\|_{L^1}.$$

As T may be chosen arbitrarily large, we conclude that $\lim_{n\to\infty}\|\chi e^{it\Delta}\nabla\phi_n\|_{L^1L^2} = 0$, and similar considerations estimate the term $\|\chi e^{it\Delta}\nabla^2\phi_n\|_{L^1L^2}$. Finally, we have

$$\|e^{it\Delta_g}\phi_n - e^{it\Delta}\phi_n\|_{L^\infty L^6} \leq \|\cdots\|_{L^\infty L^2}^{\frac{1}{3}}\|\cdots\|_{L^\infty L^\infty}^{\frac{2}{3}},$$

and the uniform norms may be estimated via Sobolev embedding:

$$\|e^{it\Delta_g}\phi_n\|_{L^\infty L^\infty} \lesssim \|(1-\Delta_g)e^{it\Delta_g}\phi_n\|_{L^\infty L^2} \lesssim \|(1-\Delta_g)\phi_n\|_{L^2} \lesssim 1.$$

Consider now the scenario where $|x_n| \to \infty$ and $\lambda_n \to 0$. We may assume that ϕ is compactly supported. Let χ be a smooth function such that $\chi(x) = 1$ when $|x| \geq 11/10$ and $\chi(x) = 0$ for $|x| \leq 1$. First we show

$$\lim_{n\to\infty}\|(1-\chi)e^{it\Delta}\phi_n\|_{L^\infty L^6} = 0. \tag{3.1}$$

The function $\chi e^{it\Delta}\phi_n$ solves the equation

$$(i\partial_t + \Delta)(\chi e^{it\Delta}\phi_n) = [\Delta,\chi]e^{it\Delta}\phi_n.$$

Thus, by Sobolev embedding and the Duhamel formula,

$$\|(1-\chi)e^{it\Delta}\phi_n\|_{L^\infty L^6} \lesssim \|\nabla[\chi,\Delta]e^{it\Delta}\phi_n\|_{L^1L^2}.$$

The right side has the form

$$\|\beta e^{it\Delta}\nabla\phi_n\|_{L^1L^2} + \|\beta e^{it\Delta}\nabla^2\phi_n\|_{L^1L^2}$$

where β is a bump function localizing to the unit ball. We focus on the potentially more dangerous second term. Fix $T > 0$ large, and split

$$\|\beta e^{it\Delta}\nabla^2\phi_n\|_{L^1L^2} \le \|\beta e^{it\Delta}\nabla^2\phi_n\|_{L^1L^2(\{|t|\le T\lambda_n\})} + \|\beta e^{it\Delta}\nabla^2\phi_n\|_{L^1L^2(\{|t|>T\lambda_n\})}.$$

By Hölder in time and a change of variable, the first term is bounded by

$$T\|\beta(x_n + \lambda_n\cdot)e^{it\Delta}\nabla^2\phi\|_{L^\infty L^2(\{|t|\le T\lambda_n^{-1}\})},$$

which goes to zero as $n \to \infty$ by approximate finite speed of propagation or, more precisely, by the Fraunhofer formula

$$\lim_{t\to\infty} \|e^{it\Delta}f - (2it)^{-\frac{3}{2}}\hat{f}(\tfrac{x}{2t})e^{\frac{|x|^2}{4t}}\|_{L^2} = 0.$$

On the other hand, by the Euclidean dispersive estimate we have

$$\|\beta e^{it\Delta}\nabla^2\phi_n\|_{L^1L^2(\{|t|>T\lambda_n\})} \lesssim \|e^{it\Delta}\nabla^2\phi_n\|_{L^1L^\infty(\{|t|>T\lambda_n\})} \lesssim \lambda_n^{\frac{1}{2}}(T\lambda_n)^{-\frac{1}{2}}\|\phi\|_{L^1} \lesssim T^{-\frac{1}{2}}.$$

Hence, choosing T arbitrarily large we conclude that

$$\lim_{n\to\infty} \|\beta e^{it\Delta}\nabla^2\phi_n\|_{L^1L^2} = 0,$$

establishing (3.1).

Since $\Delta = \Delta_g$ on the support of the cutoff χ, we also have

$$(i\partial_t + \Delta_g)(\chi e^{it\Delta}\phi_n) = [\Delta, \chi]e^{it\Delta}\phi_n,$$

so by the Duhamel formula, Sobolev embedding, and the equivalence of $\dot{H}^1$ Sobolev norms,

$$\begin{aligned}\|e^{it\Delta_g}\phi_n - \chi e^{it\Delta}\phi_n\|_{L^\infty L^6} &= \Big\|\int_0^t e^{i(t-s)\Delta_g}[\Delta,\chi]e^{is\Delta}\phi_n\,ds\Big\|_{L^\infty L^6} \lesssim \|(-\Delta_g)^{1/2}[\chi,\Delta]e^{it\Delta}\phi_n\|_{L^1L^2} \\ &\lesssim \|\nabla[\chi,\Delta]e^{it\Delta}\phi_n\|_{L^1L^2}\end{aligned}$$

which was just estimated.

Finally, consider the last case where the profile ϕ_n is concentrating at a point. For $T > 0$, split

$$\|e^{it\Delta_g}\phi_n - e^{it\Delta}\phi_n\|_{L^\infty L^6} \le \|\cdots\|_{L^\infty L^6(\{|t|\le T\lambda_n^2\})} + \|\cdots\|_{L^\infty L^6(\{|t|>T\lambda_n^2\})}. \tag{3.2}$$

For the short time contribution, let χ be a bump function centered at the origin, fix $0 < \theta < 1$, and define

$$\chi_n = \chi\Big(\frac{\cdot - x_n}{\lambda_n^\theta}\Big),\ v_n = e^{it\Delta}\phi_n.$$

Then

$$(i\partial_t + \Delta)(\chi_n v_n) = [\Delta, \chi_n]v_n = 2\langle\nabla_\infty\chi_n, \nabla v_n\rangle_\infty + (\Delta\chi_n)v_n,$$

where the inner product on the right is respect to the metric $g(x_\infty)$, hence

$$\begin{aligned}\|(1-\chi_n)v_n\|_{L^\infty L^6(\{|t|\le T\lambda_n^2\})} &\lesssim \|(1-\chi_n)\phi_n\|_{\dot{H}^1} + \|\nabla[\Delta,\chi_n]v_n\|_{L^1L^2(\{|t|\le T\lambda_n^2\})} \\ &\lesssim o(1) + T\lambda_n\|\phi\|_{H^2}.\end{aligned}$$

Further, writing $(i\partial_t + \Delta) = (i\partial_t + \Delta_g) + (\Delta - \Delta_g)$, we obtain by the Duhamel formula and Sobolev embedding

$$\begin{aligned}\|e^{it\Delta_g}\phi_n - \chi_n e^{it\Delta}\phi_n\|_{L^\infty L^6(\{|t|\le T\lambda_n^2\})} &\lesssim \|(1-\chi_n)\phi_n\|_{\dot{H}^1} + \|\nabla[\Delta,\chi_n]v_n\|_{L^1L^2(\{|t|\le T\lambda_n^2\})} \\ &\quad + \|\nabla(\Delta_g - \Delta)(\chi_n v_n)\|_{L^1L^2(\{|t|\le T\lambda_n^2\})}.\end{aligned}$$

The first two terms were estimated before. Writing out $\Delta_g - \Delta$ explicitly and using the Leibniz rule, we see that the worst contributions to the last term are quantities of the form

$$\|(g - g(x_\infty))\chi_n \nabla^3 v_n\|_{L^1 L^2(\{|t| \le T\lambda_n^2\})} \lesssim T\lambda_n^2 \lambda_n^{-2}(|x_n - x_\infty| + \lambda_n^\theta)\|e^{it\Delta}\nabla^3\phi\|_{L^\infty L^2},$$

which is acceptable.

The long time contribution to (3.2) is bounded by

$$\|e^{it\Delta_g}\phi_n\|_{L^\infty L^6(\{|t|>T\lambda_n^2\})} + \|e^{it\Delta}\phi_n\|_{L^\infty L^6(\{|t|>T\lambda_n^2\})},$$

which are dealt with respectively by the extinction lemma in the next section and the usual dispersive estimate

$$\|e^{it\Delta}\phi_n\|_{L^\infty L^6(\{|t|>T\lambda_n^2\})} \lesssim T^{-1}\|\phi\|_{L^{6/5}}.$$

□

The proof of the last case yields the following corollary, which asserts that on short time intervals, the convergence in Case (c) of the theorem occurs in the energy norm as well.

Corollary 3.2. *Let (λ_n, x_n) be a sequence such that $x_n \to x_\infty$ and $\lambda_n \to 0$. Then for any $T > 0$*

$$\lim_{n\to\infty} \|e^{it\Delta_g}\phi_n - e^{it\Delta}\phi_n\|_{L^\infty \dot{H}^1([-T\lambda_n^2, T\lambda_n^2]\times\mathbf{R}^3)} = 0.$$

4. An extinction lemma

The purpose of this section is to prove a long-time weak dispersion estimate for linear profiles concentrating within a bounded distance of the origin, which arise in the last case of Theorem 3.1. For profiles with width h, we want to establish decay for times $t \ge Th^2$ as $h \to 0$ and $T \to \infty$. We consider the times $t \le O(h)$ and $t \gg h$ separately. Semiclassical techniques are used for the first regime, while for longer times we invoke the global geometry to see that the solution is essentially Euclidean. Our ingredients consist of the frequency-localized dispersion estimate of Burq-Gerard-Tzvetkov [BGT04], a wavepacket parametrix, and a non-concentration estimate for the geodesic flow.

Proposition 4.1. *Let $d \ge 3$, and suppose $x_h \to x_0 \in \mathbf{R}^d$ as $h \to 0$. For any $\phi \in \dot{H}^1$, denoting $\phi_h = h^{-\frac{d-2}{2}}\phi(h^{-1}(\cdot - x_h))$, we have*

$$\lim_{T\to\infty} \limsup_{h\to 0} \|e^{it\Delta_g}\phi_h\|_{L^\infty L^{\frac{2d}{d-2}}(([Th^2,\infty)\times\mathbf{R}^d)} = 0.$$

Proof. We begin with several reductions. After a translation we may assume that $x_0 = 0$. Also, letting $\rho = |g|^{\frac14}$ be the square root of the Riemannian density, we have $e^{it\Delta_g} = \rho^{-1}e^{-itA}\rho$, where

$$A = \rho(-\Delta_g)\rho^{-1} = -\partial_j g^{jk}\partial_k + V$$

is self-adjoint on $L^2(dx)$ and V is a compactly supported potential. Thus

$$\begin{aligned}\|e^{it\Delta_g}\phi_h\|_{L^\infty L^{\frac{2d}{d-2}}} &\lesssim \|e^{-itA}\rho\phi_h\|_{L^\infty L^{\frac{2d}{d-2}}} \lesssim \rho(x_h)\|e^{-itA}\phi_h\|_{L^\infty L^{\frac{2d}{d-2}}} + \|e^{-itA}(\rho - \rho(x_h))\phi_h\|_{L^\infty L^{\frac{2d}{d-2}}}\\ &\lesssim \|e^{-itA}\phi_h\|_{L^\infty L^{\frac{2d}{d-2}}} + o(1) \text{ as } h \to 0,\end{aligned}$$

and it suffices to show

$$\lim_{T\to\infty} \limsup_{h\to 0} \|e^{-itA}\phi_h\|_{L^\infty L^{\frac{2d}{d-2}}([Th^2,\infty)\times\mathbf{R}^d)} = 0. \tag{4.1}$$

Compared to the Laplacian, the conjugated operator A leads to better error terms when we later consider the dynamics driven by the principal symbol $a(x,\xi) = g^{jk}\xi_j\xi_k$; the Weyl quantization $a^w(X,D)$ differs from A by a zero order potential whereas $a^w(X,D) + \Delta_g$ is first order.

Further, we shall assume that ϕ is Schwartz and that its Fourier transform $\hat{\phi}$ is supported in a frequency annulus

$$\operatorname{supp}\hat{\phi} \subset \{\varepsilon < |\xi| < \varepsilon^{-1}\} \tag{4.2}$$

for some $\varepsilon > 0$; the rescaled initial data ϕ_h are therefore frequency-localized to the region $\{h^{-1}\varepsilon < |\xi| < h^{-1}\varepsilon^{-1}\}$.

By the semiclassical dispersion estimate of Burq-Gerard-Tzvetkov [BGT04, Lemma A3] (see also [KT05, Proposition 4.7]), there exists $c > 0$ such that

$$\|e^{-itA}\phi_h\|_{L^\infty L^{\frac{2d}{d-2}}([Th^2,ch]\times\mathbf{R}^d)} \lesssim |Th^2|^{-1}\|\phi_h\|_{L^{\frac{2d}{d+2}}} = T^{-1}\|\phi\|_{L^{\frac{2d}{d+2}}}.$$

Hence, it remains to prove the long-time extinction

$$\lim_{h\to 0}\|e^{-itA}\phi_h\|_{L^\infty L^{\frac{2d}{d-2}}([ch,\infty)\times\mathbf{R}^d)} = 0. \tag{4.3}$$

Wavepacket decomposition. Using the notation of Section 2.4, we apply the FBI transform to decompose the initial data

$$\phi_h = \int \psi^h_{(x,\xi)} T_h\phi_h(x,\xi)\, dxd\xi$$

into wavepackets localized to $(h^{1/2})^d \times (h^{-1/2})^d$ boxes in phase space. We may restrict attention to just the wavepackets from the region

$$B = \{(x,\xi) : |x - x_h| \le h^\theta,\ \tfrac{\varepsilon}{10} \le \xi \le \tfrac{10}{\varepsilon}\} \tag{4.4}$$

for any $\theta < \frac{1}{2}$. Indeed, if $|x - x_h| > h^\theta$ then

$$\begin{aligned}
|T_h\phi_h(x,\xi)| &\lesssim h^{-\frac{3d}{4}} h^{-\frac{d-2}{2}} \int e^{-\frac{(x-x_h-y)^2}{2h}} |\phi(\tfrac{y}{h})|\, dy \\
&\lesssim h^{1-\frac{5d}{4}} \int_{|y|\le|x-x_h|/4} + h^{1-\frac{5d}{4}} \int_{|y|>|x-x_h|/4} \\
&\lesssim_N h^{1-\frac{5d}{4}+\theta d} e^{-\frac{(x-x_h)^2}{ch}} + h^{1-\frac{5d}{4}} h^N |x-x_h|^{-N} \\
&\lesssim_{M,N} h^M |x-x_h|^{-N}
\end{aligned}$$

for any $M, N \ge 0$. Similarly,

$$|T_h\phi_h(x,\xi)| \lesssim h^{1-\frac{3d}{4}} \int e^{-\frac{(\eta-\xi)^2}{2h}} |\hat{\phi}(\eta)|\, d\eta \lesssim \begin{cases} h^{1-\frac{3d}{4}} e^{-\frac{\varepsilon^2}{ch}}, & |\xi| < \varepsilon/10 \\ h^{1-\frac{3d}{4}} e^{-\frac{\xi^2}{ch}}, & |\xi| > 10/\varepsilon \end{cases}$$

In view of these bounds, we write

$$\phi_h = T_h^* 1_B T_h\phi_h + T_h^*(1 - 1_B)T_h\phi_h = f_h^1 + f_h^2, \tag{4.5}$$

where by the triangle inequality we obtain, for any $k \ge 0$,

$$\|\partial^k f_h^2\|_{L^2} \lesssim \int_{B^c} (h^{-\frac{d+k}{2}} + h^{-\frac{d}{2}}|h^{-1}\xi|^k)|T_h\phi_h(x,\xi)|\, dxd\xi = O(h^\infty).$$

By Sobolev embedding, it therefore suffices to show

$$\lim_{h\to 0}\|e^{-itA}f_h^1\|_{L^\infty L^{\frac{2d}{d-2}}((ch,\infty)\times\mathbf{R}^3)} = 0. \tag{4.6}$$

To prove this, we fix a large $T > 0$ and consider separately the time intervals $[ch, Th]$ and $[Th, \infty)$. On semiclassical time scales, the quantum evolution of wavepackets is modeled by the geodesic flow. More precisely, if $\psi^h_{(x,\xi)}$ is a typical wavepacket, then for $|t| \le Th$ its Schrödinger evolution $e^{-itA}\psi^h_{(x,\xi)}$ will have width $C_T h^{1/2}$ and travel along the geodesic starting at x with initial momentum $h^{-1}\xi$ (that is, with velocity $h^{-1}g^{ab}\xi_b$).

If T is sufficiently large, then by the nontrapping assumption on the metric, all the wavepackets $e^{-iThA}\psi^h_{(x,\xi)}$ with $(x,\xi) \in B$ will have exited the curved region (this is why it is convenient to assume that ϕ is frequency-localized away from 0), and for $t \geq Th$ the solution $e^{-itA}\psi^h_{(x,\xi)}$ will radiate to infinity while dispersing essentially as a Euclidean free particle. The decay for $e^{-itA}f^1_h$ will then be a consequence of the dispersive properties of the Euclidean propagator $e^{it\Delta_{\mathbf{R}^3}}$.

It will be notationally convenient in the sequel to rescale time semiclassically, that is, replace t by th, so that each wavepacket $\psi^h_{(x,\xi)}$ travels at speed $O(1)$ under the propagator e^{-ithA}. The desired estimate then becomes

$$\lim_{h\to 0} \|e^{-ithA}f^1_h\|_{L^\infty L^{\frac{2d}{d-2}}((c,\infty)\times\mathbf{R}^3)} = 0.$$

Frequency-localization. We show next that the operator A may be replaced, up to acceptable errors, by a frequency-localized version. This will bring us into the framework of Koch and Tataru [KT05] for studying the evolution of wavepackets at fixed frequency.

Choose frequency cutoffs $\chi_j \in C_0^\infty(\mathbf{R}^d \setminus \{0\})$ such that

$$\{\xi : \varepsilon \leq |\xi| \leq \varepsilon^{-1}\} \prec \chi_1 \prec \chi_2 \prec \chi_3;$$

that is, $\chi_1(\xi) = 1$ on the annulus $\varepsilon \leq |\xi| \leq \varepsilon^{-1}$ and $\chi_j = 1$ near the support of χ_{j-1}. Set $A(h) = h^2 A$, let $a = g^{ij}\xi_i\xi_j$ be the principal symbol of A, and define the localized operator $A'(h)$ by

$$A'(h)f(x) = (\chi_3 a)^w(X, hD)f(x) = (2\pi h)^{-d}\iint e^{\frac{i(x-y)\xi}{h}} a\Big(\frac{x+y}{2},\xi\Big)\chi_3(\xi)f(y)\,dyd\xi.$$

Lemma 4.2. *The propagator $e^{-\frac{itA'(h)}{h}}$, which preserves L^2, is also bounded on $\dot{H}^1$ when restricted to frequency h^{-1}:*

$$\|e^{-\frac{itA'(h)}{h}}\chi_1(hD)\|_{\dot{H}^1\to\dot{H}^1} \lesssim (1+|t|).$$

Proof. Let $u_h = e^{-\frac{itA'(h)}{h}}\chi_1(hD)$ be the solution to the evolution equation

$$[hD_t + A'(h)]u_h = 0, u_h(0) = \chi_1(hD)\phi.$$

Differentiating this equation, we obtain

$$[hD_t + A'(h)](hD)u_h = [hD, A'(h)]u_h.$$

By the pseudo-differential calculus (see e.g. [Zwo12]), $\|[hD, A'(h)]\|_{L^2\to L^2} \leq ch$, so

$$\begin{aligned}\|hDu_h(t)\|_{L^2} &\leq \|hDu_h(0)\|_{L^2} + h^{-1}\int_0^t \|[hD, A'(h)]u_h(s)\|_{L^2}\,ds \\ &\leq \|hDu_h(0)\|_{L^2} + c|t|\|\chi_1(hD)\phi\|_{L^2} \\ &\leq c(1+|t|)\|hD\chi_1(hD)\phi\|_{L^2}.\end{aligned}$$

□

Lemma 4.3. *For each $T > 0$ and for all $|t| \leq T$,*

$$\|(e^{-\frac{itA(h)}{h}} - e^{-\frac{itA'(h)}{h}})\chi_1(hD)\|_{\dot{H}^1\to\dot{H}^1} \leq c_T h|t|$$

Proof. Write

$$e^{-\frac{itA(h)}{h}} - e^{-\frac{itA'(h)}{h}} = (e^{-\frac{itA(h)}{h}} - e^{-\frac{it\tilde{A}(h)}{h}}) + (e^{-\frac{it\tilde{A}(h)}{h}} - e^{-\frac{itA'(h)}{h}}),$$

where

$$\tilde{A}(h) = a^w(X, hD) = -h^2\partial_j g^{jk}\partial_k - \frac{h^2}{4}(\partial_j\partial_k g^{jk}).$$

By the Duhamel formula,

$$\|e^{-\frac{itA(h)}{h}}\phi - e^{-\frac{it\tilde{A}(h)}{h}}\phi\|_{L^2\to L^2} \le h\int_0^t \big|\frac{1}{4}\partial_j\partial_k g^{jk} + V\big| \|e^{-\frac{isA(h)}{h}}\phi\|_{L^2}\, ds \le ch|t|\|\phi\|_{L^2}.$$

Introducing the frequency-localization, we see from semiclassical functional calculus (see [BGT04]) that

$$\|(1-\chi_2(hD))e^{-\frac{itA(h)}{h}}\chi_1(hD)\|_{L^2\to H^\sigma} = O(h^\infty)$$

and similarly with A replaced by $\tilde{A}$. That is, the linear evolutions $e^{-\frac{itA(h)}{h}}$ and $e^{-\frac{it\tilde{A}(h)}{h}}$ preserve frequency-support.

Thus

$$\begin{aligned}
\|D(e^{-\frac{itA(h)}{h}} - e^{-\frac{it\tilde{A}(h)}{h}})\chi_1(hD)\phi\|_{L^2} &\le h^{-1}\|(e^{-\frac{itA(h)}{h}} - e^{-\frac{\tilde{A}(h)}{h}})\chi_1(hD)\phi\|_{L^2} + O(h^\infty)\\
&\lesssim |t|h\|\chi_1(hD)h^{-1}\phi\|_{L^2} + O(h^\infty)\\
&\lesssim |t|h\|D\phi\|_{L^2} + O(h^\infty).
\end{aligned} \tag{4.7}$$

Now we show that

$$\|(e^{-\frac{it\tilde{A}(h)}{h}} - e^{-\frac{itA'(h)}{h}})\chi_1(hD)\|_{\dot{H}^1\to\dot{H}^1} \le ch|t|.$$

For each $\phi \in \dot{H}^1$, the function $u_h = e^{-\frac{it\tilde{A}(h)}{h}}\chi_1(hD)\phi$ solves the equation $[hD_t + A'(h)]u_h = r_h$, where

$$r_h = [(\chi_3 - 1)a]^w(X, hD)\chi_2(hD)u_h + [(\chi_3-1)a]^w(X, hD)(1-\chi_2(hD))u_h.$$

As the symbols $(\chi_3 - 1)a$ and χ_2 have disjoint supports, the first term on the right is $O(h^\infty)$ in any Sobolev norm. The frequency localization of u_h implies that the second term is similarly negligible. By the Duhamel formula and Lemma 4.2, for any $T > 0$ and $|t| \le T$ we have

$$\|(e^{-\frac{itA'(h)}{h}} - e^{-\frac{it\tilde{A}(h)}{h}})\chi_1(hD)\phi\|_{\dot{H}^1} \le c_T\int_0^t \|r_h(s)\|_{\dot{H}^1}\, ds \le |t|O(h^\infty). \tag{4.8}$$

(4.7) and (4.8) complete the proof. □

Returning to the decomposition (4.5) and recalling that $\phi_h = \chi_1(hD)\phi_h$, we have

$$\|(1-\chi_1(hD))f_h^1\|_{\dot{H}^1} = O(h^\infty),$$

By the previous lemma and Sobolev embedding, (4.6) will follow from the claims

$$\lim_{h\to 0}\|e^{-\frac{itA'(h)}{h}}f_h^1\|_{L_t^\infty L^{\frac{2d}{d-2}}((c,T)\times\mathbf{R}^d)} = 0 \tag{4.9}$$

$$\lim_{h\to 0}\|e^{-\frac{i(t-T)A(h)}{h}}e^{-\frac{iTA'(h)}{h}}f_h^1\|_{L^\infty L^{\frac{2d}{d-2}}((T,\infty)\times\mathbf{R}^d)} = 0. \tag{4.10}$$

Evolution of a wavepacket. We describe first the short-time behavior of the Schrödinger flow on a wavepacket. Let $(x,\xi)\mapsto(x^t(x,\xi),\xi^t(x,\xi))$ be the Hamilton flow on $T^*\mathbf{R}^d$ generated by the symbol $a(x,\xi) = g^{jk}(x)\xi_j\xi_k$, that is, defined by the ODE

$$\begin{cases} \dot{x}^t = a_\xi = 2g(x^t)\xi^t, \\ \dot{\xi}^t = -a_x = -(\xi^t)^*(\partial_x g)(x^t)\xi^t \end{cases} \qquad (x^0,\xi^0) = (x,\xi). \tag{4.11}$$

The curve $t \mapsto x^t(x,\xi)$ is the geodesic starting at x with tangent vector $g^{ab}\xi_b$, and for fixed y the mapping $\eta \mapsto x^1(y,\eta)$ is essentially the exponential map with basepoint y. A standard fact from geometry is the identity

$$x^t(x,\xi) = x^1(x, t\xi), \tag{4.12}$$

which follows from the fact that $s \mapsto (x^{ts}(x,\xi), t\xi^{ts}(x,\xi))$ is the bicharacteristic with initial data $(x, t\xi)$.

Proposition 4.4. *Let $\psi^h_{(x_0,\xi_0)}$ be a wavepacket. Then*

$$e^{-\frac{itA'(h)}{h}}\psi^h_{(x_0,\xi_0)}(x) = h^{-\frac{3d}{4}} v\Big(x_0, \xi_0, t, \frac{x - x_0^t}{h^{1/2}}\Big) e^{\frac{i}{h}[\xi_0^t(x-x_0^t)+\gamma(t,x_0,\xi_0)]},$$

where $\gamma(t,x_0,\xi_0) = \int_0^t (\xi a_\xi - a)(x_0^s, \xi_0^s)\,ds$, and $v(x_0,\xi_0,t,\cdot)$ is Schwartz uniformly in (x_0,ξ_0) and locally uniformly in t.

Proof. This was proved in [KT05] when $h = 1$. We reduce to that case by a change of variable.

For fixed (x_0,ξ_0), let u be the solution to

$$[hD_t + A'(h)]u = 0, \;\; u(0) = \psi^h_{(x_0,\xi_0)},$$

and define the profile v by

$$u(t,x) = h^{-\frac{3d}{4}} v\Big(t, \frac{x - x_0^t}{h^{1/2}}\Big) e^{\frac{i}{h}[\xi_0^t(x-x_0^t)+\gamma(t,x_0,\xi_0)]}.$$

Then v solves the equation $[D_t + (a^h_{(x_0,\xi_0)})^w(t,X,D)v = 0, = v(0) = \psi^1_{(0,0)}$, where

$$a^h_{(x_0,\xi_0)}(t,x,\xi) = h^{-1}[a(t, h^{1/2}x + x_0^t, h^{1/2}\xi + \xi_0^t) - h^{1/2}\xi a_\xi(x_0^t,\xi_0^t) - h^{1/2}x a_x(x_0^t,\xi_0^t) - a(x_0^t,\xi_0^t)].$$

As $a^h_{(x_0,\xi_0)}$ vanishes to second order at $(0,0)$ and satisfies $|\partial_x^\alpha\partial_\xi^\beta a^h_{(x_0,\xi_0)}| \le c_{\alpha\beta}$ for all $|\alpha|+|\beta| \ge 2$, the claim follows from Lemma 4.5 below. $\square$

The following lemma was the key step in the proof of [KT05, Proposition 4.3]

Lemma 4.5. *Let $a(t,\cdot,\cdot)$ be a time-dependent symbol which vanishes to second order at $(0,0)$ and satisfies $|\partial_x^\alpha\partial_\xi^\beta a| \le c_{\alpha\beta}$ whenever $|\alpha|+|\beta| \ge 2$, and $S(t,s)$ be the solution operator for the evolution equation*

$$[D_t + a^w(t,X,D)]u = 0.$$

Then $S(t,s) : \mathcal{S}(\mathbf{R}^d) \to \mathcal{S}(\mathbf{R}^d)$ locally uniformly in $t-s$.

Next we consider the long-time dynamics. As the metric is assumed to be nontrapping, there exists $T = O(\varepsilon^{-1})$ such that for each $(x,\xi) \in B$, $|x^t| \ge 10$ for all $t \ge T$.

Proposition 4.6. *For each $(x_0,\xi_0) \in B$ and $t \ge T$, we have a decomposition*

$$e^{-\frac{i(t-T)A(h)}{h}} e^{-\frac{iTA'(h)}{h}} \psi^h_{(x_0,\xi_0)} = v_1 + v_2,$$

where

$$|\partial^k v_1(t,x)| \le C_{k,N} h^{-\frac{3d}{4}-k}|t|^{-d/2}\Big(1 + \frac{|x - x_0^t|}{h^{1/2}|t|}\Big)^{-N}$$

and $\|v_2\|_{H^k} = O(h^\infty)$ for all k.

Proof. It suffices to verify the following two assertions:

$$|\partial_x^k e^{i(t-T)h\Delta} e^{-\frac{iTA'(h)}{h}} \psi^h_{(x_0,\xi_0)}(x)| \le C_{k,N} h^{-\frac{3d}{4}-k}|t|^{-d/2}\Big(1+\frac{|x-x_0^t|}{h^{1/2}|t|}\Big)^{-N} \tag{4.13}$$

$$\|(e^{-i(t-T)hA} - e^{i(t-T)h\Delta})e^{-\frac{iTA'(h)}{h}}\psi^h_{(x_0,\xi_0)}\|_{H^k} \le C_{k,N}h^N. \tag{4.14}$$

For ϕ Schwartz, by a stationary phase argument we have

$$|e^{it\Delta}\phi(x)| \le C_N \langle t\rangle^{-d/2}\Big\langle \frac{x}{2t}\Big\rangle^{-N}.$$

Indeed, let χ be a smoothed characteristic function of the unit ball, and partition

$$e^{it\Delta}\phi(x) = \int e^{i(x\xi - t|\xi|^2)}\chi\Big(\xi-\frac{x}{2t}\Big)\hat\phi(\xi)\,d\xi + \int e^{i(x\xi-t|\xi|^2)}[1-\chi\Big(\xi-\frac{x}{2t}\Big)]\hat\phi(\xi)\,d\xi.$$

Integrating by parts in ξ, the second term is bounded, for any $N \ge 0$, by $|t|^{-N}\langle\frac{x}{2t}\rangle^{-N}$. By the stationary phase expansion, the first term equals

$$(2\pi i t)^{-d/2} e^{\frac{i|x|^2}{4t}}\hat\phi\Big(\frac{x}{2t}\Big) + t^{-\frac{d}{2}-1}R(t,x),$$

$$|R(t,x)| \le c\|(1+|D_\xi|^2)^d\chi\Big(\xi-\frac{x}{2t}\Big)\hat\phi(\xi)\|_{L^2_\xi} \le C_N\Big\langle\frac{x}{2t}\Big\rangle^{-N}.$$

By the previous proposition and standard identities for the Euclidean propagator,

$$e^{i(t-T)h\Delta}e^{-\frac{iTA'(h)}{h}}\psi^h_{(x_0,\xi_0)} = h^{-\frac{3d}{4}}e^{\frac{i}{h}[\xi_0^t(x-x_0^t)+\gamma(t,x_0,\xi_0)]}e^{i(t-T)h\Delta}\Psi_{(x_0,\xi_0)}\Big(\frac{x-x_0^t}{h^{1/2}}\Big),$$

where $\Psi_{(x_0,\xi_0)}$ is Schwartz uniformly in (x_0,ξ_0), and we have used the fact that $(x_0^t,\xi_0^t) = (x_0^T + 2(t-T)\xi_0^T, \xi_0^T)$ for all $t \ge T$. This settles (4.13) for $k=0$. When $k>0$, we note that differentiating the above equation brings down at worst a factor of $|h^{-1}\xi_0| \le c\varepsilon^{-1}h^{-1}$.

To prove (4.14), we note first that e^{-itA} is uniformly bounded on each Sobolev space H^k. Indeed, for a sufficiently large $C>0$ the operators $(1-\Delta)^k(C+A)^{-k}$ and $(C+A)^k(1-\Delta)^{-k}$ are pseudodifferential operators of order 0, which implies that

$$\|u\|_{H^k} = \|(1-\Delta)^{k/2}u\|_{L^2} \sim \|(C+A)^{k/2}u\|_{L^2}.$$

Using the Duhamel formula, triangle inequality, and the above pointwise estimates, we can therefore bound the left side of (4.14) by

$$\int_T^\infty \sum_{m=0}^2 \|\chi D^m e^{i(t-T)h\Delta}e^{-\frac{iTA'(h)}{h}}\psi^h_{(x_0,\xi_0)}\|_{H^k} \le C_N h^N \int_T^\infty |t|^{-d/2}\,dt \le C_N h^N.$$

□

The semiclassical regime. By Proposition 4.4, on bounded time intervals each wavepacket may be regarded essentially as a particle moving under the geodesic flow. We will obtain the short-time decay (4.9) by showing that not too many wavepackets pile up near any point at any time. Heuristically, by the uncertainty principle the wavepackets have a broad distribution of initial momenta, and slower wavepackets will lag behind faster ones along each geodesic.

Lemma 4.7. *Let g be a nontrapping metric on $\mathbf{R}^d$. Then, for all $x, z \in \mathbf{R}^d$ with $|x| \le 1$ and all $0 \le r \le 1$,*

$$m(\{\xi \in \mathbf{R}^d : |x^1(x,\xi) - z| \le r\}) \le c_{x,z}r,$$

where m denotes Lebesgue measure on $\mathbf{R}^d_\xi$, and the constant $c_{x,z}$ is locally uniformly bounded in x and z. If also g is Euclidean outside a compact set, then

$$m(\{\xi \in \mathbf{R}^d : |x^1(x,\xi) - z| \le r\}) \le c_x(1+|z|)^{d-1}r.$$

The basic idea is that the preimage of a small ball under the exponential map will always be thin in the radial direction, though not necessarily in the other directions. This is a consequence of the fact that the exponential map always has nontrivial radial derivative. Note that for z near x, the above bound can be improved to $O(r^d)$ as the map $\xi \mapsto x^1(x,\xi)$ is a diffeomorphism for ξ near 0.

Proof. Fix x and z. For each ξ we have

$$x^1(x,\xi+\zeta) = x^1(x,\xi) + (\partial_\xi x^1)\zeta + r(\zeta),\ |r(\zeta)| = O(|\zeta|^2).$$

Differentiating equation (4.12) in t, we have

$$\partial_\xi x^1(x,\xi)\xi = \dot{x}^1(x,\xi) = g(x^1)\xi^1(x,\xi)$$

which implies that

$$|\xi^1||\partial_\xi x^1(x,\xi)\xi| \geq \xi^1 \cdot (\partial_\xi x^1)\xi = 2g(x^1)^{jk}(\xi^1)^j(\xi^1)^k \gtrsim |\xi^1|^2.$$

Using also the fact that $g^{jk}(x)\xi_j\xi_k$ is conserved along the flow, it follows that

$$|\partial_\xi x^1(x,\xi)\xi| \geq c|\xi|.$$

Thus, if ζ_0 is such that that $|r(\zeta)| \leq \frac{c}{2}|\zeta|$ for $|\zeta| \leq \zeta_0$, then

$$\frac{c}{2}|\zeta| \leq |x^1(x,\xi+\zeta) - x^1(x,\xi)| \leq 2c|\zeta|$$

for all ζ parallel to ξ with length at most ζ_0.

Let $S_{x,z}$ denote the set on the left side in the lemma; the nontrapping hypothesis implies that $S_{x,z}$ is compact. By the preceding considerations, the intersection of each ray $t \mapsto \frac{t\xi}{|\xi|}$ with $S_{x,z}$ has measure $O(r)$. The first inequality now follows by integrating in polar coordinates.

Under the additional hypothesis that g is flat outside a compact set, observe that for each x there exists $R > 0$ such that

$$\sup_{|\xi|_g=1} |x^t(x,\xi)| - R < 2t < \inf_{|\xi|_g=1} |x^t(x,\xi)| + R, \tag{4.15}$$

where $|\xi|_g^2 = g^{jk}(x)\xi_j\xi_k$. Indeed, for T sufficiently large and $t \geq T$,

$$x^t(x,\xi) = x^T(x,\xi) + 2(t-T)\xi^T(x,\xi),$$

and $|\xi^T| = |\xi^T|_g = |\xi^0|_g = |\xi|_g = 1$. Set $r = \sup_{|\xi|_g=1} |x^T(x,\xi)|$ to get

$$2|t-T| - r \leq |x^t(x,\xi)| \leq 2|t-T| + r.$$

Therefore, $S_{x,z}$ is contained in an annulus $\{|z| - R_x \leq |\xi| \leq |z| + R_x\}$ with thickness $O(r)$ as before, which implies that $m(S_{x,z}) \leq c(1+|z|)^{d-1}r$. □

Remark. It is clear that nontrapping hypothesis may be dispensed with from the first part provided that one restricts to a compact set of ξ. That is, if g is any metric, then for each $R > 0$ and $x, z \in \mathbf{R}^d$ with $|x| \leq 1$ and $0 \leq r < 1$, we have

$$m(\{\xi \in \mathbf{R}^d : |x^1(x,\xi) - z| \leq r\} \cap \{|\xi| \leq R\}) \leq c_{R,x,z}r.$$

We are ready to establish the short-time extinction (4.9). We have

$$|e^{-\frac{itA'(h)}{h}} f_h^1(x)| \leq \int_B |e^{-\frac{itA'(h)}{h}} \psi^h_{(x_0,\xi_0)}(x)||T_h\phi_h(x_0,\xi_0)|\, dx_0 d\xi_0.$$

By Proposition 4.4, each $e^{-\frac{itA'(h)}{h}}\psi^h_{(x_0,\xi_0)}$ concentrates in a radius $h^{1/2}$ ball at $x^t(x_0,\xi_0)$, so the integral is $O(h^\infty)$ for all $|x| \gg T\varepsilon^{-1}$.

For $|x| \lesssim T\varepsilon^{-1}$, modulo $O(h^\infty)$ we may restrict the integral to the region

$$\{(x_0,\xi_0) \in B : |x^t(x_0,\xi_0) - x| < h^\alpha\}$$

for any $\alpha < \frac{1}{2}$. By the rapid decay of ϕ_h,

$$|T_h\phi_h(x_0,\xi_0)| \le ch^{1-\frac{5d}{4}} \int e^{\frac{-(y-x_0)^2}{2h}} |\phi(h^{-1}y)|\, dy \sim c_\phi h^{1-\frac{d}{4}}$$

Combining this with Proposition 4.7 and the definition (4.4) of B,

$$\int_B |e^{-\frac{itA'(h)}{h}} \psi^h_{(x_0,\xi_0)}(x)||T_h\phi_h(x_0,\xi_0)|\, dx_0 d\xi_0 \lesssim h^{1-\frac{d}{4}} h^{-\frac{3d}{4}} h^{d\theta} h^{\alpha} = h^{1+\alpha-d(1-\theta)}.$$

As θ and α may be chosen arbitrarily close to $\frac{1}{2}$, it follows that

$$|e^{-\frac{itA'(h)}{h}} f^1_h(x)| \lesssim_\varepsilon h^{\frac{3}{2}-\frac{d}{2}-\varepsilon}$$

for any $\varepsilon > 0$. Therefore

$$\|e^{-\frac{itA'(h)}{h}} f^1_h\|_{L^{\frac{2d}{d-2}}} \le \|e^{-\frac{itA'(h)}{h}} f^1_h\|^{1-\frac{2}{d}}_{L^2} \|e^{-\frac{itA'(h)}{h}} f^1_h\|^{\frac{2}{d}}_{L^\infty} \lesssim_\varepsilon h^{1-\frac{2}{d}+\frac{3}{d}-1-\varepsilon} \lesssim_\varepsilon h^{\frac{1}{d}-\varepsilon}.$$

Remarks. (1) When g is the Euclidean metric, the exponential map is a diffeomorphism, so the bound in Lemma 4.7 is $O(r^d)$. Consequently,

$$\int_B |e^{-\frac{itA'(h)}{h}} \psi^h_{(x_0,\xi_0)}(x)||T_h\phi_h(x_0,\xi_0)|\, dx_0 d\xi_0 \lesssim h^{1-\frac{d}{4}} h^{-\frac{3d}{4}} h^{d(\theta+\alpha)} \lesssim_\varepsilon h^{1-\varepsilon}$$

for any $\varepsilon > 0$, and we find that

$$\|e^{-\frac{itA'(h)}{h}} f^1_h\|_{L^{\frac{2d}{d-2}}} \lesssim_\varepsilon h^{1-\varepsilon},$$

recovering modulo an arbitrarily small loss the $O(h)$ decay rate predicted by the $L^{\frac{2d}{d+2}} \to L^{\frac{2d}{d-2}}$ dispersive estimate for the Euclidean propagator $e^{it\Delta}$. The epsilon loss can be avoided if, instead of truncating crudely in phase space as in (4.4), we account for the contribution from each dyadic annulus $\{2^{k-1}h^{1/2} \le |x| < 2^k h^{1/2}\}$, using the rapid decay of each wavepacket on the $h^{1/2}$ scale. See the appendix for a more precise analysis.

(2) While the $L^{\frac{2d}{d-2}}$ decay was obtained by interpolating between L^2 and L^∞, this argument may be adapted to yield honest $L^1 \to L^\infty$ bounds that more directly parallel the Burq-Gerard-Tzvetkov semiclassical dispersive estimate [BGT04]. We relegate these details to the appendix as they are not presently essential.

(3) Instead of counting wavepackets, one can arrive at the preceding decay rates by appealing to the general parametrix of Hassell-Wunsch [HW05] for the Schrödinger propagator on asymptotically conic manifolds. In fact, their analysis shows that the bounds we obtain are saturated whenever there exist conjugate points z, w of order $d-1$ (that is, the highest possible). However, the wavepacket approach does let us treat both the semiclassical and long-time regimes in a unified and fairly concrete manner, and is also more robust if one is only interested in semiclassics as it uses little information about the geometry at infinity.

The long-time regime. Whereas the preceding discussion was essentially local due to finite speed of propagation, long-time decay necessarily hinges on the global geometry. Recall that the time parameter T was chosen so that all the wavepackets are far from the curved region and radiate to infinity essentially under the Euclidean Schrödinger flow.

To prove (4.10), use Proposition 4.6 to write

$$e^{-\frac{i(t-T)A(h)}{h}} e^{-\frac{iTA'(h)}{h}} f^1_h = \int_B v^h_{(x_0,\xi_0)}(t) T_h\phi_h(x_0,\xi_0)\, dx_0 d\xi_0 + \int_B r^h_{(x_0,\xi_0)}(t) T_h\phi_h(x_0,\xi_0)\, dx_0 d\xi_0,$$

where

$$|v_{(x_0,\xi_0)}(t,x)| \le C_N h^{-\frac{3d}{4}} |t|^{-d/2} \Big(1 + \frac{|x-x^t_0|}{h^{1/2}|t|}\Big)^{-N}, \quad \|r_{(x_0,\xi_0)}\|_{H^1} = O(h^\infty)$$

for any $N > 0$. The second integral is clearly negligible in $L^\infty L^{\frac{2d}{d-2}}$.

To estimate the first integral, we proceed as in the short-time estimate, interpolating between L^2 and L^∞ to exhibit decay in $L^{\frac{2d}{d-2}}$. For fixed x, modulo $O(h^\infty)$ we may restrict the integral to the region

$$B' = \{(x_0, \xi_0) \in B : |x^t(x_0, \xi_0) - x| \le h^\alpha(1 + |t|)\}$$

for any $\alpha < \frac{1}{2}$. As $x^t = x^T + 2(t-T)\xi^T$ when $t \ge T$, for each $(x_0, \xi_0) \in B$ with $|\xi_0|_g = 1$, the ray $r \mapsto (x_0, r\xi_0)$ intersects the above set in an interval of width $O(h^\alpha)$. The region B' therefore has measure at most $O(h^{d\theta}h^\alpha)$, and we obtain

$$\int_{B'} |v^h_{(x_0,\xi_0)}(t,x)||T_h\phi_h(x_0,\xi_0)|\, dx_0 d\xi_0 \le c_\varepsilon h^{1-\frac{d}{4}} h^{-\frac{3d}{4}} |t|^{-d/2} h^{d\theta+\alpha} = h^{1+\alpha-d(1-\theta)}|t|^{-d/2}.$$

Hence, recalling that θ may be chosen arbitrarily close to $\frac{1}{2}$, for any $\varepsilon > 0$ we have

$$\|e^{-\frac{i(t-T)A(h)}{h}} e^{-\frac{iTA'(h)}{h}} f_h^1\|_{L^\infty L^{\frac{2d}{d-2}}} \lesssim T^{-1} h^{1-\frac{2}{d}} h^{\frac{2(1+\alpha)}{d} - 2(1-\theta)} \lesssim_\varepsilon T^{-1} h^{\frac{1}{d}-\varepsilon}.$$

This completes the proof of Proposition 4.1. ☐

5. Linear profile decomposition

The profile decomposition will follow from repeated application of the following inverse Strichartz theorem.

Proposition 5.1. *Let $\{f_n\} \subset \dot{H}^1$ be a sequence such that $\|f_n\|_{\dot{H}^1} \le A$ and $\|e^{it\Delta_g} f\|_{L^\infty L^6} \ge \varepsilon$. Then there exist a function $\phi \in \dot{H}^1$ and parameters t_n, x_n, λ_n such that after passing to a subsequence,*

$$\lim_{n\to\infty} G_n^{-1} e^{it_n\Delta_g} \rightharpoonup \phi \text{ in } \dot{H}^1(g), \tag{5.1}$$

where $G_n\phi = \lambda_n^{-\frac{1}{2}}\phi(\frac{\cdot - x_n}{\lambda_n})$. Setting $\phi_n = e^{-it_n\Delta_g} G_n\phi$, we have

$$\liminf_n \|\phi_n\|_{\dot{H}^1(g)} \gtrsim \varepsilon^{\frac{9}{4}} A^{-\frac{5}{4}}. \tag{5.2}$$

$$\lim_n \|f_n\|^2_{\dot{H}^1} - \|f_n - e^{-it_n\Delta_g} G_n\phi\|^2_{\dot{H}^1} - \|e^{-it_n\Delta_g} G_n\phi\|^2_{\dot{H}^1} = 0. \tag{5.3}$$

$$\lim_n \|f_n\|^6_{L^6} - \|f_n - \phi_n\|^6_{L^6} - \|\phi_n\|^6_{L^6} = 0 \tag{5.4}$$

Finally, the t_n may be chosen so that either $t_n \equiv 0$ or $\lambda_n^{-2} t_n \to \infty$.

Proof. We use the following inverse Sobolev lemma:

Lemma 5.2. *If $\|f\|_{\dot{H}^1} \le A$ and $\|e^{it\Delta_g} f\|_{L^\infty L^6} \ge \varepsilon$, then there exist t, x, N, such that*

$$|(\tilde{P}_N)^2 e^{it\Delta_g} f)(x)| \gtrsim N^{\frac{1}{2}} \varepsilon^{\frac{9}{4}} A^{1-\frac{9}{4}}. \tag{5.5}$$

Proof. A Littlewood-Paley theory argument yields the following Besov refinement of Sobolev embedding (see [KV13])

$$\|e^{it\Delta_g} f\|_{L^\infty L^6} \lesssim \|f\|^{\frac{1}{3}}_{\dot{H}^1} \sup_N \|(\tilde{P}_N)^2 e^{it\Delta_g} f\|^{\frac{2}{3}}_{L^\infty L^6}.$$

Then, using the elementary inequality $\|(\tilde{P}_N)^2 f\|_2 \lesssim N^{-1}\|\tilde{P}_N f\|_{\dot{H}^1}$, which follows from the corresponding pointwise inequality for symbols, we have

$$\begin{aligned}\varepsilon^{\frac{3}{2}} A^{-\frac{1}{2}} &\lesssim \|(\tilde{P}_N)^2 e^{it\Delta_g} f\|_{L^\infty L^6} \lesssim \|(\tilde{P}_N)^2 e^{it\Delta_g} f\|^{\frac{1}{3}}_{L^\infty L^2} \|(\tilde{P}_N)^2 e^{it\Delta_g} f\|^{\frac{2}{3}}_{L^\infty L^\infty} \\ &\lesssim N^{-\frac{1}{3}} A^{\frac{1}{3}} \|(\tilde{P}_N)^2 e^{it\Delta_g} f\|^{\frac{2}{3}}_{L^\infty L^\infty}.\end{aligned}$$

☐

Select (t_n, x_n, N_n) according to this lemma, and set $\lambda_n = N_n^{-1}$. After passing to a subsequence, we may assume $\lambda_n \to \lambda_\infty \in [0, \infty]$ and $x_n \to x_\infty \in \mathbf{R}^d \cup \{\infty\}$. We may extract a weak limit

$$G_n^{-1} e^{it_n \Delta_g} f_n \rightharpoonup \phi \text{ in } \dot{H}^1(g).$$

As $\dot{H}^1(g)$ and $\dot{H}^1(\delta)$ have equivalent norms, their duals may be identified; hence the weak limit also holds in $\dot{H}^1(\delta)$.

Define $\phi_n = e^{-it_n\Delta_g} G_n \phi$.

We verify that this profile has positive energy. From Theorem 2.6 and the facts that $d_g(x, y) \sim |x - y|$, $dg = \sqrt{|g|}\, dx \sim dx$, there exist constants $c_1, c_2 > 0$ such that

$$N_n^{\frac{1}{2}} \varepsilon^{\frac{9}{4}} A^{-\frac{5}{4}} \le c_1 N_n^3 \int e^{-c_2 N_n^2 |x_n - y|^2} |e^{it_n \Delta_g} f_n|(y) dy.$$

Thus

$$c\varepsilon^{\frac{9}{4}} A^{-\frac{5}{4}} \le \int e^{-|y|^2} G_n^{-1} |e^{it_n\Delta_g} f_n|(y)\, dy.$$

As $G_n^{-1} e^{it_n\Delta_g} f_n \rightharpoonup \phi$ in $\dot{H}^1$, $|G_n^{-1} e^{it_n\Delta_g} f_n| \rightharpoonup |\phi|$ in $\dot{H}^1$. Indeed, by the Rellich-Kondrashov theorem, the sequences $G_n^{-1} e^{it_n\Delta_g}$ and $|G_n^{-1} e^{it_n \Delta_g}|$ converge to their $\dot{H}^1$ weak limits in L^2_{loc}.

Taking $n \to \infty$ in the above inequality and bounding $e^{-|y|^2}$ in $\dot{H}^{-1}$ by its $L^{6/5}$ norm, we get

$$\varepsilon^{\frac{9}{4}} A^{-\frac{5}{4}} \le \int e^{-|y|^2} |\phi|\, dy \lesssim \| |\phi| \|_{\dot{H}^1} \lesssim \|\phi\|_{\dot{H}^1}.$$

The claim (5.2) follows from the equivalence of $\dot{H}^1(\delta)$ and $\dot{H}^1(g)$.

To prove the decoupling (5.3), write

$$\begin{aligned} \|f_n\|_{\dot{H}^1}^2 - \|f_n - e^{-it_n\Delta_g} G_n\phi\|_{\dot{H}^1}^2 - \|e^{-it_n\Delta_g}\phi\|_{\dot{H}^1} &= 2\operatorname{Re}\langle e^{it_n\Delta_g} f_n - G_n\phi, G_n\phi\rangle_{\dot{H}^1} \\ &= 2\operatorname{Re}\langle G_n^{-1} e^{it_n\Delta_g} f_n - \phi, \phi\rangle_{\dot{H}^1(g_n)}, \end{aligned}$$

where $g_n(x) = g(x_n + \lambda_n x)$.

To see that the right side goes to 0, we consider two cases. If $\lambda_\infty < \infty$, then by Arzelà-Ascoli, after passing to a subsequence the metrics g_n converge boundedly and locally uniformly to some metric g_∞. If on the other hand $\lambda_\infty = \infty$, then g_n converges weakly to the Euclidean metric as $g_n(x) = \delta$ outside the shrinking balls $|x - x_n| \le \lambda_n^{-1}$.

To streamline the presentation, in the sequel we let g_∞ denote the locally uniform limit in the first case and $g_\infty = \delta$ in the second case.

When $\lambda_\infty < \infty$, then

$$\langle G_n^{-1} e^{it_n\Delta_g} f_n - \phi, \phi\rangle_{\dot{H}^1(g_n)} = \langle G_n^{-1} e^{it_n\Delta_g} f_n - \phi, \phi\rangle_{\dot{H}^1(g_\infty)} + o(1) \to 0$$

by dominated convergence.

If $\lambda_\infty = \infty$, writing

$$\langle u, v\rangle_{\dot{H}^1(g_n)} = \int \nabla u \cdot \overline{\nabla v}\, dx + \int_{|x - x_n| \le \lambda_n^{-1}} \langle du, dv\rangle_{g_n}\, dg_n - \int_{|x - x_n| \le \lambda_n^{-1}} \nabla u \cdot \overline{\nabla v}\, dx,$$

we have

$$\langle G_n^{-1} e^{it_n\Delta_g} f_n - \phi, \phi\rangle_{\dot{H}^1(g_n)} = \langle G_n^{-1} e^{it_n\Delta_g} f_n - \phi, \phi\rangle_{\dot{H}^1(\delta)} + o(1),$$

which vanishes since $G_n^{-1} e^{it_n\Delta_g} f_n - \phi \rightharpoonup 0$ in $\dot{H}^1(\delta)$.

We show next that if $t_n \lambda_n^{-2}$ is bounded, then after modifying the profile ϕ slightly we may arrange for $t_n \equiv 0$.

Suppose that $t_n \lambda_n^{-2} \to t_\infty$. By Theorem 3.1 and its corollary, we have

$$\phi_n = e^{-it_n\Delta_g} G_n \phi = G_n e^{-it_\infty \Delta}\phi + r_n, \quad \|r_n\|_{\dot{H}^1} = o(1).$$

Define the modified profile $\tilde{\phi} = e^{-it_\infty \Delta}\phi$, $\tilde{\phi}_n = G_n\tilde{\phi}$. Clearly (5.2) holds with $\tilde{\phi}_n$ in place of ϕ_n. We claim that

$$G_n^{-1} f_n \rightharpoonup \tilde{\phi} \text{ in } \dot{H}^1(g). \tag{5.6}$$

Suppose λ_n is bounded above. Passing to a subsequence, the metrics g_n converge locally uniformly to some metric g_∞. Then as

$$\begin{aligned}
\langle G_n^{-1} f_n - e^{-it_\infty\Delta}\phi, \psi\rangle_{\dot{H}^1(g_n)} &= \langle f_n - G_n e^{-it_\infty\Delta}\phi, G_n\psi\rangle)_{\dot{H}^1(g)} + o(1)\\
&= \langle e^{it_n\Delta_g} f_n - G_n\phi, e^{it_n\Delta_g} G_n\psi\rangle_{\dot{H}^1(g)} + o(1)\\
&= \langle G_n^{-1} e^{it_n\Delta_g} f_n - \phi, G_n^{-1} e^{it_n\Delta_g} G_n\psi\rangle_{\dot{H}^1(g_n)} + o(1)\\
&= \langle G_n^{-1} e^{it_n\Delta_g} f_n - \phi, e^{it_\infty\Delta}\psi\rangle_{\dot{H}^1(g_n)} + o(1)\\
&= \langle G_n^{-1} e^{it_n\Delta_g} f_n - \phi, e^{it_\infty\Delta}\psi\rangle_{\dot{H}^1(g_\infty)} + o(1)\\
&= o(1),
\end{aligned}$$

we have for all $\psi \in \dot{H}^1$

$$\langle G_n^{-1} f_n - e^{-it_\infty\Delta}\phi, \psi\rangle_{\dot{H}^1(g_\infty)} = o(1)$$

which implies weak convergence in $\dot{H}^1(g)$ since the norms defined by g_∞ and g are equivalent.

If instead $\lambda_n \to \infty$, then as before

$$\langle G_n^{-1} f_n - e^{-it_\infty\Delta}\phi, \psi\rangle_{\dot{H}^1(\delta)} = \langle G_n^{-1} f_n - e^{-it_\infty\Delta}\phi, \psi\rangle_{\dot{H}^1(g_n)} + o(1) \to 0.$$

Having verified the weak limit (5.6), the same argument as before establishes the decoupling of kinetic energies (5.3).

To establish the asymptotic additivity of nonlinear energy (5.4), we use the refined Fatou lemma of Brezis and Lieb:

Lemma 5.3 ([BL83])**.** *Suppose $f_n \in L^p(\mu)$ converge a.e. to some $f \in L^p(\mu)$ and $\sup_n \|f_n\|_{L^p} < \infty$. Then*

$$\int_{\mathbf{R}^d} \Big| |f_n|^p - |f_n - f|^p - |f|^p \Big| \, d\mu \to 0.$$

Assume $t_n \equiv 0$. Then $\phi_n = G_n\phi$ and $G_n^{-1} f_n$ converges weakly in $\dot{H}^1$ to ϕ. By Rellich-Kondrashov and a diagonalization argument, after passing to a subsequence we have $G_n^{-1} f_n \to \phi$ pointwise a.e. By a change of variable, the left side of (5.4) is bounded by

$$\int \Big| |G_n^{-1} f_n|^6 - |G_n^{-1} f_n| - \phi|^6 - |\phi|^6 \Big| \, dg_n \le \int dg_\infty + \int d|g_n - g_\infty|,$$

where we write $d|g_n - g_\infty| = |\sqrt{|g_n|} - \sqrt{|g_\infty|}|\, dx$. The first term vanishes by the Brezis-Lieb lemma, while for the second integral we note that $\int |\phi|^6 \, d|g_n - g_\infty| \to 0$ and argue as in the proof of that lemma.

Suppose $t_n \lambda_n^{-2} \to \infty$ (the case $t_n\lambda_n^{-2} \to -\infty$ is similar). Different arguments are required depending on the behavior of the parameters, but in each case we conclude that

$$\lim_{n\to\infty} \|\phi_n\|_{L^6} = 0,$$

which clearly implies (5.4).

If $\lambda_\infty = \infty$ or $x_\infty = \infty$, then by Theorem 3.1 we have

$$\phi_n = e^{-it_n\Delta} G_n\phi + r_n, \;\; \|r_n\|_{L^6} = o(1),$$

and the decay in L^6 follows from the dispersive estimate for the Euclidean propagator.

If $0 < \lambda_\infty < \infty$ and $x_\infty \in \mathbf{R}^3$, then $G_n\phi \to \phi'$, and we appeal to Lemma 5.4 below to find $\tilde{\phi} \in \dot{H}^1$ such that

$$\lim_{t\to\infty} \|e^{it\Delta_g}\phi' - e^{it\Delta}\tilde{\phi}\|_{\dot{H}^1} \to 0.$$

We bound by the triangle inequality

$$\|e^{it_n\Delta_g} G_n\phi\|_{L^6} \le \|e^{it\Delta_g}(G_n\phi - \phi')\|_{L^6} + \|e^{it\Delta_g}\phi' - e^{it\Delta}\tilde{\phi}\|_{L^6} + \|e^{it_n\Delta}\tilde{\phi}\|_{L^6}$$

and use the Euclidean dispersive estimate and Sobolev embedding.

For the remaining case where $\lambda_\infty = 0$ and $x_\infty \in \mathbf{R}^3$, we invoke the extinction lemma.

Lemma 5.4 (Linear asymptotic completeness)**.** *The limits* $\lim_{t\to\pm\infty} e^{-it\Delta_\delta} e^{it\Delta_g}$ *exist strongly in* $\dot{H}^1$*.*

Proof. Suppose first that $\phi \in C_0^\infty$. By the Duhamel formula,

$$e^{-it\Delta} e^{it\Delta_g}\phi = \phi + i\int_0^t e^{-is\Delta}(\Delta_g - \Delta)e^{is\Delta_g}\phi\, ds,$$

and we need to show that

$$\lim_{t\to\infty} \int_0^t e^{-is\Delta}(\Delta_g - \Delta)e^{is\Delta_g}\phi\, ds$$

exists in $\dot{H}^1$. We use (the dual of) the endpoint Strichartz estimate $e^{it\Delta_g} : L^2 \to L^2L^6$. For $t_1 < t_2$, we have

$$\begin{aligned}\Big\|\int_{t_1}^{t_2} e^{-is\Delta}(\Delta_g - \Delta)e^{is\Delta_g}\, ds\Big\|_{\dot{H}^1} &\lesssim \|\nabla(\Delta_g - \Delta)e^{it\Delta_g}\phi\|_{L^2L^{6/5}([t_1,t_2])} \\ &\lesssim \|\chi\nabla e^{it\Delta_g}\phi\|_{L^2L^{6/5}} + \|\chi\nabla^2 e^{it\Delta_g}\phi\|_{L^2L^{6/5}} + \|\chi\nabla^3 e^{it\Delta_g}\phi\|_{L^2L^{6/5}}\end{aligned}$$

for some bump function χ. Using Hölder, the equivalence of Sobolev spaces, and the Strichartz inequality, each term is bounded by

$$\|\chi\|_{L^{3/2}} \|(1-\Delta)^{3/2} e^{it\Delta_g}\phi\|_{L^2L^6([t_1,t_2])} \lesssim \|(1-\Delta_g)^{3/2} e^{it\Delta_g}\phi\|_{L^2L^6([t_1,t_2])} \lesssim \|\phi\|_{H^3}.$$

As $t_1, t_2 \to \infty$, the left side goes to 0. Thus

$$\lim_{t\to\infty} e^{-it\Delta_\delta} e^{it\Delta_g}\phi$$

exists in $\dot{H}^1$ for any $\phi \in C_0^\infty$.

For general $\phi \in \dot{H}^1$, select for each $\varepsilon > 0$ some $\phi_\varepsilon \in C_0^\infty$ with $\|\phi - \phi_\varepsilon\|_{\dot{H}^1} < \varepsilon$. Write $W(t) = e^{-it\Delta_\delta} e^{it\Delta_g}$,

$$W(t)\phi = W(t)\phi_\varepsilon + W(t)(\phi - \phi_\varepsilon).$$

As $W(t)$ are bounded on $\dot{H}^1$ uniformly in t, we have for all $t_1 < t_2$

$$\|W(t_2)\phi - W(t_1)\phi\|_{\dot{H}^1} \le \|W(t_2)\phi_\varepsilon - W(t_1)\phi_\varepsilon\|_{\dot{H}^1} + c\varepsilon;$$

so $W(t)\phi$ also converges in $\dot{H}^1$. ☐

This completes the proof of Proposition 5.1. ☐

We now prepare to introduce the linear profile decomposition.

Definition 5.1. Two frames $(\lambda_n^1, t_n^1, x_n^1)$ and $(\lambda_n^2, t_n^2, x_n^2)$ are *orthogonal* if

$$\lim_{n\to\infty} \frac{\lambda_n^1}{\lambda_n^2} + \frac{\lambda_n^2}{\lambda_n^1} + \frac{|t_n^1 - t_n^2|}{\lambda_n^1\lambda_n^2} + \frac{|x_n^1 - x_n^2|}{\sqrt{\lambda_n^1\lambda_n^2}} = \infty.$$

They are *equivalent* if

$$\frac{\lambda_n^1}{\lambda_n^2} \to \lambda_\infty \in (0,\infty), \ \ \frac{t_n^1 - t_n^2}{\lambda_n^1 \lambda_n^2} \in \mathbf{R}, \ \ \frac{x_n^1 - x_n^2}{\sqrt{\lambda_n^1 \lambda_n^2}} \to x_\infty \in \mathbf{R}^3.$$

Lemma 5.5. *If frames $(\lambda_n^1, t_n^1, x_n^1)$ and $(\lambda_n^2, \lambda_n^2, \lambda_n^2)$ are orthogonal, then*

$$(e^{-it_n^2\Delta_g} G_n^2)^{-1} e^{-it_n^1 \Delta_g} G_n^1$$

converges in weak $\dot H^1$ to zero. If they are equivalent, then $(e^{-it_n^2\Delta_g} G_n^2)^{-1} e^{-it_n^1 \Delta_g} G_n^1$ converges strongly to some injective $U_\infty : \dot H^1 \to \dot H^1$.

Proof. Assume the frames are orthogonal, and put $t_n = t_n^2 - t_n^1$. Suppose first that $|(\lambda_n^1)^{-2} t_n| \to \infty$. By passing to a subsequence, we may assume $\lambda_n^1 \to \lambda_\infty^1 \in [0,\infty]$ and $x_n^1 \to x_\infty^1 \in \mathbf{R}^3 \cup \{\infty\}$. Then

$$\|(G_n^2)^{-1} e^{i(t_n^2 - t_n^1))\Delta_g} G_n^1 \phi\|_{L^6} \to 0 \text{ for each } \phi \in \dot H^1.$$

Indeed, if $\lambda_\infty^1 \in (0,\infty)$ and $x_\infty^1 \in \mathbf{R}^3$ this follows from by Lemma 5.4 and the Euclidean dispersive estimate. For all other configurations of λ_∞^1 and x_∞^1, we appeal to Theorem 3.1 to see that

$$\|e^{i(t_n^2 - t_n^1)\Delta_g} G_n^1 \phi - e^{i(t_n^2 - t_n^1)} G_n^1 e^{it\Delta}\phi\|_{L^6} \to 0.$$

where Δ is, up to a linear change of variable, the Euclidean Laplacian. The decay in L^6 therefore follows from the Euclidean dispersive estimate.

As $(G_n^2)^{-1} e^{i(t_n^2 - t_n^1)\Delta_g} G_n^1 \phi$ forms a bounded sequence in $\dot H^1$, to determine its weak limit it suffices to test against compactly supported functions. For $\psi \in C_0^\infty$, we have

$$|\langle (G_n^2)^{-1} e^{i(t_n^2 - t_n^1)\Delta_g} G_n^1 \phi, \psi\rangle_{L^2}| \le \|(G_n^2)^{-1} e^{i(t_n^2 - t_n^1)\Delta_g} G_n^1 \phi\|_{L^6} \|\psi\|_{L^{6/5}} \to 0.$$

Assume now that $(\lambda_n^1)^{-2}(t_n^2 - t_n^1) \to t_\infty \in \mathbf{R}$. This implies that

$$\frac{\lambda_n^1}{\lambda_n^2} + \frac{\lambda_n^2}{\lambda_n^1} + \frac{|x_n^1 - x_n^2|}{\sqrt{\lambda_n^1 \lambda_n^2}} \to \infty. \tag{5.7}$$

As before, we may assume that $\lambda_n^1 \to \lambda_\infty^1 \in [0,\infty]$ and $x_n^1 \to x_\infty \in \mathbf{R}^3 \cup \{\infty\}$.

If $\lambda_\infty^1 \in (0,\infty)$ and $x_\infty^1 \in \mathbf{R}^3$, then it must be the case that

$$\lim_{n\to\infty} \frac{\lambda_n^1}{\lambda_n^2} \in \{0,\infty\},$$

Since the functions $f_n := e^{i(t_n^2 - t_n^1)\Delta_g} G_n^1 \phi$ form a precompact subset of $\dot H^1$, the sequences ∇f_n and $\xi \hat f_n(\xi)$ are tight in L^2. It follows that

$$\langle (G_n^2)^{-1} e^{i(t_n^2 - t_n^1)\Delta_g} G_n^1 \phi, \psi\rangle_{\dot H^1(\delta)} = \langle e^{i(t_n^2 - t_n^1)\Delta_g} G_n^1 \phi, G_n^2 \psi\rangle_{\dot H^1(\delta)} \to 0.$$

From the equivalence of $\dot H^1(\delta)$ and $\dot H^1(g)$ we conclude weak convergence to zero in $\dot H^1(g)$.

For all other configurations of the limiting parameters λ_∞^1 and x_∞^1, we appeal to Theorem 3.1 and Corollary 3.2 to see that

$$\|(G_n^2)^{-1} e^{i(t_n^2 - t_n^2)\Delta_g} G_n^1 \phi - (G_n^2)^{-1} e^{i(t_n^2 - t_n^1)\Delta} G_n^1 \phi\|_{\dot H^1} \to 0,$$

where Δ is the Euclidean Laplacian modulo a linear change of variable. Thus

$$\langle (G_n^2)^{-1} e^{i(t_n^2 - t_n^1)\Delta_g} G_n^1 \phi, \psi\rangle_{\dot H^1(\delta)} = \langle (G_n^2)^{-1} G_n^1 e^{it_\infty \Delta}\phi, \psi\rangle_{\dot H^1(\delta)} + o(1),$$

and under the assumption (5.7), the operator $(G_n^2)^{-1} G_n^1$ converges in weak $\dot H^1$ to zero.

Now suppose the frames are equivalent. This implies that $(\lambda_n^1)^{-2}(t_n^2 - t_n^1) \to t_\infty \in \mathbf{R}$. If $\lambda_\infty^1 \in (0,\infty)$ and $x_\infty^1 \in \mathbf{R}^3$, then $t_n \to (\lambda_\infty^1)^{-2} t_\infty \in \mathbf{R}$, $\lambda_n^2 \to \lambda_\infty^2 \in (0,\infty)$, $x_n^2 \to x_\infty^2 \in \mathbf{R}^3$, and $(G_n^2)^{-1} e^{i(t_n^2 - t_n^1)\Delta_g} G_n^1$ converges strongly to $(G_\infty^2)^{-1} e^{it_\infty \Delta_g} G_\infty^1 \phi$ where G_∞^j is the scaling and translation operator corresponding to $(\lambda_\infty^j, x_\infty^j)$. For all other values of λ_∞^1 and x_∞^1, we appeal to Theorem 3.1 to see that

$$(G_n^2)^{-1} e^{i(t_n^2 - t_n^1)\Delta_g} G_n^1 \to G_\infty e^{it_\infty \Delta}$$

where G_∞ is the scaling and translation operator associated to the parameters $(\lambda_\infty, \sqrt{\lambda_\infty} x_\infty)$, and

$$\lambda_\infty = \lim_{n\to\infty} \frac{\lambda_n^1}{\lambda_n^2}, \quad x_\infty = \lim_{n\to\infty} \frac{x_n^1 - x_n^2}{\sqrt{\lambda_n^1 \lambda_n^2}}.$$

In both cases the limiting operator is clearly invertible. □

Proposition 5.6 (Linear profile decomposition). *Let f_n be a bounded sequence in $\dot H^1$. After passing to a subsequence, there exist $J^* \in \{1, 2, \dots\} \cup \{\infty\}$, profiles ϕ^j, and parameters $(\lambda_n^j, t_n^j, x_n^j)$ such that for each finite J we have a decomposition*

$$f_n = \sum_{j=1}^J e^{-it_n^j \Delta_g} G_n^j \phi^j + r_n^J,$$

where $G_n^j \phi(x) = (\lambda_n^j)^{-\frac12} \phi(\frac{\cdot - x_n}{\lambda_n^j})$, which satisfies the following properties:

$$(G_n^J)^{-1} r_n^J \rightharpoonup 0 \text{ in } \dot H^1. \tag{5.8}$$

$$\lim_{J \to J^*} \limsup_{n\to\infty} \|e^{it\Delta_g} r_n^J\|_{L^\infty L^6} = 0. \tag{5.9}$$

$$E(f_n) = \sum_{j=1}^J E(\phi_n^j) + E(r_n^J) + o(1) \text{ as } n \to \infty. \tag{5.10}$$

$$\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \frac{|x_n^j - x_n^k|}{\sqrt{\lambda_n^j \lambda_n^k}} + \frac{|t_n^j - t_n^k|}{\lambda_n^j \lambda_n^k} \to \infty \text{ for all } j \neq k. \tag{5.11}$$

Moreover, the times t_n^j may be chosen for each j so that either $t_n^j \equiv 0$ or $\lim_{n\to\infty} (\lambda_n^j)^{-2} t_n^j \to \pm\infty$.

Proof. We iteratively apply Proposition 5.1 to construct the profiles. Let $r_n^0 = f_n$. Passing to a subsequence, we may assume the existence of the limits

$$A_J = \lim_{n\to\infty} \|r_n^J\|_{\dot H^1}, \quad \varepsilon_J = \lim_{n\to\infty} \|e^{it\Delta_g} r_n^J\|_{L^\infty L^6}.$$

If $\varepsilon_J = 0$ then stop and set $J* = J$. Otherwise, apply Proposition 5.1 to the sequence r_n^J to obtain a set of parameters $(t_n^{J+1}, x_n^{J+1}, \lambda_n^{J+1}$ and a profile

$$\phi^{J+1} = \text{w-lim} (G_n^{J+1})^{-1} e^{it_n^{J+1}\Delta} r_n^J, \quad \phi_n^{J+1} = G_n^{J+1} \phi^{J+1}. \tag{5.12}$$

Set $r_n^{J+1} = r_n^J - G_n^{J+1} \phi^{J+1}$, and continue the procedure replacing J by $J+1$.

If ε^J never equals zero, then set $J^* = \infty$. In this case, the kinetic energy decoupling (5.3), the lower bound (5.2) imply

$$A_{J+1}^2 \le A_J^2 \Big[1 - c\big(\frac{\varepsilon_J}{A_J}\big)^{\frac92}\Big]$$

which in view of the Sobolev embedding $\varepsilon_J \le cA_J$ compels $\varepsilon_J \to 0$ as $J \to \infty$.

It remains to verify the decoupling of parameters.

Suppose (5.11) failed. Choose $j < k$ with k minimal such that the frames $(\lambda_n^j, t_n^j, x_n^j)$ and $(\lambda_n^k, t_n^k, x_n^k)$ are not orthogonal. After passing to a subsequence, we may arrange for the frames $(\lambda_n^j, t_n^j, x_n^j)$, $(\lambda_n^\ell, t_n^\ell, x_n^\ell)$ to be equivalent when $\ell = k$ and orthogonal for $j < \ell < k$. By construction,

$$r_n^{j-1} = e^{-it_n^j \Delta_g} G_n^j \phi^j + e^{-it_n^k \Delta_g} G_n^k \phi^k + \sum_{j<\ell<k} e^{-it_n^\ell \Delta_g} G_n^\ell \phi^\ell,$$

hence

$$(G_n^j)^{-1}e^{it_n^j\Delta_g}r_n^{j-1} = \phi^j + (e^{-it_n^j\Delta_g}G_n^j)^{-1}e^{-it_n^k\Delta_g}G_n^k\phi^k + \sum_{j<\ell<k}(e^{-it_n^j\Delta_g}G_n^j)^{-1}e^{-it_n^\ell\Delta_g}G_n^\ell\phi^\ell.$$

By Lemma 5.5, $U_\infty = \lim_{n\to\infty}(e^{-it_n^j\Delta_g}G_n^j)^{-1}e^{-it_n^k\Delta_g}G_n^k$ is an invertible operator on $\dot{H}^1$, and we obtain

$$\phi^j = \phi^j + U_\infty\phi^k.$$

Thus $\phi^k = 0$, contrary to the nontriviality of the profile guaranteed by (5.2). □

6. Euclidean nonlinear profiles

Proposition 6.1. *Let (λ_n, t_n, x_n) be a frame such that $\lambda_n \to \lambda_\infty \in [0,\infty]$, $x_n \to x_\infty \in \mathbf{R}^3 \cup \{\infty\}$, and either $t_n \equiv 0$ or $\lambda_n^{-2}t_n \to \pm\infty$. Assume that the limiting parameters conform to one of the following scenarios:*

(i) $\lambda_\infty = \infty$.
(ii) $x_\infty = \infty$.
(iii) $x_\infty \in \mathbf{R}^3$, $\lambda_\infty = 0$.

Then, for n sufficiently large, there exists a unique global solution u_n to the equation (1.1) *with $u_n(0) = e^{-it_n\Delta_g}G_n\phi$ and which also has finite global Strichartz norm*

$$\|\nabla u_n\|_{L^{10}L^{\frac{30}{13}}(\mathbf{R}\times\mathbf{R}^3)} \le C(E(u_n(0))).$$

Moreover, for any $\varepsilon > 0$ there exists $\psi^\varepsilon \in C_0^\infty(\mathbf{R}\times\mathbf{R}^3)$ such that

$$\limsup_{n\to\infty}\|\nabla\big[u_n - G_n\psi^\varepsilon(\lambda_n^{-2}(t-t_n))\big]\|_{L^{10}L^{\frac{30}{13}}(\mathbf{R}\times\mathbf{R}^3)} < \varepsilon.$$

In particular, by Sobolev embedding the spacetime bound and approximation statement hold in $Z = L^{10}L^{10}$ as well.

Proof. In each regime, for n large one expects the solution to the variable-coefficient equation (1.1) to resemble a solution to a constant coefficient NLS

$$i\partial_t u = -\Delta u + |u|^4 u$$

where Δ is the Laplacian for a limiting geometry. In the first two cases, the metric is the standard one on $\mathbf{R}^3$, while in the last case the geometry is given by the constant metric $g(x_\infty)$. At any rate, all finite-energy solutions to the limiting equations are known to scatter [CKS+08]. We use these solutions to build good approximate solutions to (1.1). As the former obey good spacetime bounds, we deduce by stability theory that the same is true of the actual solutions to (1.1).

Let $g_\infty = g(x_\infty)$ in the last case and $g_\infty = \delta$ in all other cases, and denote by Δ the associated Laplacian.

If $t_n \equiv 0$, let v be the global scattering solution to the constant coefficient defocusing NLS

$$(i\partial_t + \Delta)v = |v|^4 v \tag{6.1}$$

with $v(0) = \phi$. If $\lambda_n^{-2}t_n \to \pm\infty$, let v instead be the unique solution to the above equation such that

$$\lim_{t\to\mp\infty}\|v(t) - e^{it\Delta}\phi\|_{\dot{H}^1} = 0.$$

In all cases, the Euclidean solution enjoys the global in time spacetime bounds

$$\|\nabla v\|_{L^2L^6\cap L^\infty L^2} \le C(E(\phi)) < \infty. \tag{6.2}$$

See [TVZ07, Lemma 3.11].

Fix a small parameter $0 < \theta \ll 1$, and let χ be a smooth bump function equal to 1 on the unit ball. Define spatial and Fourier space cutoffs χ_n and P_n as follows.

If $\lambda_n \to 0$ and $x_n \to x_\infty \in \mathbf{R}^3$, let $d_n = |x_n - x_\infty|$ and define

$$\chi_n = \chi\Big(\frac{(d_n+\lambda_n)^{1/3}(x-x_n)}{\lambda_n}\Big),\ P_n = \chi(\lambda_n(\lambda_n+d_n)^{1/6}D).$$

If $\lambda_n \to 0$ and $|x_n| \to \infty$, let

$$\chi_n = \chi\Big(\frac{x-x_n}{\lambda_n^{2/3}}\Big),\ P_n = \chi(\lambda_n^{4/3}D).$$

If $\lambda_n \to \lambda_\infty \in (0,\infty)$ and $d_n = |x_n| \to \infty$, set

$$\chi_n = \chi\Big(\frac{x-x_n}{d_n^{1/2}}\Big),\ P_n = \chi(d_n^{1/2}D).$$

If $\lambda_n \to \infty$, set

$$\chi_n = \chi\Big(\frac{x-x_n}{\lambda_n^{4/3}}\Big),\ P_n = \chi(\lambda_n^{5/6}D).$$

There is of course some latitude in the choice of exponents. Define the rescaled Euclidean solutions

$$v_n(t) = \lambda_n^{-1/2}v(\lambda_n^{-2}t, \lambda_n^{-1}(\cdot - x_n)) = G_n v(\lambda_n^{-2}t).$$

For $T > 0$ to be chosen later, set

$$\tilde{u}_n = \begin{cases} \chi_n P_n v_n, & |t| \le T\lambda_n^2 \\ e^{i(t-T\lambda_n^2)\Delta_g}\tilde{u}_n(T\lambda_n^2), & t \ge T\lambda_n^2 \\ e^{i(t+T\lambda_n^2)\Delta_g}\tilde{u}_n(-T\lambda_n^2), & t \le -T\lambda_n^2 \end{cases}$$

In the next two lemmas we prepare to invoke Proposition 2.11.

Lemma 6.2.

$$\lim_{T\to\infty}\limsup_{n\to\infty} \|\nabla[(i\partial_t + \Delta_g)\tilde{u}_n - F(\tilde{u}_n)]\|_{L^1L^2 + L^2L^{6/5}} \to 0.$$

Lemma 6.3.

$$\lim_{T\to\infty}\limsup_{n\to\infty} \|\tilde{u}_n(-t_n) - e^{-it_n\Delta_g}G_n\phi\|_{\dot{H}^1} = 0$$

Proof of Lemma 6.2. We estimate separately the contributions on $\{|t| \le T\lambda_n^2\}$ and $\{|t| > T\lambda_n^2\}$.

The Euclidean window. When $|t| \le T\lambda_n^2$, write

$$\begin{aligned} &(i\partial_t + \Delta_g)\tilde{u}_n - F(\tilde{u}_n) \\ &= \chi_n P_n(i\partial_t + \Delta)v_n + (\Delta_g - \Delta)\chi_n P_n v_n + [i\partial_t + \Delta, \chi_n P_n]v_n - F(\chi_n P_n v_n) \\ &= (\Delta_g - \Delta)\chi_n P_n v_n + [\Delta, \chi_n P_n]v_n + \chi_n P_n F(v_n) - F(\chi_n P_n v_n) \\ &= (a) + (b) + (c). \end{aligned}$$

Consider first the scenario where $\lambda_\infty = 0$ and $x_\infty \in \mathbf{R}^3$. We have

$$\begin{aligned} &\|\nabla(\Delta_g - \Delta)\chi_n P_n v_n\|_{L^1L^2} \\ &\lesssim \|(g^{jk} - g^{jk}(x_\infty))\nabla\partial_j\partial_k\chi_n P_n v_n\|_{L^1L^2} + \|(\partial g^{jk})\partial_j\partial_k\chi_n P_n v_n\|_{L^1L^2} \\ &+ \|\nabla g^{jk}\Gamma_{jk}^m\partial_m\chi_n P_n v_n\|_{L^1L^2} \end{aligned}$$

By Hölder in time and the definition of the cutoffs, the first term is bounded by

$$\Big(\frac{\lambda_n}{(\lambda_n + d_n)^{1/3}} + d_n\Big)(T\lambda_n^2)\lambda_n^{-2}(\lambda_n + d_n)^{-1/3}\|v_n\|_{L^\infty\dot{H}^1} \le T(\lambda_n + d_n)^{1/3}\|v\|_{L^\infty\dot{H}^1} \to 0.$$

Similarly, the second and third terms are at most

$$(T\lambda_n^2)\lambda_n^{-1}(\lambda_n + d_n)^{-1/6}\|v_n\|_{L^\infty \dot{H}^1} \leq T\lambda_n(\lambda_n + d_n)^{-1/6}\|v\|_{L^\infty \dot{H}^1} \to 0.$$

Hence (a) is acceptable.

Next, we have by Hölder and Sobolev embedding

$$\begin{aligned}\|\nabla[\Delta, \chi_n P_n]v_n\|_{L^1L^2} &\leq \|\nabla(\Delta\chi_n)P_n v_n\|_{L^1L^2} + \|\nabla\langle\nabla\chi_n, \nabla P_n v_n\rangle\|_{L^1L^2} \\ &\lesssim T(\lambda_n + d_n)^{\frac{2}{3}}\|\nabla v\|_{L^\infty L^2} + T(d_n + \lambda_n)^{\frac{1}{6}}\|\nabla v\|_{L^\infty L^2}.\end{aligned}$$

Thus (b) is also acceptable.

To bound the nonlinear commutator (c), write

$$\chi_n P_n F(v_n) - F(\chi_n P_n v_n) = (\chi_n P_n - 1)F(v_n) + F(v_n) - F(\chi_n P_n v_n).$$

Estimate

$$\begin{aligned}&\|\nabla(1 - \chi_n P_n)F(v_n)\|_{L^2L^{6/5}} \\ &\leq \|\nabla(1 - \chi_n)F(v_n)\|_{L^2L^{6/5}} + \|(\nabla\chi_n)(1 - P_n)F(v_n)\|_{L^2L^{6/5}} \\ &+ \|(1 - \chi_n)(1 - P_n)\nabla F(v_n)\|_{L^2L^{6/5}}.\end{aligned}$$

By a change of variable, the last term is at most

$$\|(1 - P_n)\nabla F(v_n)\|_{L^2L^{6/5}([-T\lambda_n^2, T\lambda_n^2])} = \|(1 - \tilde{P}_n)\nabla F(v)\|_{L^2L^{6/5}([-T,T])},$$

where $\tilde{P}_n = \chi((\lambda_n + d_n)^{\frac{1}{6}}D)$, which goes to zero by the estimate

$$\|\nabla F(v)\|_{L^2L^{6/5}} \lesssim \|v\|_{L^{10}L^{10}}^4\|\nabla v\|_{L^{10}L^{\frac{30}{13}}} < C(E(\phi))$$

and dominated convergence.

By Hölder and Sobolev embedding, the first two terms are bounded by

$$(\lambda_n + d_n)^{1/3}\lambda_n^{-1}(T\lambda_n^2)^{\frac{1}{2}}\|v_n\|_{L^\infty L^6}^5 \lesssim T^{\frac{1}{2}}(\lambda_n + d_n)^{1/3}\|\nabla v\|_{L^\infty L^2} \to 0$$

Also, as

$$\begin{aligned}F(v_n) - F(\chi_n P_n v_n) &= (1 - \chi_n P_n)v_n \int_0^1 F_z((1-\theta)\chi_n P_n v_n + \theta v_n)\, d\theta \\ &+ \overline{(1 - \chi_n P_n)v_n} \int_0^1 F_{\overline{z}}((1-\theta)\chi_n P_n v_n + \theta v_n)\, d\theta,\end{aligned}$$

we obtain by the Leibniz rule, Hölder, Sobolev embedding, and the L^p continuity of the Littlewood-Paley projections

$$\begin{aligned}&\|\nabla[F(v_n) - F(\chi_n P_n v_n)]\|_{L^2L^{\frac{6}{5}}} \\ &\lesssim \|\nabla(1 - \chi_n P_n)v_n\|_{L^{10}L^{\frac{30}{13}}}\|v\|_{L^{10}L^{10}}^4 + \|(1 - \chi_n P_n)v_n\|_{L^{10}L^{10}}\|v_n\|_{L^{10}}^3\|\nabla\chi_n P_n v_n\|_{L^{10}L^{\frac{30}{13}}} \\ &\lesssim \|\nabla v\|_{L^{10}L^{\frac{30}{13}}}^4\|\nabla(1 - \tilde{\chi}_n\tilde{P}_n v)\|_{L^{10}L^{\frac{30}{13}}},\end{aligned}$$

where $\tilde{P}_n = \chi((\lambda_n + d_n)^{\frac{1}{6}}D)$ and $\chi_n = \chi((\lambda_n + d_n)x)$. By dominated convergence, this also vanishes as $n \to \infty$.

Now we consider the case where $\lambda_n \to \infty$, and estimate the errors (a), (b), and (c) as before.

Since $\Delta_g - \Delta = (g^{jk} - \delta^{jk})\partial_j\partial_k - g^{jk}\Gamma_{jk}^m\partial_m$, we have

$$\|\nabla(\Delta_g - \Delta)\chi_n P_n v_n\|_{L^2L^{6/5}} \leq \sum_{j=1}^3 \|\tilde{\chi}\nabla^j\chi_n P_n v_n\|_{L^2L^{6/5}}$$

where $\tilde{\chi}$ is a spatial cutoff. As the v_n are being rescaled to low frequencies, the terms with the fewest derivatives applied to v_n are least favorable. Estimate

$$\begin{aligned}\|\tilde{\chi}\nabla\chi_n P_n v_n\|_{L^2 L^{6/5}} &\le \|\tilde{\chi}(\nabla\chi_n)P_n v_n\|_{L^2L^{6/5}} + \|\tilde{\chi}\nabla P_n v_n\|_{L^2L^{6/5}} \\ &\lesssim \lambda_n^{-\frac{4}{3}}(T\lambda_n^2)^{\frac{2}{5}}\|\chi\|_{L^{\frac{15}{11}}}\|P_n v_n\|_{L^{10}L^{10}} + \|\tilde{\chi}\|_{L^{\frac{6}{5}}}\|\nabla P_n v_n\|_{L^2L^\infty} \\ &\lesssim T^{\frac{2}{5}}\lambda_n^{-\frac{8}{15}}\|v\|_{L^{10}L^{10}} + \lambda_n^{-\frac{5}{12}}\|\nabla v\|_{L^2L^6},\end{aligned}$$

which is acceptable by (6.2). Also,

$$\begin{aligned}&\|\nabla[\Delta,\chi_n P_n]v_n\|_{L^1L^2} \le \|\nabla(\Delta\chi_n)P_n v_n\|_{L^1L^2} + \|\nabla\langle\nabla\chi_n,\nabla P_n v_n\rangle\|_{L^1L^2} \\ &\lesssim \|(\nabla^3\chi_n)P_n v_n\|_{L^1L^2} + \|(\nabla^2\chi_n)\nabla P_n v_n\|_{L^1L^2} + \|(\nabla\chi_n)\nabla^2 P_n v_n\|_{L^1L^2} \\ &\lesssim (\lambda_n^{-\frac{4}{3}})^2(T\lambda_n^2)\|\nabla P_n v_n\|_{L^\infty L^2} + \chi_n^{-\frac{4}{3}}(T\lambda_n^2)\lambda_n^{-\frac{5}{6}}\|\nabla P_n v_n\|_{L^\infty L^2} \\ &\lesssim T(\lambda_n^{-\frac{2}{3}} + \lambda_n^{-\frac{1}{6}})\|\nabla v\|_{L^\infty L^2}.\end{aligned}$$

Finally, the same argument as above yields

$$\|\nabla[\chi_n P_n F(v_n) - F(\chi_n P_n v_n)]\|_{L^2L^{\frac{6}{5}}} \to 0.$$

The remaining cases $\lambda_\infty < \infty$, $|x_n| \to \infty$ are dealt with similarly.

The long-time contribution. When $t \ge T\lambda_n^2$,

$$\|\nabla[(i\partial_t + \Delta_g)\tilde{u}_n - F(\tilde{u}_n)]\|_{L^2L^{6/5}} \lesssim \|\tilde{u}_n\|^4_{L^{10}L^{10}((T\lambda_n^2,\infty))}\|(-\Delta_g)^{1/2}\tilde{u}_n\|_{L^{10}L^{\frac{30}{13}}}.$$

The last norm on the right is bounded by Strichartz and energy conservation. To estimate the L^{10} norm, let $v_+ \in \dot{H}^1$ be the forward scattering state for the Euclidean solution v, defined by

$$\lim_{t\to\infty}\|v(t) - e^{it\Delta}v_+\|_{\dot{H}^1} = 0,$$

and write $v_{+n} = G_n v_+$. Then

$$\begin{aligned}\tilde{u}_n(t) &= e^{i(t-T\lambda_n^2)\Delta_g}\chi_n P_n v_n(T\lambda_n^2) \\ &= e^{it\Delta_g}(v_{+n}) + e^{i(t-T\lambda_n^2)\Delta_g}[e^{iT\lambda_n^2\Delta}(v_{+n}) - e^{iT\lambda_n^2\Delta_g}(v_{+n})] \\ &+ e^{i(t-T\lambda_n^2)\Delta_g}(\chi_n P_n - 1)v_n(T\lambda_n^2) + e^{i(t-T\lambda_n^2)\Delta_g}[v_n(T\lambda_n^2) - e^{iT\lambda_n^2\Delta}(v_{+n})],\end{aligned}$$

and we see that if T is sufficiently large, each term becomes acceptably small for n large. Indeed, by interpolating Theorem 3.1 or Proposition 4.1 with a Strichartz estimate,

$$\lim_{T\to\infty}\lim_{n\to\infty}\|e^{it\Delta_g}v_{+n}\|_{L^{10}L^{10}((T\lambda_n^2,\infty)} = 0.$$

The remaining terms are also acceptable due to Strichartz, Theorem 3.1, dominated convergence, and the definition of the scattering state v_{+n}. □

Proof of Lemma 6.3. If $t_n \equiv 0$ then there is nothing to prove. So suppose $\lambda_n^{-2}t_n \to \infty$. Recall that by definition,

$$\lim_{t\to-\infty}\|v(t) - e^{it\Delta}\phi\|_{\dot{H}^1} = 0.$$

Referring to the definition of $\tilde{u}_n$, for n large enough

$$\begin{aligned}\tilde{u}_n(-t_n) &= e^{-it_n\Delta_g}e^{iT\lambda_n^2\Delta_g}\chi_n P_n v_n(-T\lambda_n^2) = e^{-it_n\Delta_g}e^{iT\lambda_n^2\Delta_g}G_n v(-T) + r_n \\ &= e^{-it_n\Delta_g}e^{iT\lambda_n^2\Delta_g}e^{-iT\lambda_n^2\Delta_g}G_n\phi + r_n \\ &= e^{-it_n\Delta_g}G_n\phi + r_n,\end{aligned}$$

where, by Theorem 3.1 and Corollary 3.2, in each line

$$\lim_{T\to\infty}\limsup_{n\to\infty}\|r_n\|_{\dot H^1}=0.$$

□

By the preceding lemmas, for T large enough and n large, the function $\tilde u_n(t-t_n,x)$ is a good approximate solution to (1.1) in the sense of Proposition 2.11. Thus for any $\varepsilon>0$ and all n sufficiently large, there is a unique global solution u_n to (1.1) with

$$\|u_n\|_{Z(\mathbf{R})}+\|\nabla u_n\|_{L^{10}L^{\frac{30}{13}}(\mathbf{R}\times\mathbf{R}^3)}\le C(E(u_n(0))).$$

Finally, for any $\varepsilon>0$ there exists $\psi^\varepsilon\in C^\infty(\mathbf{R}\times\mathbf{R}^3)$ such that $\|\nabla(v-\psi^\varepsilon)\|_{L^{10}L^{\frac{30}{13}}(\mathbf{R}\times\mathbf{R}^3)}<\varepsilon$. In view of the definition of $\tilde u_n$ and the fact that, as proved above,

$$\lim_{T\to\infty}\limsup_{n\to\infty}\|\tilde u_n\|_{Z([-T\lambda_n^2,T\lambda_n^2]^c)}=0,$$

another application of Proposition 2.11 yields

$$\lim_{T\to\infty}\limsup_{n\to\infty}\|\nabla u_n\|_{L^{10}L^{\frac{30}{13}}([-T\lambda_n^2,T\lambda_n^2]^c\times\mathbf{R}^3)}=0.$$

Therefore

$$\lim_{T\to\infty}\limsup_{n\to\infty}\|\nabla[\tilde u_n-G_n\psi^\varepsilon(\lambda_n^{-2}t)]\|_{L^{10}L^{\frac{30}{13}}(\mathbf{R}\times\mathbf{R}^3)}\lesssim\varepsilon,$$

as required. □

7. Nonlinear profile decomposition and global wellposedness

In this section, we show that failure of Theorem (1.1) would imply the existence of an "almost-periodic" solution in the sense that it remains in a precompact subset of $\dot H^1$. This will already preclude finite time blowup and hence prove the global existence part of the theorem. In the next section, we rule out almost-periodic solutions under a smallness assumption on the metric and obtain global spacetime bounds in that setting.

Although we have worked mainly with the $Z=L^{10}L^{10}$ norm, in the sequel we shall also need the stronger norm

$$Y=L^{10}\dot H^{1,\frac{30}{13}}.$$

Let

$$\Lambda(E):\sup\{\|u\|_{Z(\mathbf{R})}:E(u)\le E,\ u \text{ solves } (1.1)\}$$
$$E_c=\sup\{E:\Lambda(E)<\infty\}.$$

$$\Lambda'(E):\sup\{\|u\|_{Z(I)}:|I|\le 1,\ E(u)\le E,\ u \text{ solves}(1.1)\}$$
$$E_c'=\sup\{E:\Lambda'(E)<\infty\}.$$

The small data theory implies that $E_c,E_c'>0$. Global existence (resp. scattering) would follow if we show that $E_c'<\infty$ (resp. $E_c<\infty$).

Proposition 7.1. *Suppose $E_c<\infty$. Let u_n be a sequence of solutions to* (1.1) *with $E(u_n)\to E_c$ such that for some sequence of times t_n, $\|u_n\|_{Z((-\infty,t_n))}\to\infty$ and $\|u_n\|_{Z((t_n,\infty))}\to\infty$. Then some subsequence of $u(t_n)$ converges in $\dot H^1$.*

The method of proof yields an analogous statement for global existence:

Proposition 7.2. *Suppose $E_c'<\infty$, and fix any $\delta>0$. Let u_n be a sequence of solutions to* (1.1) *with $E(u_n)\to E_c$ such that for some sequence of times t_n, $\|u_n\|_{Z((t_n-\delta,t_n))}\to\infty$ and $\|u_n\|_{Z((t_n,t_n+\delta))}\to\infty$. Then some subsequence of $u(t_n)$ converges in $\dot H^1$.*

We prove the global-in-time proposition; as the reader may verify, a nearly identical argument yields the local-in-time version.

Proof of Prop. 7.1. By translating in time, we may assume without loss that $t_n \equiv 0$. After passing to a subsequence, we obtain a decomposition

$$u_n(0) = \sum_{j=1}^{J} e^{-it_n^j \Delta_g} G_n^j \phi^j + r_n^J \tag{7.1}$$

into asymptotically independent profiles with the properties described in Proposition 5.6. In particular,

$$\lim_{n\to\infty} \Big[E(u_n(0)) - \sum_{j=1}^{J} E(e^{-it_n^j\Delta_g} G_n^j \phi_n^j) - E(r_n^J) \Big] = 0, \tag{7.2}$$

$$\lim_{J\to J^*} \limsup_{n\to\infty} \|e^{it\Delta_g} r_n^J\|_{L^\infty L^6} = 0. \tag{7.3}$$

Lemma 7.3. *There exists j such that* $\limsup_{n\to\infty} E(e^{-it_n^j\Delta_g} G_n^j \phi^j) = E_c$.

This will be proved below using a nonlinear profile decomposition. For the moment, we assume the result and observe how it yields the proposition. By the lemma, $u_n(0)$ takes the form

$$u_n(0) = e^{-it_n\Delta_g} G_n \phi + r_n$$

where $\|r_n\|_{\dot{H}^1} \to 0$ and G_n is associated to some frame (λ_n, x_n). After passing to a subsequence, we may assume that $\lambda_n \to \lambda_\infty \in [0,\infty]$, $x_n \to x_\infty \in \mathbf{R}^3 \cup \{\infty\}$, and $\lambda_n^{-2} t_n \to t_\infty \in \mathbf{R} \cup \{\pm\infty\}$.

We claim that $\lambda_\infty \in (0,\infty)$, $x_\infty \in \mathbf{R}^3$, and $t_\infty = 0$, which would clearly imply that $u_n(0)$ converges in $\dot{H}^1$. If either of the first two statements failed, Proposition 6.1 would imply that $\limsup_{n\to\infty} \|u_n\|_{Z(\mathbf{R})} < \infty$, contrary to the assumptions on u_n. Thus $\lambda_\infty \in (0,\infty)$ and $x_\infty \in \mathbf{R}^3$. If $t_n \to \infty$, then, writing G_∞ for the operator associated to the parameters $(\lambda_\infty, x_\infty)$, we have by the Strichartz estimate

$$\|(-\Delta_g)^{1/2} e^{it\Delta_g} u_n(0)\|_{L^{10} L^{\frac{30}{13}}((-\infty,0)\times\mathbf{R}^3)} \le \|(-\Delta_g)^{1/2} e^{it\Delta} G_\infty \phi\|_{L^{10} L^{\frac{30}{13}}((-\infty,-t_n)\times\mathbf{R}^3)} + o(1) \to 0,$$

which implies by the small data theory that $\lim_{n\to\infty} \|u_n\|_{Z(-\infty,0)} = \infty$, contrary to the hypothesis that u_n blows up forwards and backwards in time. □

Corollary 7.4. *If $E_c < \infty$, then there exists a global solution u_c to* (1.1) *with $E(u_c) = E_c$ and $\|u\|_{Z((-\infty,0])} = \|u\|_{Z([0,\infty))} = \infty$. Moreover, u is* almost-periodic *in the sense that $\{u_c(t) : t \in \mathbf{R}\}$ is precompact in $\dot{H}^1$.*

Proof. Let u_n be a sequence of solutions with $E(u_n) \to E_c$ and $\|u_n\|_Z \to \infty$. Choose t_n such that $\|u_n\|_{(-\infty,t_n]} = \|u_n\|_{[t_n,\infty)}$. By the previous proposition, there exists $\phi \in \dot{H}^1$ such that after passing to a subsequence, $u_n(t_n) \to \phi$ in $\dot{H}^1$. Let u_c be the maximal solution with $u_c(0) = 0$. Proposition 7.1 and the stability theory imply that u_c is global and blows up forwards and backwards in time. Another application of the previous proposition yields precompactness of the orbit $\{u(t) : t \in \mathbf{R}\}$ in $\dot{H}^1$. □

An immediate consequence of Proposition 7.2 and the stability theory is that the equation (1.1) is globally wellposed.

Corollary 7.5. *Under the hypotheses of Theorem 1.1, solutions of* (1.1) *are global in time.*

Proof. If $E_c' < \infty$, then there exists a sequence of solutions u_n with

$$E(u_n) \to E_c', \quad \|u_n\|_{Z((-\frac{1}{2},0))}, \ \|u_n\|_{Z((0,\frac{1}{2}))} \to \infty.$$

By Proposition 7.2, some subsequence of $u_n(0)$ converges to some $\phi \in \dot H^1$. Let $u_c : (T_-, T_+) \times \mathbf{R}^3 \to \mathbf{C}$ be the maximal-lifespan solution with $u_c(0) = \phi$. By the Proposition 2.11, u_c has infinite Z-norm on $(-\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, so the interval of definition for u_c is contained in $(-\frac{1}{2}, \frac{1}{2})$. As in the previous corollary, the solution curve $u_c(t)$ is precompact in $\dot H^1$, so along some sequence of times $t_n \to T_+$ the functions $u_c(t_n)$ converge to some ϕ_+ in $\dot H^1$. But then we may use a local solution u_+ with $u_+(0) = \phi_+$ and the stability theory to continue u_c to a larger time interval $(-T, T_+ + \delta)$, contradicting its maximality. □

We prove Proposition 7.1 in the remainder of this section. While the argument is fairly standard, involving a nonlinear profile decomposition, some remarks are warranted concerning the interaction between nonlinear profiles and the linear evolution of the remainder in the decomposition. This is normally controlled using local smoothing, which prevent high-frequency linear solutions from lingering in a confined region. In Euclidean space, the local smoothing estimate takes the form

$$\|\nabla e^{it\Delta_{\mathbf{R}^3}}\phi\|_{L^2(\mathbf{R}\times\{|x|\le R\})} \lesssim R^{1/2}\|\phi\|_{\dot H^{1/2}}.$$

In the present setting of a nontrapping compact metric perturbation, Rodnianski and Tao [RT07] proved that

$$\|\langle x\rangle^{-\frac{1}{2}-\sigma}\nabla e^{it\Delta_g}\phi\|_{L^2(\mathbf{R}\times\mathbf{R}^3)} + \|\langle x\rangle^{-\frac{3}{2}-\sigma}u\|_{L^2(\mathbf{R}\times\mathbf{R}^3)} \lesssim_\sigma \|\phi\|_{\dot H^{1/2}},\ \sigma > 0, \tag{7.4}$$

which strengthens an earlier local-in-time result of Doi [Doi96]. However, this and other local smoothing estimates on manifolds work at a fixed spatial scale, and since the metric is not scale-invariant, it is not obvious how the constants depend on the size of the physical localization.

The following lemma is analogous to Lemma 7.1 of Ionescu-Pausader concerning NLS on the torus [IP12], although the proof there is quite different due to trapping.

Let $\chi(\lambda)$ be a smooth function on the real line equal to 1 when $\lambda \le 1$ and vanishing when $\lambda \ge 1.2$, and define the spectral multipliers $P_{\le N} = \chi(\sqrt{-\Delta_g}/N)$. By Theorem 2.7, these satisfy the Littlewood-Paley estimates of Proposition 2.8 except when $p = 1$ or $p = \infty$ (which will not be needed).

Lemma 7.6. *For any $R, N, T > 0$, $B \ge 1$, and $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}^3$,*

$$\|\nabla e^{it\Delta_g} P_{>BN}\phi\|_{L^2(|t-t_0|\le TN^{-2},\ |x-x_0|\le RN^{-1})} \le CB^{-1/2}N^{-1}\|\phi\|_{\dot H^1}.$$

Proof. By invariance under time translation, we may take $t_0 = 0$. We adapt the standard proof of local smoothing on Euclidean space via a Morawetz multiplier but need to deal with error terms arising from the background curvature. These will be controlled by the estimate (7.4) adapted to the metric.

Let $a(x) = \langle x\rangle$. We compute (all derivatives are partial derivatives)

$$\partial a = \frac{x}{\langle x\rangle}, \qquad \partial^2 a = \langle x\rangle^{-3}P_r + \langle x\rangle^{-1}P_\theta,$$
$$\Delta a \ge \frac{3}{\langle x\rangle^3}, \qquad \Delta^2 a = -\frac{15}{\langle x\rangle^7}$$
$$|\partial^k a| \le \frac{c_k}{\langle x\rangle^{k-1}},$$

where P_r and $P_\theta = I - P_r$ are the radial and tangential projections, respectively.

Now write $D = d + \Gamma$ for the Levi-Civita covariant derivative, where

$$\Gamma = \Gamma^m_{ij} = \frac{1}{2}g^{mk}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij})$$

are the Christoffel symbols for the metric g; by our assumptions on g, Γ is supported in the unit ball.

If u is a solution to the equation

$$(i\partial_t + \Delta_g)u = \mu|u|^4 u, \ \mu \in \mathbf{R},$$

define the Morawetz action

$$M(t) = \int_{\mathbf{R}^3} a(x)|u(t,x)|^2 \, dg.$$

Then as in the Euclidean setting, we have

$$\partial_t M(t) = 2 \operatorname{Im} \int \overline{u} D^\alpha a D_\alpha u \, dg,$$

and the Morawetz identity

$$\partial_t^2 M = 4 \operatorname{Re} \int (D^2_{\alpha\beta} a) D^\alpha \overline{u} D^\beta u \, dg - \int (\Delta_g^2 a)|u|^2 \, dg + \frac{4\mu}{3} \int (\Delta_g a)|u|^6 \, dg. \tag{7.5}$$

We apply this identity with $\mu = 0$ and $u = e^{it\Delta_g} P_{>BN}\phi$; later on we will use this when $\mu = 1$. For $N > 0$ and $x_0 \in \mathbf{R}^3$, let $a_{N,x_0}(x) = a\big(N(x - x_0)\big)$. We compute

$$\begin{aligned} Da_{N,x_0} &= N(\partial a)\Big(N(x-x_0)\Big) \\ D^2_{\alpha\beta} a_{N,x_0} &= N^2 (\partial_\alpha \partial_\beta a)\Big(N(x-x_0)\Big) - N\Gamma^\mu_{\alpha\beta} \partial_\mu a\Big(N(x-x_0)\Big). \end{aligned}$$

Then

$$\begin{aligned} \Delta_g^2 a_{N,x_0} &= g^{\alpha\beta}(\partial_\alpha\partial_\beta - \Gamma^\mu_{\alpha\beta}\partial_\mu) g^{\alpha'\beta'}(\partial_{\alpha'}\partial_{\beta'} - \Gamma^{\mu'}_{\alpha'\beta'}\partial_{\mu'}) a_{N,x_0} \\ &= N^4 g^{\alpha\beta} g^{\alpha'\beta'} (\partial_\alpha\partial_\beta\partial_{\alpha'}\partial_{\beta'} a)\Big(N(x-x_0)\Big) + (N^3 P_3 a + N^2 P_2 a + N^1 P_1 a)\Big(N(x-x_0)\Big), \end{aligned}$$

where P_k is a differential operator of order k with coefficients supported in the unit ball; hence

$$\Big| P_k a\Big(N(x-x_0)\Big)\Big| \le c_k \, 1_{\{|x|\le 1\}}(x) \Big\langle N(x-x_0)\Big\rangle^{1-k}.$$

Inserting these bounds into the Morawetz identity and integrating in time over the interval $|t| \le TN^{-2}$, we obtain

$$\begin{aligned} \iint_{|t|\le TN^{-2}} \langle N(x-x_0)\rangle^{-3} |\nabla u|^2 \, dxdt &\lesssim N^{-1} \iint 1_{\{|t|\le TN^{-2}, |x|\le 1\}}(t,x)(|\nabla u|^2 + |u|^2) \, dxdt \\ &\quad + N^2 \iint_{|t|\le TN^{-2}} |u|^2 \, dxdt + N^{-1} \|u\|_{L^\infty L^2} \|u\|_{L^\infty \dot H^1}. \end{aligned}$$

By the unitarity of the propagator and the spectral localization of u, we have

$$\|u\|_{L^\infty L^2} \|u\|_{L^\infty \dot H^1} \lesssim (BN)^{-1} \|\phi\|^2_{\dot H^1}.$$

Also, by Hölder in time, the second term on the right may be bounded by

$$T\|u\|^2_{L^\infty L^2} \lesssim T(BN)^{-2} \|\phi\|^2_{\dot H^1}.$$

Finally, the first term on the right is controlled by (7.4).

Summing up, we obtain

$$\|\nabla u\|^2_{L^2(\{|t|\le TN^{-2}, |x-x_0|\le RN^{-1}\})} \lesssim (B^{-1}N^{-2} + T(BN)^{-2}) \|\phi\|^2_{\dot H^1}.$$

□

Proof of Lemma 7.3. Assuming that the claim fails, the asymptotic additivity of energy implies the existence of some $\delta > 0$ such that $\limsup_{n\to\infty} E(e^{-it_n^j\Delta_g} G_n^j \phi^j) \le E_c - \delta$ for all j. We shall deduce that

$$\limsup_{n\to\infty} \|u_n\|_{Z(\mathbf{R})} \le C(E_c, \delta) < \infty, \tag{7.6}$$

which contradicts the hypotheses on u_n.

For each $j \le J$, let u_n^j be the maximal-lifespan nonlinear solution with $u_n^j(0) = e^{-it_n^j\Delta_g} G_n^j \phi^j$; by the definition of E_c, for all n sufficiently large we have $\|u_n^j\|_{Z(\mathbf{R})} \le C$, hence $\|u_n^j\|_{Y(\mathbf{R})} \le C'$. Define

$$\tilde{u}_n^J = \sum_{j=1}^J u_n^j + e^{it\Delta_g} r_n^J.$$

The bound (7.6) will be a consequence of Proposition 2.11 and the following three assertions:

(1) $\limsup_{J\to J^*} \limsup_{n\to\infty} \|\tilde{u}_n^J\|_{Y(\mathbf{R})} \le C(E_c, \delta) < \infty$.
(2) $\lim_{J\to J^*} \limsup_{n\to\infty} \|u_n(0) - \tilde{u}_n^J(0)\|_{\dot{H}^1} = 0$.
(3) $\lim_{J\to J^*} \limsup_{n\to\infty} \|\nabla[(i\partial_t + \Delta_g)\tilde{u}_n^J - F(\tilde{u}_n^J)]\|_{N(\mathbf{R})} = 0$, where $F(z) = |z|^4 z$.

Proof of claim (1). As the Strichartz estimate and the hypothesis of bounded energy imply that the remainder $e^{it\Delta_g} r_n^J$ is bounded in Y, it suffices to show that

$$\limsup_{J\to J^*} \limsup_{n\to\infty} \Big\| \sum_{j=1}^J u_n^j \Big\|_{Y(\mathbf{R})} < \infty.$$

For each J, we have

$$\Big\|\sum_{j=1}^J u_n^j\Big\|_Y^2 = \Big\|\big(\sum_{j=1}^J \nabla u_n^j\big)^2\Big\|_{L^5 L^{\frac{15}{13}}} \le \sum_{j=1}^J \|\nabla u_n^j\|^2_{L^{10} L^{\frac{30}{13}}} + c_J \sum_{j\ne k} \|(\nabla u_n^j)(\nabla u_n^k)\|_{L^5 L^{\frac{15}{13}}}. \tag{7.7}$$

By Lemma (7.7), the cross-terms vanish as $n \to \infty$. By the asymptotic additivity of energy, there is some J_0 such that $\limsup_{n\to\infty} \|\nabla u_n^j(0)\|_{L^2}$ is smaller than the small-data threshold in Proposition 2.10 for all $j \ge J_0$. In view of the small-data estimate (2.7), for any $J > J_0$ we have

$$\limsup_{n\to\infty} \Big\|\sum_{j=1}^J u_n^j\Big\|_Y^2 \le C_{J_0}(E_c) + \limsup_{n\to\infty} \sum_{j=J_0}^J E(u_n^j) \le C_{J_0}(E_c) + E_c.$$

For future reference, we observe this also proves that for any $\varepsilon > 0$, there exists $J'(\varepsilon, E_c)$ with

$$\limsup_{n\to\infty} \Big\| \sum_{J' \le j \le J} u_n^j \Big\|_Y < \eta. \tag{7.8}$$

for all J.

Lemma 7.7. *For all $j \ne k$,*

$$\lim_{n\to\infty} \|u_n^j u_n^k\|_{L^5 L^5} + \|u_n^j \nabla u_n^k\|_{L^5 L^{\frac{15}{8}}} + \|\nabla u_n^j \nabla u_n^k\|_{L^5 L^{\frac{15}{13}}} = 0.$$

Proof of Lemma 7.7. The argument is well-known, and we will just illustrate it by estimating the middle term. By Proposition 6.1, for each $\varepsilon > 0$ there exist $\psi^j, \psi^k \in C_0^\infty(\mathbf{R} \times \mathbf{R}^3)$ such that

$$\|\nabla[u_n^j(t) - G_n^j \psi^j((\lambda_n^j)^{-2}(t - t_n^j))]\|_{L^{10} L^{\frac{30}{13}}} + \|\nabla[u_n^k(t) - G_n^k \psi^k((\lambda_n^k)^{-2}(t - t_n^k))]\|_{L^{10} L^{\frac{30}{13}}} < \varepsilon$$

for all n sufficiently large. Letting v_n^j, v_n^k denote the compactly supported approximations, we have by Hölder

$$\begin{aligned}\|u_n^j(\nabla u_n^k)\|_{L^5 L^{\frac{15}{8}}} &\le \|u_n^j - v_n^j\|_{L^{10}L^{10}} \|\nabla u_n^k\|_{L^{10}L^{\frac{30}{13}}} + \|v_n^j\|_{L^{10}L^{10}} \|\nabla(u_n^k - v_n^k)\|_{L^{10}L^{\frac{30}{13}}} \\ &\quad + \|v_n^j \nabla v_n^k\|_{L^5 L^{\frac{15}{8}}}.\end{aligned}$$

The last term vanishes due to the pairwise orthogonality of the frames $(\lambda_n^j, t_n^j, x_n^j)$ and $(\lambda_n^k, t_n^k, x_n^k)$. Thus

$$\limsup_{n\to\infty} \|u_n^j \nabla u_n^k\|_{L^5 L^{\frac{15}{8}}} \le C(E_c, \delta)\varepsilon$$

for any $\varepsilon > 0$. □

Claim (2) is immediate.
Proof of Claim (3). Write

$$\begin{aligned}(i\partial_t + \Delta_g)\tilde{u}_n^J - F(\tilde{u}_n^J) &= \sum_{j=1}^J F(u_n^j) - F\Big(\sum_{j=1}^J u_n^j\Big) \\ &\quad + F\Big(\sum_{j=1}^J u_n^j\Big) - F\Big(\sum_{j=1}^J u_n^j + e^{it\Delta_g} r_n^J\Big),\end{aligned}$$

and expand

$$\begin{aligned}F\Big(\sum_{j=1}^J u_n^j\Big) - \sum_{j=1}^J F(u_n^j) &= \Big|\sum_{j=1}^J u_n^j\Big|^4 \Big(\sum_{j=1}^J u_n^j\Big) - \sum_{j=1}^J |u_n^j|^4 u_n^j \\ &= \sum_{j=1}^J \Big(\Big|\sum_{j=1}^J u_n^j\Big|^4 - |u_n^j|^4\Big) u_n^j \\ &= \sum_{j=1}^J \sum_{k\ne j} \Big(u_n^j u_n^k \int_0^1 G_z\Big(\sum_{\ell=1}^J u_n^\ell - \theta u_n^j\Big)\, d\theta + u_n^j \overline{u_n^k} \int_0^1 G_{\overline{z}}\Big(\sum_{\ell=1}^J u_n^\ell - \theta u_n^j\Big)\, d\theta\Big),\end{aligned}$$

where $G(z) = |z|^4$. By the Leibniz rule, Hölder, and Lemma 7.7,

$$\begin{aligned}\|F\Big(\sum_{j=1}^J u_n^j\Big) - \sum_{j=1}^J F(u_n^j)\|_{L^2 L^{\frac{6}{5}}} &\le \sum_{j=1}^J \sum_{j\ne k} \|\nabla(u_n^j u_n^k)\|_{L^5 L^{\frac{15}{8}}} \Big\| \int_0^1 G'\Big(\sum_{\ell=1}^J u_n^\ell - \theta u_n^j\Big)\, d\theta \Big\|_{L^{\frac{10}{3}} L^{\frac{10}{3}}} \\ &\le c_J \sum_{j=1}^J \sum_{j\ne k} \|u_n^j \nabla u_n^k\|_{L^5 L^{\frac{15}{8}}} \Big(\Big\|\sum_{\ell=1}^J u_n^\ell\Big\|_{L^{10}L^{10}}^3 + \|u_n^j\|_{L^{10}L^{10}}^3 \Big) \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Similarly, write

$$\begin{aligned}F\Big(\sum_{j=1}^J u_n^j + e^{it\Delta_g} r_n^J\Big) - F\Big(\sum_{j=1}^J u_n^j\Big) &= \Big(\Big|\sum_{j=1}^J u_n^j + e^{it\Delta_g} r_n^J\Big|^4 - \Big|\sum_{j=1}^J u_n^j\Big|^4\Big) \sum_{j=1}^J u_n^j \\ &\quad + \Big|\sum_{j=1}^J u_n^j + e^{it\Delta_g} r_n^J\Big|^4 e^{it\Delta_g} r_n^J \\ &= (I) + (II).\end{aligned}$$

First consider (I). As before, writing $G(z) = |z|^4$, we have by the Leibniz rule, Hölder, and Sobolev embedding

$$\begin{aligned}\Big\|\nabla(I)\Big\|_{L^2L^{\frac{6}{5}}} &\leq \Big\|\nabla(e^{it\Delta_g}r_n^J)\int_0^1 G'\Big(\sum_{j=1}^J u_n^j + \theta e^{it\Delta_g}r_n^J\Big)\sum_j u_n^j\Big\|_{L^2L^{\frac{6}{5}}} \\ &+ \Big\|(e^{it\Delta_g}r_n^J)\nabla\int_0^1 G'\Big(\sum_{j=1}^J u_n^j + \theta e^{it\Delta_g}r_n^J\Big)\sum_{j=1}^J u_n^j\Big\|_{L^2L^{\frac{6}{5}}} \\ &\lesssim \|\nabla(e^{it\Delta_g}r_n^J)\Big|\sum_{j=1}^J u_n^j\Big|^4\Big\|_{L^2L^{\frac{6}{5}}} + \|\nabla(e^{it\Delta_g}r_n^J)\|_{L^{10}L^{\frac{30}{13}}}\|e^{it\Delta_g}r_n^J\|_{L^{10}L^{10}}^3\Big\|\sum_{j=1}^J u_n^j\Big\|_{L^{10}L^{10}} \\ &+ \|e^{it\Delta_g}r_n^J\|_{L^{10}L^{10}}(\|\nabla u_n^J\|_{L^{10}L^{\frac{30}{13}}}^4 + \|\nabla e^{it\Delta_g}r_n^J\|_{L^{10}L^{\frac{30}{13}}}^4).\end{aligned}$$

By (7.3) and interpolation, $\lim_{J\to J^*}\limsup_{n\to\infty}\|e^{it\Delta_g}r_n^J\|_{L^{10}L^{10}} = 0$; therefore, all but the first term are acceptable.

To deal with the first term, we recall that for any ε, there exists by (7.8) a threshold $J'(\varepsilon)$ such that for all n large,

$$\Big\|\sum_{j=J'}^J u_n^j\Big\|_{L^{10}L^{10}} < \varepsilon.$$

With ε fixed but arbitrary, this implies that

$$\begin{aligned}&\|\nabla(e^{it\Delta_g}r_n^J)\Big|\sum_{j=1}^J u_n^j\Big|^4\Big\|_{L^2L^{\frac{6}{5}}} \\ &\leq c_{J'}\sum_{j=1}^{J'}\|(u_n^j)^4\nabla e^{it\Delta_g}r_n^J\|_{L^2L^{\frac{6}{5}}} + \varepsilon^4 E_c^{1/2}.\end{aligned}$$

It therefore remains to show that

$$\limsup_{J\to J^*}\limsup_{n\to\infty}\|(u_n^j)^4\nabla e^{it\Delta_g}r_n^J\|_{L^2L^{\frac{6}{5}}} \lesssim \varepsilon \text{ for each } j \leq J'. \tag{7.9}$$

Select $\psi^j \in C_0^\infty(\mathbf{R}\times\mathbf{R}^3)$ so that $\|u_n^j - v_n^j\|_Y < \varepsilon$, where $v_n^j = G_n^j\psi^j((\lambda_n^j)^{-2}(t-t_n^j))$. Then we may replace u_n^j by v_n^j in the above sum since for all n sufficiently large, since

$$\begin{aligned}\|(u_n^j)^4\nabla e^{it\Delta_g}r_n^J\|_{L^2L^{\frac{6}{5}}} &\leq c\|u_n^j - v_n^j\|_{L^{10}L^{10}}(\|u_n^j\|_{L^{10}L^{10}}^3 + \|v_n^j\|_{L^{10}L^{10}}^3)\|\nabla e^{it\Delta_g}r_n^J\|_{L^{10}L^{\frac{30}{13}}} \\ &\leq C(E_c)\varepsilon.\end{aligned}$$

Let χ_n^j denote the characteristic function of $\operatorname{supp}(v_n^j)$. Putting $N_n^j = (\lambda_n^j)^{-1}$, we estimate using Hölder, Littlewood-Paley theory, and Lemma 7.6

$$\begin{aligned}\|(v_n^j)^4\nabla e^{it\Delta_g}r_n^J\|_{L^2L^{\frac{6}{5}}} &\lesssim (N_n^j)^2\|\chi_n^j\nabla e^{it\Delta_g}P_{\leq BN_n^j}r_n^J\|_{L^2L^{\frac{6}{5}}} + (N_n^j)^2\|\chi_n e^{it\Delta_g}P_{>BN_n^j}r_n^J\|_{L^2L^{\frac{6}{5}}} \\ &\lesssim (N_n^j)^{-1}\|\nabla e^{it\Delta_g}P_{\leq BN_n^j}r_n^J\|_{L^\infty L^6} + N_n^j\|\chi_n e^{it\Delta_g}P_{>BN_n^j}r_n^J\|_{L^2L^2} \\ &\lesssim B\|e^{it\Delta_g}r_n^J\|_{L^\infty L^6} + B^{-1/2}.\end{aligned}$$

As the remainder vanishes in $L^\infty L^6$ and B is arbitrary, it follows that

$$\lim_{J\to J^*}\limsup_{n\to\infty}\|(v_n^j)^4\nabla e^{it\Delta_g}r_n^J\|_{L^2L^{\frac{6}{5}}} = 0.$$

Altogether, we obtain (7.9), hence (I) is acceptable.

The contribution of (II) is estimated similarly. By the Leibniz rule,

$$\|\nabla(II)\|_{L^2L^{\frac{6}{5}}} \lesssim \|e^{it\Delta_g} r_n^J (u_n^J)^3 \nabla u_n^J\|_{L^2L^{\frac{6}{5}}} + \|(u_n^J)^4 \nabla e^{it\Delta_g} r_n^J\|_{L^2L^{\frac{6}{5}}}.$$

The first term is acceptable due to the undifferentiated $e^{it\Delta_g} r_n^J$, while the second term is handled is above. This completes the proof of Claim 3, and therefore finishes the proof of Lemma 7.3 asserting the existence of a critical profile. Consequently, Proposition 7.1 is proved. □

8. Scattering for small metric perturbations

In this final section we prove scattering for metrics g with $\|g-\delta\|_{C^3} \le \varepsilon$ for some ε depending on the diameter of $\operatorname{supp}(g-\delta)$. If the curvature is sufficiently mild, we can adapt the one-particle Bourgain-Morawetz inequality [Bou99] for the Euclidean nonlinear Schrödinger equation to preclude the existence of almost-periodic solutions, which, when combined with Corollary 7.4, yields scattering. We do not attempt to optimize ε here as we believe the smallness assumption is artificial, but do not know how to prove that at this time.

Proposition 8.1. *There exists $\varepsilon > 0$ such that if $\|g-\delta\|_{C^3} \le \varepsilon$, then for any solution u to the nonlinear equation* (1.1) *and any time interval I,*

$$\int_I \int_{|x|\le |I|^{1/2}} \frac{|u|^6}{\langle x\rangle}\, dxdt \le c|I|^{1/2} E(u).$$

Proof. Let $a = \langle x\rangle$ as in the proof of Lemma 7.6, and write $a_R = a(x)\chi(\frac{\cdot}{R})$ where χ is a smooth cutoff equal to 1 on the ball $|x|\le 1$ and supported in $|x|\le 2$. Then

$$\partial a_R = O(1), \qquad \Delta a_R = \Big(\frac{2}{\langle x\rangle} + \frac{1}{\langle x\rangle^3}\Big) 1_{\{|x|\le R\}} + O(R^{-1} 1_{\{|x|\sim R\}})$$

$$\partial^2 a_R = \partial^2 a 1_{\{|x|\le R\}} + O(R^{-1} 1_{\{|x|\sim R\}}), \quad \Delta^2 a_R = -\frac{15}{\langle x\rangle^7} 1_{\{|x|\le R\}} + O(R^{-3} 1_{\{|x|\sim R\}}).$$

Let $D = d + \Gamma$ denote the covariant derivative, where Γ is supported in the unit ball and $\|\Gamma\|_{C^2} = O(\varepsilon)$. It follows that if ε is sufficiently small, the above formulas continue to hold with the partial derivatives replaced by the covariant derivative D and Δ by the metric Laplacian Δ_g. Applying the Morawetz identity (7.5) with action $M(t) = \int a_R(x)|u(t,x)|^2\, dg$, we obtain $|\partial_t M| \le cR\|\nabla u\|_{L^2}^2$ and

$$\begin{aligned}\int_{|x|\le R} \frac{|u|^6}{\langle x\rangle}\, dx &\le \partial_t^2 M + cR^{-3}\int_{|x|\sim R} |u|^2\, dx + cR^{-1}\int_{|x|\sim R} |\nabla u|^2 + |u|^6\, dx\\ &\le \partial_t^2 M + cR^{-1}E(u).\end{aligned}$$

Setting $R = |I|^{1/2}$ and integrating in time, we obtain

$$\int_I \int_{|x|\le |I|^{1/2}} \frac{|u|^6}{\langle x\rangle}\, dxdt \le \sup_t 2|\partial_t M| + c|I|R^{-1}E(u) \lesssim |I|^{1/2}E(u).$$

□

By Corollary 7.4, if there is a finite energy solution to (1.1) that failed to scatter, then there exists a nonzero almost-periodic solution u_c, i.e. which remains in a precompact subset of $\dot H^1$.

Corollary 8.2. *If $\|g-\delta\|_{C^3} \le \varepsilon$, the equation* (1.1) *does not admit nonzero almost-periodic solutions. Hence, all finite-energy solutions to* (1.1) *scatter.*

Proof. Suppose $0 \ne u_c$ is almost-periodic. Then there exists $\eta > 0$ and a radius R such that

$$\|u_c(t)\|_{L^6(\{|x|\le R\})} \ge \eta \text{ for all } t.$$

For if not, there would exist radii $R_n \to \infty$ and times t_n such that $\|u_c(t_n)\|_{L^6(\{|x|\le R_n\})} \to 0$. By compactness, it follows that some subsequence of $u_c(t_n)$ converges in $\dot{H}^1$ to 0. But this yields the contradiction that $E(u_c) = 0$.

We now apply Proposition 8.1 on time intervals I with $|I|^{1/2} > R$, and deduce

$$\eta |I| R^{-1} \le \int_I \int_{|x|\le |I|^{1/2}} \frac{|u_c|^6}{\langle x \rangle}\, dxdt \lesssim |I|^{1/2} E(u_c).$$

But this yields a contradiction for I sufficiently large. □

9. Appendix: A semiclassical $L^1 \to L^\infty$ estimate

We demonstrate how the techniques from the proof of Proposition 4.1 yield a refinement of the Burq-Gerard-Tzvetkov dispersive estimate [BGT04].

Proposition 9.1. *Let g be a smooth metric on $\mathbf{R}^d$ with all derivatives bounded, set $a(x,\xi) = g^{jk}\xi_j\xi_k$ and denote by*

$$A(h) = a^w(x, hD) = (2\pi h)^{-d} \int_{\mathbf{R}^d} e^{\frac{i(x-y)\xi}{h}} a\left(\frac{x+y}{2}, \xi\right) d\xi$$

its semiclassical Weyl quantization. Let $\chi \in C_0^\infty(\mathbf{R}^d)$ be a frequency cutoff. Then,

(1) *There exists a constant $c > 0$, depending on g,*

$$\|e^{-\frac{itA(h)}{h}} \chi(hD)\|_{L^1(\mathbf{R}^d)\to L^\infty(\mathbf{R}^d)} \lesssim |ht|^{-d/2} \text{ for all } |t| \le c.$$

(2) *For each $T > 0$, we have*

$$\|e^{-\frac{itA(h)}{h}} \chi(hD)\|_{L^1(\mathbf{R}^d)\to L^\infty(\mathbf{R}^d)} \lesssim_T h^{-d+\frac{1}{2}} \text{ for all } c \le |t| \le T.$$

Remark. If g is nontrapping and sufficiently flat at infinity (i.e. is a scattering metric), the second part follows immediately from the Hassell-Wunsch parametrix [HW05] which also provides more precise information on the possible decay rates. They show that if x and y are conjugate points of order $k \le d-1$, then $|e^{it\Delta_g}(x,y)| \lesssim t^{-\frac{d}{2}-\frac{k}{2}}$. In particular, the case $k = d-1$ coincides with the decay rate we obtain.

Proof. Only the second part is new, as the first part follows essentially from the WKB analysis of [BGT04]. We use the basic argument for Proposition 4.1 but do the accounting more carefully to avoid the epsilon loss. That is, we frequency-localize, decompose into wavepackets, and track their evolution along the geodesic flow.

By introducing a spatial cutoff $\eta(h^{-1}X)$, where $\eta \in C_0^\infty(\mathbf{R}^d)$, we further assume the initial data $u_h(0) = \chi(hD)\eta(h^{-1}X)\phi$ is localized to a minimum uncertainty box in phase space. This will be convenient for the wavepacket analysis below. As the domain is L^1, the full bound may be recovered via a partition of unity and the triangle inequality (note that the hypotheses are translation-invariant).

Write $u_h = e^{-\frac{itA(h)}{h}} \chi(hD)\eta(X)\phi$, where by linearity we normalize $\|\phi\|_{L^1} = 1$; thus u_h solves the equation

$$[hD_t + A(h)]u_h = 0, \; u_h(0) = \chi(hD)\eta(h^{-1}X)\phi.$$

Select $\chi_1, \chi_2 \in C_0^\infty(\mathbf{R}^d \setminus \{0\})$ such that $\chi \prec \chi_1 \prec \chi_2$, and define the localized operator $A'(h) = (\chi_2 a)^w(x, hD)$. By the semiclassical functional calculus $\|[1-\chi_1(hD)]u_h\|_{H^s} \le O(h^\infty)\|u_h\|_{L^2}$ for any $s > 0$, and we deduce that

$$[hD_t + A'(h)]u_h = r_h$$

where $\|r_h\|_{H^s} \le O(h^\infty)\|u_h\|_{L^2} \le O(h^\infty)\|\phi\|_{L^1}$. By the Duhamel formula and Sobolev embedding, the remainder may safely be ignored, therefore we shall just study the free evolution under $A'(h)$.

To make the notation less cumbersome, write $\phi_h = \chi(hD)\eta(h^{-1}X)\phi$. From the convolution representation of $\chi(hD)$ (and recalling the normalization of ϕ), we have the pointwise bound

$$|\phi_h(x)| \lesssim_N h^{-d} \left\langle \frac{x}{h} \right\rangle^{-N}.$$

Let

$$T_h\phi_h(x,\xi) = c_d h^{-\frac{3d}{4}} \int e^{\frac{i\xi(x-y)}{h}} e^{-\frac{(x-y)^2}{2h}} \phi_h(y)\, dy = c_d' h^{-\frac{5d}{4}} \int e^{\frac{ix\eta}{h}} e^{-\frac{(\eta-\xi)^2}{2h}} \widehat{\phi_h}\left(\frac{\eta}{h}\right) d\eta$$

denote the FBI transform of ϕ_h. When $|x| \sim 2^k h^{1/2}$, we have

$$\begin{aligned} |T_h\phi_h| &\lesssim h^{-\frac{3d}{4}} \int_{|y|\le \frac{1}{2}|x|} e^{-\frac{(x-y)^2}{2h}} h^{-d} \left\langle \frac{y}{h} \right\rangle^{-N} dy + h^{-\frac{3d}{4}} \int_{|y|\ge\frac{1}{2}|x|} h^{-d} \left\langle \frac{y}{h}\right\rangle^{-N} dy \\ &\lesssim h^{-\frac{3d}{4}} (e^{-c2^{2k}} + h^{100d} 2^{-100dk}). \end{aligned}$$

Similarly, for $|\xi| \gg 1$,

$$\begin{aligned} |T_h\phi_h| &\lesssim h^{-\frac{5d}{4}} \int e^{-\frac{c|\xi|^2}{h}} \chi(\eta) |\eta(\widehat{h^{-1}X})\phi(\eta)|\, d\eta \\ &\lesssim h^{-\frac{5d}{4}} e^{-\frac{c|\xi|^2}{h}}. \end{aligned}$$

Choose $R > 0$ so that $\mathrm{supp}(\chi) \subset \{|\xi| \le R/4\}$, and partition phase space as $T^*\mathbf{R}^d = B' \cup \bigcup_{k\ge 0} B_k$, where

$$\begin{gathered} B' = \{|\xi| > R\} \\ B_0 = \{|x| \le h^{1/2},\ |\xi| \le R\},\ B_k = \{2^{k-1}h^{1/2} < |x| \le 2^k h^{1/2},\ |\xi| \le R\}. \end{gathered}$$

Decompose

$$\phi_h = \int_{B'} T_h\phi_h(x,\xi) \psi^h_{(x,\xi)}\, dxd\xi + \sum_{k\ge 0} \int_{B_k} T_h\phi_h(x,\xi)\psi^h_{x,\xi}\, dxd\xi,$$

where $\psi^h_{x,\xi}(y) = 2^{-\frac{d}{2}} \pi^{-\frac{3d}{4}} h^{-\frac{3d}{4}} e^{\frac{i\xi(y-x)}{h}} e^{-\frac{(y-x)^2}{2h}}$. By the preceding bounds and Proposition 4.4, we estimate

$$\begin{aligned} \left| \int_{B'} T_h\phi_h e^{-\frac{itA'(h)}{h}} \psi^h_{x,\xi}\, dxd\xi \right| &\lesssim \sum_k \int_{|x|\sim 2^k h^{1/2},\ |\xi|\ge R} h^{-\frac{3d}{4}} h^{-d} (e^{-c2^{2k}} + 2^{-50dk}) e^{-c\frac{|\xi|^2}{h}}\, dxd\xi \\ &= O(h^\infty). \end{aligned}$$

Next, let $0 < \alpha < \frac{1}{2}$ be a small parameter, and estimate

$$\begin{aligned} \sum_{2^k > h^{-\alpha}} \left| \int_{B_k} T_h\phi_h e^{-\frac{itA'(h)}{h}} \psi^h_{x,\xi}\, dxd\xi \right| &\lesssim \sum_{2^k > h^{-\alpha}} h^{-\frac{3d}{2}} (e^{-c2^{2k}} + h^{100d} 2^{-100dk}) \\ &\lesssim h^{10d}, \end{aligned}$$

so this contribution is acceptable. It remains to bound the terms

$$\left| \int_{B_k} T_h\phi_h e^{-\frac{itA'(h)}{h}} \psi^h_{x,\xi}(y)\, dxd\xi \right| \lesssim \int_{B_k} h^{-\frac{3d}{4}} \left\langle \frac{y - x^t}{h^{1/2}} \right\rangle^{-100d} |T_h\phi_h(x,\xi)|,$$

where $2^k h^{1/2} \le h^{1/2-\alpha} < 1$. We bound the right side by

$$\sum_j \int_{B_{k,j}} h^{-\frac{3d}{4}} \left\langle \frac{y - x^t}{h^{1/2}} \right\rangle^{-100d} |T_h \phi_h(x,\xi)|\, d\xi dx \le \sum_{2^j \le h^{-\alpha}} + \sum_{2^j > h^{-\alpha}}.$$

where $B_{k,j} = B_k \cap \{(x,\xi) : 2^{j-1} < |x^t - y| \le 2^j h^{1/2}\}$ and $B_{k,0} = B_k \cap \{(x,\xi) : |x^t - y| \le h^{1/2}\}$. The second sum is negligible as before. For the first, we invoke the exponential map estimate of Lemma (4.7) to bound the jth member by

$$\begin{aligned} &h^{-\frac{3d}{4}}(2^{-100jd})h^{-\frac{3d}{4}}(e^{-c2^{2k}} + h^{100d}2^{-100dk})(2^k h^{1/2})^d(2^j h^{1/2}) \\ &\le h^{-d+\frac{1}{2}} 2^{-99jd}(e^{-c2^{2k}} + h^{100d}2^{-100dk}). \end{aligned}$$

The proof is finished by summing over j and k. □

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