University of Toronto – MAT246H1-S – LEC0201/9201

Concepts in Abstract Mathematics

Homework questions – Week 2

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Exercise 1.

1. Prove that $\forall n \in \mathbb{N},\ \exists k \in \mathbb{N},\ n^3 + 2n = 3k$.
2. Prove that $\forall n \in \mathbb{N},\ \sum_{k=0}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$.

Exercise 2.

We define a sequence $(u_n)_{n\geq 1}$ by $u_1 = 3$ and $\forall n \in \mathbb{N} \setminus \{0\},\ u_{n+1} = \frac{2}{n}\sum_{k=1}^{n} u_k$.

Prove that $\forall n \in \mathbb{N} \setminus \{0\},\ u_n = 3n$.

Exercise 3. *Bernoulli's inequality.*
Prove that $\forall x \in [-1, +\infty),\ \forall n \in \mathbb{N},\ (1+x)^n \geq 1 + nx$.
(*here we consider the usual order* $\geq$ *on* $\mathbb{R}$)

Exercise 4.
For $n \in \mathbb{N}$, we define the statement $P(n)$ by $2^n > n^2$.

1. Prove that $\forall n \geq 3,\ P(n) \implies P(n+1)$.
2. For which $n \in \mathbb{N}$, is $P(n)$ true?

Exercise 5.
What do you think about the following proof by induction?

We want to prove that for any $n \geq 2$, n distinct points of the plane are always on the same line.
Proof:

- Base case: when $n = 2$ the property is known to be true.
- Induction step: we assume that the property is true for some $n \geq 2$ and we want to show that it also holds for $n + 1$.
  Let $A_1, A_2, \dots, A_{n+1}$ be $n+1$ distinct points of the plane. By the induction hypothesis, we have
  - $A_1, A_2, \dots, A_n$ are on the same line L.
  - $A_2, A_3, \dots, A_{n+1}$ are on the same line L'.

  Then $A_2, A_3, \dots, A_n$ are at the same time on L and L' so that $L = L'$.
  Thus $A_1, \dots, A_{n+1}$ are on the same line. Which ends the induction step. □

Exercise 6.
Given $n \in \mathbb{N} \setminus \{0\}$, prove that there exists a unique couple $(a, b) \in \mathbb{N}$ such that $n = 2^a(2b+1)$.

Exercise 7.
Find all the increasing functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(2) = 2$ and $\forall p, q \in \mathbb{N},\ f(pq) = f(p)f(q)$.
Recall that a function $f : \mathbb{N} \to \mathbb{N}$ *is increasing if* $\forall x, y \in \mathbb{N},\ x < y \implies f(x) < f(y)$.

Exercise 8.

1. Prove that if $S \subset \mathbb{Z}$ admits a greatest element then it is unique.
2. Prove that a non-empty finite subset of $\mathbb{Z}$ admits a greatest element.

Exercise 9.
Let $n \in \mathbb{N} \setminus \{0\}$. Prove that if one square of a $2^n \times 2^n$ chessboard is removed, then the remaining squares can be covered by L-shaped trominoes.

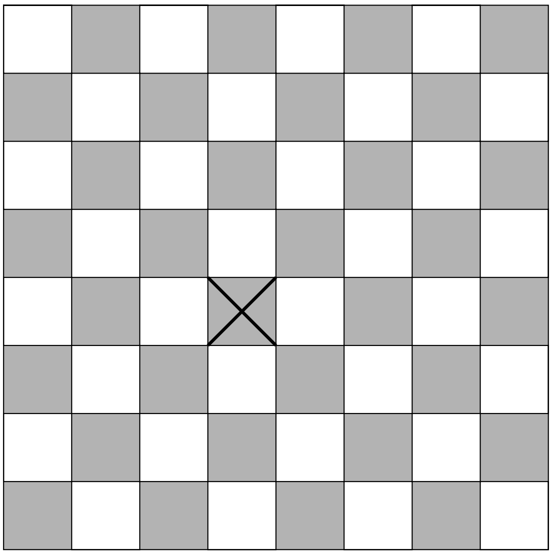


Figure 1: A 8×8 chessboard with a removed square.

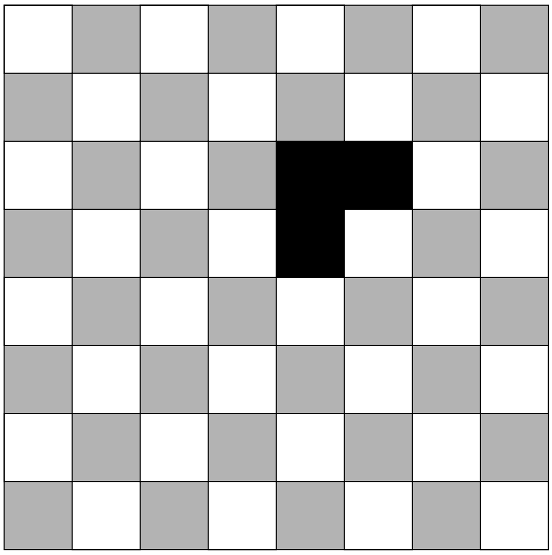


Figure 2: An L-shaped tromino on a chessboard.

Sample solutions to Exercise 1.

1. We are going to prove by induction that $\forall n \in \mathbb{N}, \exists k \in \mathbb{N}, n^3 + 2n = 3k$.
   - *Base case at $n = 0$:* $0^3 + 2 \times 0 = 3 \times 0$.
   - *Induction step:* assume that for some $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $n^3 + 2n = 3k$. Then

   $$\begin{aligned}(n+1)^3 + 2(n+1) &= n^3 + 3n^2 + 3n + 1 + 2n + 2 \\ &= 3k + 3n^2 + 3n + 3 \quad \text{by the induction hypothesis} \\ &= 3(k + n^2 + n + 1)\end{aligned}$$

   The induction step is proved since $k + n^2 + n + \in \mathbb{N}$.

2. We are going to prove by induction that $\forall n \in \mathbb{N}, \displaystyle\sum_{k=0}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$.
   - *Base case at $n = 0$:* $\displaystyle\sum_{k=0}^{0} \frac{k}{2^k} = 0$ and $2 - \frac{0+2}{2^0} = 2 - 2 = 0$.
   - *Induction step:* assume that $\displaystyle\sum_{k=0}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$ for some $n \in \mathbb{N}$.

   $$\begin{aligned}\sum_{k=0}^{n+1} \frac{k}{2^k} &= \sum_{k=0}^{n} \frac{k}{2^k} + \frac{n+1}{2^{n+1}} \\ &= 2 - \frac{n+2}{2^n} + \frac{n+1}{2^{n+1}} \quad \text{by the induction hypothesis} \\ &= 2 - \frac{2n+4-n-1}{2^{n+1}} = 2 - \frac{(n+1)+2}{2^{n+1}}\end{aligned}$$

   which ends the induction step.

Sample solutions to Exercise 2.

We are going to prove by (strong) induction that $\forall n \geq 1, u_n = 3n$.

- *Base case at $n = 1$:* $u_1 = 3 \times 1$.
- *Induction step:* assume that $u_k = 3k$ for $k = 1, \dots, n$ where $n \geq 1$. Then

$$\begin{aligned}u_{n+1} &= \frac{2}{n+1} \sum_{k=1}^{n} u_k \\ &= \frac{2}{n} \sum_{k=1}^{n} 3k \quad \text{by the induction hypothesis} \\ &= \frac{6}{n} \sum_{k=1}^{n} k = \frac{6}{n} \frac{n(n+1)}{2} = 3(n+1)\end{aligned}$$

which ends the induction step.

Sample solutions to Exercise 3.

Let $x \in [-1, +\infty)$. We are going to prove by induction that $\forall n \in \mathbb{N}, (1+x)^n \geq 1 + nx$.

- *Base case at $n = 0$:* $(1+x)^0 = 1$ and $1 + 0 \times x = 1$.
- *Induction step:* assume that $(1+x)^n \geq 1 + nx$ for some $n \in \mathbb{N}$. Then

$$\begin{aligned}(1+x)^{n+1} &= (1+x)^n(1+x) \\ &\geq (1+nx)(1+x) \quad \textit{by the induction hypothesis since } 1 + x \geq 0 \\ &= 1 + x + nx + nx^2 \\ &\geq 1 + x + nx = 1 + (n+1)x\end{aligned}$$

which ends the induction step.

Sample solutions to Exercise 4.

1. Let $n \geq 3$. Assume that $P(n)$ is true, i.e. $2^n > n^2$, and let's prove $P(n+1)$, i.e. $2^{n+1} > (n+1)^2$.
   From the assumption, we get that $2^{n+1} = 2 \times 2^n \geq 2n^2$. Hence it is enough to prove that $2n^2 > (n+1)^2$ which is equivalent to $n^2 - 2n - 1 > 0$.
   We study the sign of the polynomial $x^2 - 2x - 1$. It is a polynomial of degree 2 with positive leading coefficient and its discriminant is $(-2)^2 - 4 \times (-1) = 8 > 0$. Therefore

x	$-\infty$		$1-\sqrt{2}$		$1+\sqrt{2}$		$+\infty$
x^2-2x-1		$+$	0	$-$	0	$+$	

   Since $n \geq 3 > 1 + \sqrt{2}$, we know that $n^2 - 2n - 1 > 0$. Hence $P(n+1)$ holds.
2. $P(3)$ and $P(4)$ are false, but $P(5)$ is true. So by induction, $\forall n \geq 5,\ P(n)$ is true.
   Beware: even if the induction step is true for $n \geq 3$, we can only start the induction proof at $n = 5$! The base case is very important in a proof by induction.

Sample solutions to Exercise 5.

The induction step is false when $n = 2$ (it only holds for $n \geq 3$). Indeed, for $n = 2$, we only have that $A_1, A_2 \in L$ and that $A_2, A_3 \in L'$. Which is not enough to get that $L = L'$ since we only know that they have one point in common (it works if they have at least two points in common).

Beware: if you start an induction proof with a base case at n_0, you have to make sure that the induction step $P(n) \implies P(n+1)$ holds for every $n \geq n_0$. Otherwise, you didn't prove anything...

Sample solutions to Exercise 6.

Existence. We are going to prove the existence of such a couple (a, b) by a strong induction on n.

- *Base case at $n = 1$:* $1 = 2^0(2 \times 0 + 1)$.
- *Induction step.* Assume that for $1, 2, \dots, n$ admit such an expression for some $n \geq 1$.
  - First case: $n + 1$ is even, i.e. $n + 1 = 2k$ for some $k \in \mathbb{N}$.
    Note that $k \neq 0$ since otherwise $1 \leq n + 1 = 0$.
    Since $1 < 2$ and $k \neq 0$, we get that $k < 2k = n + 1$, so that $k \leq n$.
    Hence, by the induction hypothesis, $k = 2^a(2b + 1)$ for some $(a, b) \in \mathbb{N}^2$.
    Then $n + 1 = 2k = 2^{a+1}(2b + 1)$.
  - Second case: $n + 1$ is odd, i.e. $n + 1 = 2k + 1$ for some $k \in \mathbb{N}$. But then $n + 1 = 2^0(2 \times k + 1)$.

  Which ends the induction step.

Uniqueness. Assume that $2^a(2b + 1) = 2^\alpha(2\beta + 1)$ for $a, b, \alpha, \beta \in \mathbb{N}$.
If $a < \alpha$ then, by cancellation, we obtain $2b + 1 = 2^{\alpha-a}(2\beta + 1)$. Which is impossible since the LHS is odd whereas the RHS is even.
If $\alpha < a$ then, by cancellation, we obtain $2^{a-\alpha}(2b + 1) = 2\beta + 1$. Which is impossible since the RHS is odd whereas the LHS is even.
Therefore $a = \alpha$, and by cancellation we obtain $2b + 1 = 2\beta + 1$, hence $2b = 2\beta$ and finally $b = \beta$.
We proved that $(a, b) = (\alpha, \beta)$.

Sample solutions to Exercise 7.

The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n$ satisfies the conditions of the question. Actually, as we are going to prove, it is the only one.
From now on, we assume that $f : \mathbb{N} \to \mathbb{N}$ satisfigies $f(2) = 2$ and $\forall p, q \in \mathbb{N},\ f(pq) = f(p)f(q)$, and we want to prove that $\forall n \in \mathbb{N},\ f(n) = n$.

- We know that $0 < 1 < 2$ hence $0 \leq f(0) < f(1) < f(2) = 2$.
  Therefore, the only possibility is that $f(0) = 0$ and $f(1) = 1$.

- Let's prove by strong induction that $\forall n \in \mathbb{N},\ f(n) = n$.
  - *Base case at $n = 0$:* $f(0) = 0$.
  - *Induction step.* Assume that $f(0) = 0, f(1) = 1, f(2) = 2, f(3) = 3, \dots, f(n) = n$ for some $n \geq 0$.
    - First case: $n + 1$ is even, i.e. there exists $k \in \mathbb{N}$ such that $n + 1 = 2k$.
      Note that $k \neq 0$ since otherwise $1 \leq n + 1 = 0$.
      Since $1 < 2$ and $k \neq 0$, we get that $k < 2k = n + 1$, so that $k \leq n$.
      Then, by the induction hypothesis, $f(n + 1) = f(2k) = f(2)f(k) = 2k = n + 1$.
    - Second case: $n + 1$ is odd, i.e. there exists $k \in \mathbb{N}$ such that $n + 1 = 2k + 1$.
      Either $k = 0$ and then $f(n + 1) = f(1) = 1 = n + 2$.
      Or $k \neq 0$ and then $k + 1 < 2k + 1 = n + 1$, i.e. $k \leq n$.
      Then $f(n + 2) = f(2(k + 1)) = f(2)f(k + 1) = n + 2$ by the induction hypothesis.
      Thus $n = f(n) < f(n + 1) \leq f(n + 2) = n + 2$.
      The only possible value is that $f(n + 1) = n + 1$.

Sample solutions to Exercise 8.

1. Let $m, m' \in S$ be two greatest elements of S.
   Since $m \in S$ and m' is a greatest element, we have $m \leq m'$.
   Similarly, since $m' \in S$ and m is a greatest element of S, we have $m' \leq m$.
   Hence $m = m'$.

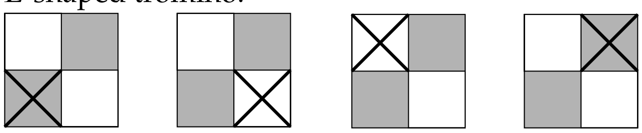
   *That's why we say **the** greatest element: if it exists, it is unique (whereas we say **an** upper bound).*

2. Let's prove that a non-empty finite subset $S \subset \mathbb{Z}$ has a greatest element, by induction on $n = \#S$.
   - *Base case at $n = 1$:* if S is a singleton, then its unique element is its greatest element.
   - *Induction step.* Assume that the statement holds for sets of cardinal n, for some $n \geq 1$.
     Let $S \subset \mathbb{Z}$ be such that $\#S = n + 1$.
     Particulalry $S \neq \varnothing$, so there exists $a \in S$.
     Set $T = S \setminus \{a\}$. Then $\#T = n$, so by the induction hypothesis T admits a greatest element $m \in T$.
     I claim that $M = \max(m, a) \in T \cup \{a\} = S$ is the greatest element of S.
     Indeed, let $n \in S$, either $n = a$ and then $a \leq M$, or $n \in T$ and then $n \leq m \leq M$.
     Which ends the inductive step.

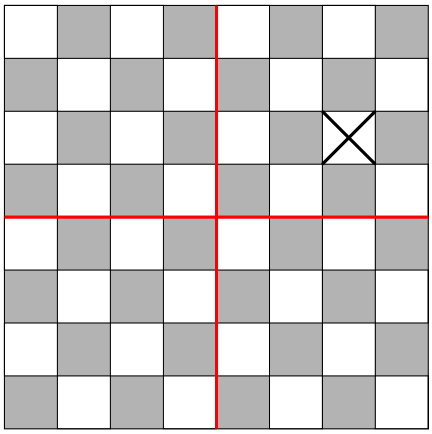
Sample solutions to Exercise 9.

We are going to prove by induction on $n \geq 1$ that if one square of a $2^n \times 2^n$ chessboard is removed, then the remaining squares can be covered with L-shaped trominoes.

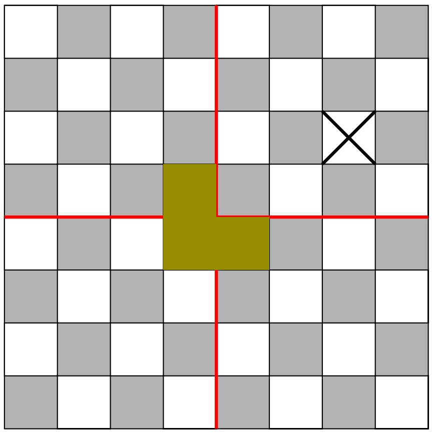
- *Base case at $n = 1$.* There are only four possible cases and for each of them the remaining is exactly one L-shaped tromino:



- Assume that the statement holds for some $n \geq 1$ and consider a $2^{n+1} \times 2^{n+1}$ chessboard with a removed square.
  We may split this chessboard into four $2^n \times 2^n$ chessboards as follows:



We may place an L-shaped tromino such that it covers the corner situated at the center for each $2^n \times 2^n$ chessboard without a removed square, see below.



Now, each of the $2^n \times 2^n$ chessboards has a removed square: we may apply the induction hypothesis in order to cover the remaining squares with L-shaped trominoes.