Countable sets & Cantor's diagonal argument

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Countable sets – 1

Notation

In what follows, we set $\aleph_0 := |\mathbb{N}|$ (pronounced *aleph nought*).

Definition

A set E is countable if either E is finite or $|E| = \aleph_0$.

Countable sets – 2

Proposition

1. $|\mathbb{N} \setminus \{0\}| = \aleph_0$
2. $|\{n \in \mathbb{N} \ : \ n \equiv 0 \mod 2\}| = \aleph_0$
3. $|\mathbb{N} \times \mathbb{N}| = \aleph_0$

Proof.

1. The function $f : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ defined by $f(n) = n + 1$ is bijective with inverse $f^{-1} : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ defined by $f^{-1}(n) = n - 1$.

2. The function $f : \mathbb{N} \to \{n \in \mathbb{N} \ : \ n \equiv 0 \mod 2\}$ defined by $f(n) = 2n$ is bijective.

3. Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $f(a, b) = 2^a 3^b$.
   Then f is injective by uniqueness of the prime decomposition. Thus $|\mathbb{N} \times \mathbb{N}| \leq \aleph_0$.
   Besides $\{0\} \times \mathbb{N} \subset \mathbb{N} \times \mathbb{N}$, thus $\aleph_0 = |\{0\} \times \mathbb{N}| \leq |\mathbb{N} \times \mathbb{N}|$.
   Hence $|\mathbb{N} \times \mathbb{N}| = \aleph_0$ by Cantor–Schröder–Bernstein theorem. ∎

Countable sets – 3

Proposition

If $S \subset \mathbb{N}$ is infinite then $|S| = \aleph_0$.

Proof. Let's define the function $f : \mathbb{N} \to S$ by induction as follows.
Set $f(0) = \min S$ *(which is well-defined by the well-ordering principle since $S \neq \varnothing$ as it is infinite)*.
And then, assuming that $f(n)$ is already defined, we set $f(n+1) = \min\{k \in S : k > f(n)\}$ *(which is well-defined by the well-ordering principle: the involved set is non-empty since otherwise S would be finite)*.
It is easy to check that f is injective (note that $\forall n \in \mathbb{N},\ f(n+1) > f(n)$), therefore $\aleph_0 \leq |S|$.

But since $S \subset \mathbb{N}$, we also have $|S| \leq \aleph_0$.

Thus, by Cantor–Schröder–Bernstein theorem, $|S| = \aleph_0$. ∎

Countable sets – 4

Proposition

A set E is countable if and only if $|E| \leq \aleph_0$ (i.e. there exists an injection $f : E \to \mathbb{N}$), otherwise stated E is countable if and only if there exists a bijection between E and a subset of $\mathbb{N}$.

Proof.

$\Rightarrow$ Assume that E is countable.

- Either E is finite and then there exists $n \in \mathbb{N}$ and a bijection $g : \{k \in \mathbb{N} : k < n\} \to E$.
  We define $f : E \to \mathbb{N}$ by $f(x) = g^{-1}(x)$ *(which is well-defined since $\{k \in \mathbb{N} : k < n\} \subset \mathbb{N}$)*.
  And f is an injection since g^{-1} is.
- Or $|E| = \aleph_0$, i.e. there exists a bijection $f : E \to \mathbb{N}$.

$\Leftarrow$ Assume there exists an injection $f : E \to \mathbb{N}$.
Assume that E is infinite. Then $|E| = |f(E)| = \aleph_0$.
Thus either E is finite or $|E| = \aleph_0$. In both cases E is countable. ■

Countable sets – 5

Theorem

A countable union of countable sets is countable,
i.e. if I is countable and if for every $i \in I$, E_i is countable then $\bigcup_{i\in I} E_i$ is countable.

Proof.
WLOG we may now assume that $I \subset \mathbb{N}$.
Let $i \in I$. Since E_i is countable, there exists an injection $f_i : E_i \to \mathbb{N}$[1].
We define $\varphi : \bigcup_{i\in I} E_i \to \mathbb{N}\times\mathbb{N}$ by $\varphi(x) = (n, f_n(x))$ where $n = \min\{i \in I \ : \ x \in E_i\}$ *(which exists by the well-ordering principle)*.
It is not difficult to check that φ is injective.
Therefore $\bigcup_{i\in I} E_i$ is countable. ∎

[1] We use the axiom of countable choice here.

Theorem

If E is an infinite set then there exists $T \subset E$ such that $|T| = \aleph_0$, i.e. $\aleph_0$ is the least infinite cardinal.

Proof.

For $n \in \mathbb{N}$, set $E_n = \{S \in \mathcal{P}(E) \ : \ |S| = n\}$.
Since E is infinite, it contains a subset of cardinal n, therefore $E_n \neq \varnothing$.
So for every $n \in \mathbb{N}$, we can pick[2] $S_n \in E_n$.
Then $T := \bigcup_{n\in\mathbb{N}} S_n$ is countable as a countable union of countable sets.
Besides, $\forall n \in \mathbb{N}$, $S_n \subset T$ and $|S_n| = n$.
Therefore T is infinite since for every $n \in \mathbb{N}$ it contains a subset of cardinal n.
Thus $|T| = \aleph_0$ as an infinite countable set. ∎

[2] We use the axiom of countable choice here.

Countable sets – 7

Theorem

$|\mathbb{Z}| = \aleph_0$

Proof 1. Since $\mathbb{N} \subset \mathbb{Z}$, we have $|\mathbb{N}| \leq |\mathbb{Z}|$.

Define $f : \mathbb{Z} \to \mathbb{N}$ by $f(n) = \begin{cases} 2^n & \text{if } n \geq 0 \\ 3^{-n} & \text{if } n < 0 \end{cases}$.

Then f is injective by uniqueness of the prime factorization. Therefore $|\mathbb{Z}| \leq |\mathbb{N}|$.
Hence $|\mathbb{Z}| = |\mathbb{N}|$ by Cantor–Schröder–Bernstein theorem. ■

Proof 2.

Define $f : \mathbb{Z} \to \mathbb{N}$ by $f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -(2n+1) & \text{if } n < 0 \end{cases}$.

Then f is bijective with inverse $f^{-1}(m) = \begin{cases} k & \text{if } \exists k \in \mathbb{N},\ m = 2k \\ -k-1 & \text{if } \exists k \in \mathbb{N},\ m = 2k+1 \end{cases}$.

Therefore $|\mathbb{Z}| = |\mathbb{N}|$. ■

Theorem

$|\mathbb{Q}| = \aleph_0$

Proof 1. Note that $\mathbb{N} \subset \mathbb{Q}$, therefore $\aleph_0 \leq |\mathbb{Q}|$.
Define $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$ by $f\left(\frac{a}{b}\right) = (a, b)$ where $\frac{a}{b}$ is in lowest form.
Then f is injective and thus $|\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{Z}|$. Since $|\mathbb{Z}| = |\mathbb{N}|$, we get $|\mathbb{Z} \times \mathbb{Z}| = |\mathbb{N} \times \mathbb{N}| = \aleph_0$.
We conclude using Cantor–Schröder–Bernstein theorem. ∎

Proof 2. Note that $\mathbb{N} \subset \mathbb{Q}$, therefore $\aleph_0 \leq |\mathbb{Q}|$.
Moreover $f : \mathbb{Z} \times \mathbb{N} \setminus \{0\} \to \mathbb{Q}$ defined by $f(a, b) = \frac{a}{b}$ is surjective. Thus $|\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{N} \setminus \{0\}|$.
Since $|\mathbb{Z}| = |\mathbb{N}|$ and $|\mathbb{N} \setminus \{0\}| = |\mathbb{N}|$, we get $|\mathbb{Z} \times \mathbb{N} \setminus \{0\}| = |\mathbb{N} \times \mathbb{N}| = \aleph_0$.
We conclude using Cantor–Schröder–Bernstein theorem. ∎

Proof 3. Note that $\mathbb{N} \subset \mathbb{Q}$, therefore $\aleph_0 \leq |\mathbb{Q}|$.
Since $\mathbb{Q} = \bigcup_{(a,b) \in \mathbb{Z} \times \mathbb{N} \setminus \{0\}} \left\{ \frac{a}{b} \right\}$, $\mathbb{Q}$ is countable as a countable union of countable sets. So $|\mathbb{Q}| \leq \aleph_0$.
We conclude using Cantor–Schröder–Bernstein theorem. ∎

Cantor's diagonal argument – 1

Theorem: $\mathbb{R}$ is not countable (Cantor 1874, the proof below dates back to 1891)

$\aleph_0 < |\mathbb{R}|$

Proof. We are going to prove that there is no surjection $\mathbb{N} \to \mathbb{R}$ (and hence no such bijection).

Let $f : \mathbb{N} \to \mathbb{R}$ be a function. Given $n \in \mathbb{N}$, we know that $f(n)$ has a unique proper decimal expansion $f(n) = \sum_{k=0}^{+\infty} a_{nk} 10^{-k}$ where $a_{n0} \in \mathbb{Z}$ and $a_{nk} \in \{0, 1, \dots, 9\}$ for $k \geq 1$, i.e.

$$\begin{array}{lllllllll} f(0) = & \color{red}{a_{00}} & . & a_{01} & a_{02} & a_{03} & a_{04} & a_{05} & \dots \\ f(1) = & a_{10} & . & \color{red}{a_{11}} & a_{12} & a_{13} & a_{14} & a_{15} & \dots \\ f(2) = & a_{20} & . & a_{21} & \color{red}{a_{22}} & a_{23} & a_{24} & a_{25} & \dots \\ f(3) = & a_{30} & . & a_{31} & a_{32} & \color{red}{a_{33}} & a_{34} & a_{35} & \dots \\ f(4) = & a_{40} & . & a_{41} & a_{42} & a_{43} & \color{red}{a_{44}} & a_{45} & \dots \\ \vdots & & & & & \vdots & & & \end{array}$$

Given $k \in \mathbb{N}$, we set $b_k = \begin{cases} 1 & \text{if } a_{kk} = 0 \\ 0 & \text{otherwise} \end{cases}$.

Then $b = \sum_{k=0}^{+\infty} b_k 10^{-k}$ is a real number written with its unique proper decimal expansion.

Note that for every $n \in \mathbb{N}$, $b \neq f(n)$ since $b_n \neq a_{nn}$ (we use the uniqueness of the proper decimal expansion).

Therefore $b \notin \operatorname{Im}(f)$ and f is not surjective. ∎

Cantor's diagonal argument – 2

Cantor's theorem

Given a set E, $|E| < |\mathcal{P}(E)|$.

Proof. *We are going to use Cantor's diagonal argument again.*
First, note that $g : E \to \mathcal{P}(E)$ defined by $g(x) = \{x\}$ is injective, therefore $|E| \leq |\mathcal{P}(E)|$.

We are going to prove that there is no surjection $E \to \mathcal{P}(E)$ *(and hence no such bijection)*.
Let $f : E \to \mathcal{P}(E)$ be a function. Define $S = \{x \in E : x \notin f(x)\}$.
Let $x \in E$.

- If $x \in f(x)$ then $x \notin S$.
- Otherwise, if $x \notin f(x)$ then $x \in S$.

Therefore $f(x) \neq S$ since one contains x but not the other one.
Thus $S \notin \mathrm{Im}(f)$ and f is not surjective. ∎

Remark

There is no greatest cardinal.

$|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$

We already know that $|\mathbb{N}| < |\mathbb{R}|$ and that $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$. Actually $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.

Theorem

$|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$

Proof.
Define $f : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ by $f(S) = \sum_{n \in S} 10^{-n}$.
Then f is injective by uniqueness of the proper decimal expansion. Thus $|\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$.

Define $g : \mathbb{R} \to \mathcal{P}(\mathbb{Q})$ by $g(x) = \{q \in \mathbb{Q} : q < x\}$.
Then g is injective. Indeed, let $x, y \in \mathbb{R}$ be such that $x < y$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $q \in \mathbb{Q}$ such that $x < q < y$. So $q \notin g(x)$ but $q \in g(y)$. Therefore $g(x) \neq g(y)$.
Hence $|\mathbb{R}| \leq |\mathcal{P}(\mathbb{Q})| = |\mathcal{P}(\mathbb{N})|$ *(prove the last equality using that $|\mathbb{Q}| = |\mathbb{N}|$)*.

We conclude thanks to Cantor–Schröder–Bernstein theorem. ∎

There is no set of all sets

Theorem

There is no set containing all sets.

Proof. Assume that such a set V exists.
Then the powerset $\mathcal{P}(V)$ exists too and $\mathcal{P}(V) \subset V$ by definition of V.
Therefore $|\mathcal{P}(V)| \leq |V|$, but $|V| < |\mathcal{P}(V)|$ by Cantor's theorem. Hence a contradiction. ∎

We may similarly prove that there is no set containing all finite sets, or even all singletons.

Theorem

There is no set containing all singletons.

Proof. Assume that the set S of all singletons exists.
Define $f : \mathcal{P}(S) \to S$ by $f(x) = \{x\}$ (which is well-defined).
Since f is one-to-one, we get that $|\mathcal{P}(S)| \leq |S|$.
Which contradicts $|S| < |\mathcal{P}(S)|$ (Cantor's theorem). ∎