CARDINAL COMPARISON

April 1st, 2021

Cardinal comparison – 1

Definition

Given two sets E and F, we write $|E| \leq |F|$ if there exists an injective function $f : E \to F$.

Remark

It is compatible with the definition in the finite case.

Cardinal comparison – 2

Cantor–Schröder–Bernstein theorem

Given two sets E and F, if $|E| \leq |F|$ and $|F| \leq |E|$ then $|E| = |F|$.

Remark

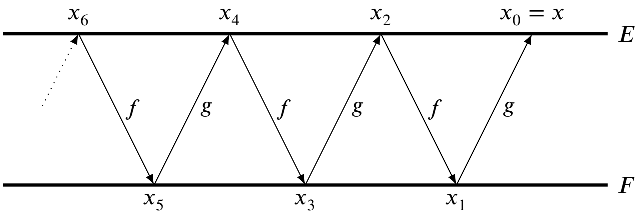
The above theorem is less trivial than it seems at first glance. It states that:
if there exist injections $f : E \to F$ and $g : F \to E$ then there exists a bijection $h : E \to F$.

- 1887: Cantor stated the theorem with no proof.
- 1887: Dedekind proved it but didn't publish his proof (it was found after his death).
- 1895: Cantor published a proof (nonetheless, the proof relies on the *trichotomoy principle* which Tarski later proved to be equivalent to the axiom of choice).
- 1897: Bernstein, Schröder and Dedekind independently found proofs of the theorem (without the axiom of choice). But Schröder's proof later appeared to be incorrect.
- Several other proofs are now known, e.g. Zermelo (1901, 1908) and König (1906).

Cantor–Schröder–Bernstein theorem: strategy of the proof – 1

We are given two injective functions $f : E \to F$ and $g : F \to E$.
Fix $x \in E$. Set $x_0 = x \in E$ and then we define the next terms inductively by

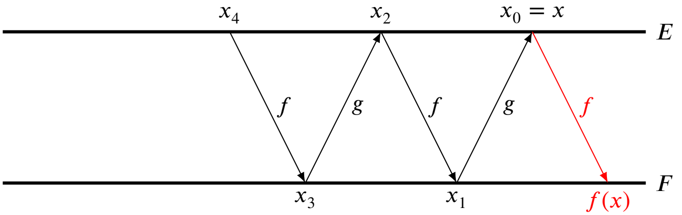
- if $x_n \in E$ then we define $x_{n+1} \in F$ as the unique antecedant of x_n by g (if it exists, otherwise we stop).
- if $x_n \in F$ then we define $x_{n+1} \in E$ as the unique antecedant of x_n by f (if it exists, otherwise we stop).



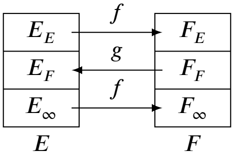
Then we face three possible cases:

1. or the chain ends with an element in F, and then we put x in E_F,
2. Either the chain ends with an element in E, and then we put x in E_E,
3. or the inductive definition of the chain doesn't stop, and then we put x in E_∞.

We have a partition $E = E_E \sqcup E_F \sqcup E_\infty$. We construct similarly $F = F_E \sqcup F_F \sqcup F_\infty$.



If $x \in E_E$ then $f(x) \in F_E$. Therefore $f_{|E_E} : E_E \to F_E$ is well-defined.
Besides, it is injective since f is, and it is surjective by definition of F_E.
Hence $f_{|E_E} : E_E \to F_E$ is a bijection.
Similarly $g_{|F_F} : F_F \to E_F$ and $f_{|E_\infty} : E_\infty \to F_\infty$ are bijections.
Finally, we glue them in order to obtain a bijection $h : E \to F$.



Cantor–Schröder–Bernstein theorem: the proof

Proof of Cantor–Schröder–Bernstein theorem.
Let $f : E \to F$ and $g : F \to E$ be two injective functions. Set

- $E_E = \left\{x \in E \ : \ \exists n \in \mathbb{N}, \exists r \in E \setminus \operatorname{Im}(g), x = (g \circ f)^n(r)\right\}$
- $E_F = \left\{x \in E \ : \ \exists n \in \mathbb{N}, \exists s \in F \setminus \operatorname{Im}(f), x = g\left((f \circ g)^n(s)\right)\right\}$
- $E_\infty = E \setminus \left(E_E \sqcup E_F\right)$
- $F_E = \left\{y \in F \ : \ \exists n \in \mathbb{N}, \exists r \in E \setminus \operatorname{Im}(g), y = f\left((g \circ f)^n(r)\right)\right\}$
- $F_F = \left\{y \in F \ : \ \exists n \in \mathbb{N}, \exists s \in F \setminus \operatorname{Im}(f), y = (f \circ g)^n(s)\right\}$
- $F_\infty = F \setminus \left(F_E \sqcup F_F\right)$

Note that if $x \in E_E$ then $f(x) \in F_E$. So $f_{|E_E} : E_E \to F_E$ is well-defined.
It is injective since f is. And it is surjective by definition of the sets. Thus it is bijective.
Similarly, $g_{|F_F} : F_F \to E_F$ and $f_{|E_\infty} : E_\infty \to F_\infty$ are well-defined and bijective.

We define $h : E \to F$ by $h(x) = \begin{cases} f(x) & \text{if } x \in E_E \\ g^{-1}(x) & \text{if } x \in E_F \\ f(x) & \text{if } x \in E_\infty \end{cases}$.

Then h is a bijection *(check it)*. ∎

Proposition

1. If E is a set then $|E| \leq |E|$.
2. Given two sets E and F, if $|E| \leq |F|$ and $|F| \leq |E|$ then $|E| = |F|$.
3. Given three sets E, F and G, if $|E| \leq |F|$ and $|F| \leq |G|$ then $|E| \leq |G|$.

Proof.

1. $id : E \to E$ is an injective function.
2. It is Cantor–Bernstein–Schröder theorem.
3. Assume that $|E| \leq |F|$ and $|F| \leq |G|$,
   i.e. that there exist injections $f : E \to F$ and $g : F \to G$.
   Then $g \circ f : E \to G$ is injective, thus $|E| \leq |G|$. ∎

Remark

Comparison of cardinals shares the characteristic properties of an order.
Nonetheless, it is not an order since it is not a binary relation on a set (as for equipotence).

Cardinal comparison – 4

Proposition

If $E \subset F$ then $|E| \leq |F|$.

Proof. Indeed, $f : E \to F$ defined by $f(x) = x$ is injective. ■

Proposition

If $|E_1| = |E_2|$ and $|F_1| = |F_2|$ then $|E_1 \times F_1| = |E_2 \times F_2|$.

Proof. Assume that $|E_1| = |E_2|$ and $|F_1| = |F_2|$ then there exist bijections $f : E_1 \to E_2$ and $g : F_1 \to F_2$.
We define $h : E_1 \times F_1 \to E_2 \times F_2$ by $h(x, y) = (f(x), g(y))$. Let's check that h is a bijection.

- *h is injective.*
  Let $(x, y), (x', y') \in E_1 \times F_1$ be such that $h(x, y) = h(x', y')$.
  Then $f(x) = f(x')$ and $g(y) = g(y')$, thus $x = x'$ and $y = y'$ since f and g are injectives.
  We proved that $(x, y) = (x', y')$.
- *h is surjective.*
  Let $(z, w) \in E_2 \times F_2$. Since f is surjective, there exists $x \in E_1$ such that $z = f(x)$.
  Since g is surjective, there exists $y \in F_1$ such that $w = g(y)$.
  Then $h(x, y) = (f(x), g(y)) = (z, w)$. ■

Cardinal comparison – 5

Theorem

Given two sets, E and F, $|E| \leq |F|$ if and only if there exists a surjective function $g : F \to E$.

Proof.

$\Rightarrow$ Assume that there exists an injective function $f : E \to F$, then $\tilde{f} = f : E \to f(E)$ is bijective.
If $E = \varnothing$, then there is nothing to prove. So we may assume that there exists $u \in E$.

Define $g : F \to E$ by $g(y) = \begin{cases} \tilde{f}^{-1}(y) & \text{if } y \in f(E) \\ u & \text{otherwise} \end{cases}$

Let $x \in E$, then $g(f(x)) = \tilde{f}^{-1}(f(x)) = x$. Thus g is surjective.

$\Leftarrow$ Assume that there exists a surjective function $g : F \to E$, then[1] $\forall x \in E, \exists y_x \in g^{-1}(x)$.
Define $f : E \to F$ by $f(x) = y_x$. Then f is injective, so $|E| \leq |F|$.
Indeed, assume that $f(x) = f(x')$ then $g(f(x)) = g(f(x'))$.
But $g(f(x)) = g(y_x) = x$ and similarly $g(f(x')) = x'$. Thus $x = x'$. ∎

[1] We use the axiom of choice here.