Cardinality: finite sets

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March 25th, 2021

Finite sets – 1

Definition: finite set

We say that a set E is finite if there exists $n \in \mathbb{N}$ and a bijection $f : \{k \in \mathbb{N} : k < n\} \to E$.
Then we write $|E| = n$.

Note that $\{k \in \mathbb{N} : k < n\} = \{0, 1, 2, \dots, n-1\}$.

Finite sets – 2

Lemma

Let $n, p \in \mathbb{N}$. If there exists an injective function $f : \{k \in \mathbb{N} : k < n\} \to \{k \in \mathbb{N} : k < p\}$ then $n \leq p$.

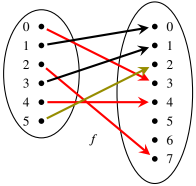
Proof. We prove the statement by induction on n.

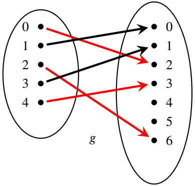
- *Base case at $n = 0$:* for any $p \in \mathbb{N}$ we have $n \leq p$.
- *Induction step.* Assume that the statement holds for some $n \in \mathbb{N}$.
  Let $p \in \mathbb{N}$. Assume that there exists an injective function $f : \{k \in \mathbb{N} : k < n+1\} \to \{k \in \mathbb{N} : k < p\}$.

  Define $g : \{k \in \mathbb{N} : k < n\} \to \{k \in \mathbb{N} : k < p-1\}$ as follows: $g(x) = \begin{cases} f(x) & \text{if } f(x) < f(n) \\ f(x) - 1 & \text{if } f(x) > f(n) \end{cases}$

  Note that $f(x) \neq f(n)$ since f is injective.

  - ⋆ Claim 1: g is well-defined, i.e. $\forall x \in \{k \in \mathbb{N} : k < n\}, g(x) \in \{k \in \mathbb{N} : k < p-1\}$.
    Let $x \in \{k \in \mathbb{N} : k < n\}$.
    So either, $f(x) < f(n)$ and then $g(x) = f(x) < f(n) < p$, therefore $0 \leq g(x) < p - 1$.
    Or, $f(x) > f(n)$ and then $g(x) = f(x) - 1 < p - 1$, therefore $0 \leq g(x) < p - 1$.
  - ⋆ Claim 2: g is injective.
    Let $x, y \in \{k \in \mathbb{N} : k < n\}$ be such that $g(x) = g(y)$.
    First case: $f(x), f(y) < f(n)$.
    Then $g(x) = f(x)$ and $g(y) = f(y)$. So $f(x) = f(y)$ and thus $x = y$ since f is injective.
    Second case: $f(x), f(y) > f(n)$.
    Then $g(x) = f(x) - 1$ and $g(y) = f(y) - 1$. So $f(x) = f(y)$ and thus $x = y$ since f is injective.
    Third case: $f(x) < f(n)$ and $f(y) > f(n)$.
    Then $g(x) = f(x) < f(n)$ and $g(y) = f(y) - 1 > f(n) - 1 \geq f(n)$. Therefore, this case is impossible.





Therefore, by the induction hypothesis, $n \leq p - 1$, i.e. $n + 1 \leq p$. ∎

Finite sets – 3

Definition: finite set

We say that a set E is finite if there exists $n \in \mathbb{N}$ and a bijection $f : \{k \in \mathbb{N} : k < n\} \to E$.
Then we write $|E| = n$.

Corollary

Let E be a finite set. If $|E| = n$ and $|E| = m$, then $m = n$.
Then we say that $|E|$ is the *cardinal* of E, which is uniquely defined.

Proof. Assume there exists a bijection $f_1 : \{k \in \mathbb{N} : k < n\} \to E$ and a bijection $f_2 : \{k \in \mathbb{N} : k < m\} \to E$.
Then $f_2^{-1} \circ f_1 : \{k \in \mathbb{N} : k < n\} \to \{k \in \mathbb{N} : k < m\}$ is a bijection, so by the above lemma, $n \leq m$.
Similarly, $f_1^{-1} \circ f_2 : \{k \in \mathbb{N} : k < m\} \to \{k \in \mathbb{N} : k < n\}$ is a bijection and thus $m \leq n$.
Therefore $n = m$. ■

Remark: the empty set

$|E| = 0 \Leftrightarrow E = \varnothing$
Indeed, if $E = \varnothing$ then $f : \{k \in \mathbb{N} : k < 0\} \to E$ is always bijective: injectiveness and surjectiveness are vacuously true. So $|E| = 0$.
Otherwise, if $E \neq \varnothing$ then $f : \{k \in \mathbb{N} : k < 0\} \to E$ is never surjective (thus never bijective), so $|E| \neq 0$.

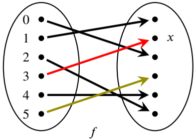
Finite sets – 4

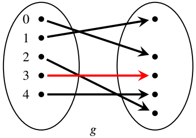
Proposition

If $E \subset F$ and F is finite then E is finite too, besides, $|E| \leq |F|$.

Proof. Let's prove by induction on $n = |F|$ that if $E \subset F$ then E is finite and $|E| \leq n$.

- *Base case at $n = 0$:* then $F = \varnothing$, so the only possible subset is $E = \varnothing$ and then $|E| = 0$.
- *Induction step.* Assume that the statement holds for some $n \in \mathbb{N}$.
  Let F be a set such that $|F| = n + 1$.
  - *First case:* $E = F$. Then the statement is obvious.
  - *Second case:* $E \neq F$. Then there exists $x \in F \setminus E$.
    There exists a bijection $f : \{k \in \mathbb{N} : k < n + 1\} \to F$.
    Since f is bijective, there exists a unique $m < n + 1$ such that $f(m) = x$.
    Define $g : \{k \in \mathbb{N} : k < n\} \to F \setminus \{x\}$ by $g(k) = f(k)$ for $k \neq m$,
    and, if $m \neq n$, $g(m) = f(n)$.
    Then g is a bijection, so $F \setminus \{x\}$ is finite and $|F \setminus \{x\}| = n$.
    Since $E \subset F \setminus \{x\}$, by the induction hypothesis, E is finite and $|E| \leq n < n + 1$. ∎





Finite sets – 5

Proposition

Let $E \subset F$ with F finite. Then $|F| = |E| + |F \setminus E|$.

Proof. Since $F \setminus E \subset F$ and $E \subset F$, we know that E and $F \setminus E$ are finite.
Denote $r = |E|$ and $s = |F \setminus E|$.
There exist bijections $f : \{k \in \mathbb{N} : k < r\} \to E$ and $g : \{k \in \mathbb{N} : k < s\} \to F \setminus E$.
Define $h : \{k \in \mathbb{N} : k < r + s\} \to F$ by $h(k) = \begin{cases} f(k) & \text{if } k < r \\ g(k-r) & \text{if } k \geq r \end{cases}$.

- h is well-defined:
  Indeed, if $0 \leq k < r$ then $f(k)$ is well-defined and $f(k) \in E \subset F$.
  If $r \leq k < r + s$ then $0 \leq k - r < s$ so that $g(k-r)$ is well-defined and $g(k-r) \in F \setminus E \subset F$.
- h is a bijection:
  - h is injective: let $x, y \in \{0, 1, \dots, r+s-1\}$ be such that $h(x) = h(y)$.
    Either $h(x) = h(y) \in E$ and then $f(x) = h(x) = h(y) = f(y)$ thus $x = y$ since f is injective.
    Or $h(x) = h(y) \in F \setminus E$ and then $g(x-r) = h(x) = h(y) = g(y-r)$ thus $x - r = y - r$ since g is injective, hence $x = y$.
  - h is surjective: let $y \in F$.
    Either $y \in E$, and then there exists $x \in \{0, 1, \dots, r-1\}$ such that $f(x) = y$, since f is surjective. Then $h(x) = f(x) = y$.
    Or $y \in F \setminus E$, and then there exists $x \in \{0, 1, \dots, s-1\}$ such that $g(x) = y$ since g is surjective. Then $h(x+r) = g(x) = y$.

Therefore $|F| = r + s = |E| + |F \setminus E|$. ∎

Finite sets – 6

Proposition

Let E and F be two finite sets. Then

1. $|E \cup F| = |E| + |F| - |E \cap F|$
2. $|E \times F| = |E| \times |F|$

Proof.

1. Using the previous proposition twice, we get

$$|E \cup F| = |E \sqcup (F \setminus (E \cap F))| = |E| + |F \setminus (E \cap F)| = |E| + |F| - |E \cap F|$$

2. We prove this proposition by induction on $n = |F| \in \mathbb{N}$.
   - *Base case at $n = 0$:* then $F = \varnothing$ so $E \times F = \varnothing$ too and $|E \times F| = 0 = |E| \times 0 = |E| \times |F|$.
   - *Case $n = 1$:* we will use this special case later in the proof.
     Assume that $F = \{*\}$ and that $|E| = n$. Then there exists a bijection $f : \{k \in \mathbb{N} : k < n\} \to E$.
     Note that $g : \{k \in \mathbb{N} : k < n\} \to E \times F$ defined by $g(k) = (f(k), *)$ is a bijection.
     Therefore $|E \times F| = n = n \times 1 = |E| \times |F|$.
   - *Induction step.* Assume that the statement holds for some $n \in \mathbb{N}$.
     Let F be a set such that $|F| = n + 1$.
     Since $|F| > 0$, there exists $x \in F$ and $|F \setminus \{x\}| = |F| - |\{x\}| = n + 1 - 1 = n$. Then

$$\begin{aligned} |E \times F| &= |(E \times (F \setminus \{x\})) \sqcup (E \times \{x\})| = |E \times (F \setminus \{x\})| + |E \times \{x\}| \\ &= |E| \times |F \setminus \{x\}| + |E| \text{ using the induction hypothesis and the case } n = 1 \\ &= |E| \times (|F| - 1) + |E| = |E| \times |F| \end{aligned}$$

∎

Proposition

Assume that $E \subset F$ with F finite. Then $E = F \Leftrightarrow |E| = |F|$.

Proof.
$\Rightarrow$ It is obvious.
$\Leftarrow$ Assume that $|E| = |F|$. Then $|F \setminus E| = |F| - |E| = 0$. Thus $F \setminus E = \varnothing$, i.e. $E = F$. ■

Proposition

Let E a finite set. Then F is finite and $|E| = |F|$ if and only if there exists a bijection $f : E \to F$.

Proof.
$\Rightarrow$ Assume that F is finite and that $|E| = |F| = n$.
Then there exist bijections $\varphi : \{k \in \mathbb{N} : k < n\} \to E$ and $\psi : \{k \in \mathbb{N} : k < n\} \to F$.
Therefore $f = \psi \circ \varphi^{-1} : E \to F$ is a bijection.
$\Leftarrow$ Assume that there exists a bijection $f : E \to F$.
Since E is finite there exists a bijection $\varphi : \{k \in \mathbb{N} : k < |E|\} \to E$.
Thus $f \circ \varphi : \{k \in \mathbb{N} : k < |E|\} \to F$ is a bijection. Therefore F is finite and $|F| = |E|$. ■

Proposition

Let E, F be two finite sets such that $|E| = |F|$. Let $f : E \to F$. Then TFAE:

1. f is injective,
2. f is surjective,
3. f is bijective.

Proof.
Assume that f is injective.
There exists a bijection $\varphi : \{k \in \mathbb{N} \, : \, k < |E|\} \to E$.
Then $f \circ \varphi : \{k \in \mathbb{N} \, : \, k < |E|\} \to f(E)$ is a bijection. Thus $|f(E)| = |E| = |F|$.
Since $f(E) \subset F$ and $|f(E)| = |F|$, we get $f(E) = F$, i.e. f is surjective.

Assume that f is surjective.
Then for every $y \in F$, $f^{-1}(y) \subset E$ is finite and non-empty, i.e. $\left|f^{-1}(y)\right| \geq 1$.
Assume by contradiction that there exists $y \in F$ such that $|f^{-1}(y)| > 1$.

Thus $|E| = \left| \bigsqcup_{y \in F} f^{-1}(y) \right| = \sum_{y \in F} \left|f^{-1}(y)\right| > |F| = |E|$. Hence a contradiction.

Finite sets – 9

Proposition

Let E and F be two finite sets. Then $|E| \leq |F|$ if and only if there exists an injection $f : E \to F$.

Proof.
$\Rightarrow$ Assume that $|E| \leq |F|$.
There exist bijections $\varphi : \{k \in \mathbb{N} : k < |E|\} \to E$ and $\psi : \{k \in \mathbb{N} : k < |F|\} \to F$.
Since $|E| \leq |F|$, $f = \psi \circ \varphi^{-1} : E \to F$ is well-defined and injective.
$\Rightarrow$ Assume that there exists an injection $f : E \to F$.
Then f induces a bijection $f : E \to f(E)$, so that $|E| = |f(E)|$.
And since $f(E) \subset F$, we have $|f(E)| \leq |F|$. ∎

Corollary: the pigeonhole principle or Dirichlet's drawer principle

Let E and F be two finite sets. If $|E| > |F|$ then there is no injective function $E \to F$.

Examples

- There are two non-bald people in Toronto with the exact same number of hairs on their heads.
- During a post-covid party with $n > 1$ participants, we may always find two people who shook hands to the same number of people.

Finite sets – 10

Remark: trichotomy principle for finite sets

Since the cardinal of a finite set is a natural number, we deduce from the fact that $\mathbb{N}$ is totally ordered, that given two finite sets E and F, exactly one of the followings occurs:

- either $|E| < |F|$
  i.e. there is an injection $E \to F$ *but no bijection* $E \to F$*,*
- or $|E| = |F|$
  i.e. there is a bijection $E \to F$*,*
- or $|E| > |F|$
  i.e. there is an injection $F \to E$ *but no bijection* $E \to F$*.*