Reviews about functions

March 23$^{\text{rd}}$, 2021

Functions – 1

(Informal) definition of a function

A *function* (or *map*) is the data of two sets A and B together with a "process" which assigns to each $x \in A$ a unique $f(x) \in B$:

$$f : \left\{ \begin{array}{ccc} A & \to & B \\ x & \mapsto & f(x) \end{array} \right.$$

Here, f is the name of the function, A is the *domain* of f, and B is the *codomain* of f.

Remark

This process can be:

- A formula: $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{x^2-\pi} + 42$.
- An exhaustive list: $f : \{1, 2, 3\} \to \mathbb{R}$ defined by $f(1) = \pi$, $f(2) = \sqrt{2}$, $f(3) = e$.
- A property characterizing f: $\log$ is the unique antiderivative of $g : (0, +\infty) \to \mathbb{R}$ defined by $g(x) = \frac{1}{x}$ such that $\log(1) = 0$.
- By induction: we define the sequence $u_n : \mathbb{N} \to \mathbb{R}$ by $u_0 = 1$ and $\forall n \in \mathbb{N}$, $u_{n+1} = u_n^2 + 1$.
- …

Functions – 2

Remark

The domain and codomain are part of the definition of a function.

For instance:

- $f : \left\{ \begin{array}{ccc} \mathbb{R} & \to & (0,+\infty) \\ x & \mapsto & e^x \end{array} \right.$ and $g : \left\{ \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & e^x \end{array} \right.$
  are not the same function (the first one is surjective but not the second one).
- $f : \left\{ \begin{array}{ccc} [0,+\infty) & \to & \mathbb{R} \\ x & \mapsto & x^2+1 \end{array} \right.$ and $g : \left\{ \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2+1 \end{array} \right.$
  are not the same function (the first one is injective but not the second one).

A function is not simply a ”formula”, you need to specify the domain and the codomain.

Injective/Surjective/Bijective functions – 1

Injective/Surjective/Bijective functions

Given a function $f : A \to B$.

- We say that f is *injective* (or *one-to-one*) if $\forall x_1, x_2 \in A,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
  or equivalently by taking the contrapositive $\forall x_1, x_2 \in A,\ f(x_1) = f(x_2) \implies x_1 = x_2$
- We say that f is *surjective* (or *onto*) if $\forall y \in B, \exists x \in A,\ y = f(x)$
- We say that f is *bijective* if it is injective and surjective, i.e. $\forall y \in B, \exists ! x \in A,\ y = f(x)$

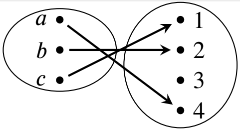

Figure: Injective

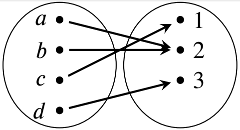

Figure: Surjective

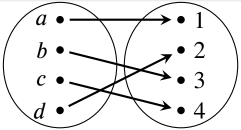

Figure: Bijective

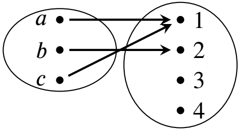

Figure: Not injective

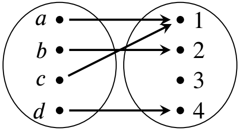

Figure: Not surjective

Injective/Surjective/Bijective functions – 2

Proposition

Let $f : E \to F$ and $g : F \to G$ be two functions.

1. If f and g are injective then so is $g \circ f$.
2. If f and g are surjective then so is $g \circ f$.
3. If $g \circ f$ is injective then f is injective too.
4. If $g \circ f$ is surjective then g is surjective too.

Proof.

1. Let $x, y \in E$ be such that $g(f(x)) = g(f(y))$.
   Then $f(x) = f(y)$ since g is injective. Thus $x = y$ since f is injective.
2. Let $z \in G$. Since g is surjective, it exists $y \in F$ such that $z = g(y)$.
   Since f is surjective, it exists $x \in E$ such that $y = f(x)$. Therefore $z = g(f(x))$.
3. Let $x, y \in E$ such that $f(x) = f(y)$.
   Then $g(f(x)) = g(f(y))$ and thus $x = y$ since $g \circ f$ is injective.
4. Let $z \in G$. Since $g \circ f$ is surjective, there exists $x \in E$ such that $z = g(f(x))$.
   Then $y = f(x) \in F$ satisfies $g(y) = z$. ∎

Inverse of a bijection

Proposition

$f : A \to B$ is bijective if and only if there exists $g : B \to A$ such that $\begin{cases} \forall x \in A,\ g(f(x)) = x \\ \forall y \in B,\ f(g(y)) = y \end{cases}$.

Then g is unique, it is called the *inverse of f* and denoted by $f^{-1} : B \to A$.

Proof.

$\Rightarrow$ Assume that f is bijective, then $\forall y \in B, \exists! x_y \in A, f(x_y) = y$.
We define $g : B \to A$ by $g(y) = x_y$. Then g satisfies the required properties.
$\Leftarrow$ Assume that there exists g as in the statement.
Then $g \circ f = id_A$ is injective, so f is too.
And $f \circ g = id_B$ is surjective, thus f is too.
Therefore f is bijective.

Uniqueness: assume there exist two such functions $g_1, g_2 : B \to A$.
Let $y \in B$. Then $f(g_1(y)) = y = f(g_2(y))$.
So $g_1(y) = g_2(y)$ since f is injective. ∎

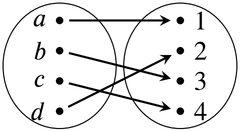

Figure: Bijective function

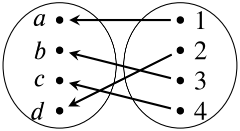

Figure: Its inverse