Real numbers – 3

March 18th, 2021

$\mathbb{Q}$ is dense in $\mathbb{R}$ – 1

Theorem: $\mathbb{Q}$ is dense in $\mathbb{R}$

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Proof. Let $x, y \in \mathbb{R}$ be such that $x < y$. Set $\varepsilon = y - x > 0$.
By the archimedean property, there exists $n \in \mathbb{N} \setminus \{0\}$ such that $n\varepsilon > 1$, i.e. $\frac{1}{n} < \varepsilon$.
Set $m = \lfloor nx \rfloor + 1$. Then $nx < m \leq nx + 1$, so $x < \frac{m}{n} \leq x + \frac{1}{n} < x + \varepsilon = y$. ∎

$\mathbb{Q}$ is dense in $\mathbb{R}$ – 2

Definition: interval

A subset $I \subset \mathbb{R}$ is an *interval* if $\forall x, y \in I, \forall z \in \mathbb{R}, (x \leq z \leq y \Rightarrow z \in I)$.

$\mathbb{Q}$ is dense in $\mathbb{R}$ – 2

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Crollary

If $I \subset \mathbb{R}$ is a non-empty interval not reduced to a singleton then $I \cap \mathbb{Q} \neq \varnothing$.

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A subset $I \subset \mathbb{R}$ is an *interval* if $\forall x, y \in I, \forall z \in \mathbb{R}, (x \leq z \leq y \Rightarrow z \in I)$.

Crollary

If $I \subset \mathbb{R}$ is a non-empty interval not reduced to a singleton then $I \cap \mathbb{Q} \neq \varnothing$.

Proof. Since I is non-empty and not reduced to a singleton, there exist $x, y \in I$ with $x < y$.
Then, since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $q \in \mathbb{Q}$ such that $x < q < y$.
Since I is an interval, $q \in I$. Hence $q \in I \cap \mathbb{Q} \neq \varnothing$. ■

$\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$

Theorem: $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$

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Proof. Let $x, y \in \mathbb{R}$ such that $x < y$.
Since , there exists $q \in \mathbb{Q}$ such that $x < q < y$.
Similarly, there exists $p \in \mathbb{Q}$ such that $x < p < q$.
Hence we obtained $p, q \in \mathbb{Q}$ such that $x < p < q < y$.
Set $s = p + \frac{\sqrt{2}}{2}(q - p)$. Then $s \in \mathbb{R} \setminus \mathbb{Q}$ (otherwise, by contradiction, $\sqrt{2}$ would be in $\mathbb{Q}$) and $p < s < q$ (notice that $0 < \frac{\sqrt{2}}{2} < 1$ so s is a number between p and q).
We obtained $s \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < s < y$. ∎

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We obtained $s \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < s < y$. ∎

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Corollary

If $I \subset \mathbb{R}$ is an interval which is non-empty and not reduced to a singleton then $I \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \varnothing$.

Decimal representation of real numbers – 1

Lemma

Let $(a_k)_{k\geq 1}$ be a sequence such that $\forall k \in \mathbb{N} \setminus \{0\}$, $a_k \in \{0, 1, \dots, 9\}$. Then the series

$$S = \sum_{k=1}^{+\infty} \frac{a_k}{10^k}$$

is convergent and $S \geq 0$.

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is convergent and $S \geq 0$.

Proof.

Note that $0 \leq \dfrac{a_k}{10^k} \leq \dfrac{9}{10^k}$ and that $\displaystyle\sum_{k=1}^{+\infty} \frac{9}{10^k}$ is convergent (geometric series with ratio $\dfrac{1}{10} < 1$).

Therefore we may conclude using the BCT. ∎

Decimal representation of real numbers – 2

Unfortunately the decimal representation may not be unique: $0.9999\ldots = \sum_{k=1}^{+\infty} \frac{9}{10^k} = \frac{9}{10} \times \frac{1}{1-\frac{1}{10}} = 1.000\ldots$

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Definition: proper decimal expansion

Let $x \in \mathbb{R}$. We say that $\lfloor x \rfloor . a_1 a_2 a_3 \ldots \coloneqq \lfloor x \rfloor + \sum_{k=1}^{+\infty} \frac{a_k}{10^k}$ is a *proper decimal expansion of* x if

1. $\forall k \in \mathbb{N} \setminus \{0\},\ a_k \in \{0, 1, \ldots, 9\}$
2. $\forall n \in \mathbb{N} \setminus \{0\},\ \sum_{k=1}^{n} \frac{a_k}{10^k} \leq x - \lfloor x \rfloor < \sum_{k=1}^{n} \frac{a_k}{10^k} + \frac{1}{10^n}$

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Proposition

If $\lfloor x \rfloor + \sum_{k=1}^{+\infty} \frac{a_k}{10^k}$ is a proper decimal expansion of $x \in \mathbb{R}$ then

1. $x = \lfloor x \rfloor + \sum_{k=1}^{+\infty} \frac{a_k}{10^k}$
2. $\forall N \in \mathbb{N} \setminus \{0\},\ \exists k > N,\ a_k \neq 9$

Decimal representation of real numbers – 2

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Definition: proper decimal expansion

Let $x \in \mathbb{R}$. We say that $\lfloor x \rfloor . a_1 a_2 a_3 \ldots \coloneqq \lfloor x \rfloor + \sum_{k=1}^{+\infty} \frac{a_k}{10^k}$ is a *proper decimal expansion of x* if

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Proposition

If $\lfloor x \rfloor + \sum_{k=1}^{+\infty} \frac{a_k}{10^k}$ is a proper decimal expansion of $x \in \mathbb{R}$ then 1. $x = \lfloor x \rfloor + \sum_{k=1}^{+\infty} \frac{a_k}{10^k}$ 2. $\forall N \in \mathbb{N} \setminus \{0\}, \exists k > N,\ a_k \neq 9$

Proof.

1. We already proved that $S = \sum_{k=1}^{+\infty} \frac{a_k}{10^k}$ is convergent. Hence we get $S \leq x - \lfloor x \rfloor \leq S$. So $x = \lfloor x \rfloor + S$.
2. Assume by contradiction that there exists $N \in \mathbb{N} \setminus \{0\}$ such that $\forall k > N,\ a_k = 9$.

   Then $x - \lfloor x \rfloor = \sum_{k=1}^{+\infty} \frac{a_k}{10^k} = \sum_{k=1}^{N} \frac{a_k}{10^k} + \sum_{k=N+1}^{+\infty} \frac{9}{10^k} = \sum_{k=1}^{N} \frac{a_k}{10^k} + \frac{1}{10^N}$. Which contradicts the definition of proper decimal expansion. ■

Decimal representation of real numbers – 3

Theorem

A real number x admits a unique proper decimal expansion.

Decimal representation of real numbers – 3

Theorem

A real number x admits a unique proper decimal expansion.

Proof. Let $x \in \mathbb{R}$. Up to replacing x with $x - \lfloor x \rfloor$, we may assume that $\lfloor x \rfloor = 0$.

Assume that $\displaystyle\sum_{k=1}^{+\infty} \frac{a_k}{10^k}$ is a proper decimal expansion of x. Then $a_n \leq 10^n \left(x - \displaystyle\sum_{k=1}^{n-1} \frac{a_k}{10^k} \right) < a_n + 1$.

So the only possible suitable sequence (a_n) is given by $a_1 = \lfloor 10x \rfloor$ and $a_{n+1} = \left\lfloor 10^{n+1} \left(x - \displaystyle\sum_{k=1}^{n} \frac{a_k}{10^k} \right) \right\rfloor$.

It proves the uniqueness, but we still need to check that it is valid.

1. Since $\lfloor x \rfloor = 0$, we have $0 \leq x < 1$. Thus $0 \leq 10x < 10$. Therefore $a_1 = \lfloor 10x \rfloor \in \{0, 1, \dots, 9\}$.
   Let $n \in \mathbb{N} \setminus \{0\}$, then $0 \leq 10^n \left(x - \displaystyle\sum_{k=1}^{n-1} \frac{a_k}{10^k} \right) - a_n < 1$. Thus $0 \leq 10^{n+1} \left(x - \displaystyle\sum_{k=1}^{n} \frac{a_k}{10^k} \right) < 10$.
   Therefore $a_{n+1} \in \{0, 1, \dots, 9\}$.
2. We have $\forall n \in \mathbb{N} \setminus \{0\}$, $\displaystyle\sum_{k=1}^{n} \frac{a_k}{10^k} \leq x < \sum_{k=1}^{n} \frac{a_k}{10^k} + \frac{1}{10^n}$ by construction.

■

Decimal representation of real numbers – 4

Theorem

A real number $x \in \mathbb{R}$ is rational if and only if its proper decimal expansion is eventually periodic, i.e. $\exists r \in \mathbb{N}, \exists s \in \mathbb{N} \setminus \{0\}, \forall k \in \mathbb{N}, a_{r+k+s} = a_{r+k}$, so that
$x = \lfloor x \rfloor . b_1 b_2 \dots b_r \underline{c_1 c_2 \dots c_s} := \lfloor x \rfloor . b_1 b_2 \dots b_r c_1 c_2 \dots c_s c_1 c_2 \dots c_s c_1 \dots$

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Proof.
$\Rightarrow$ Let $x = \frac{a}{b}$ where $a \in \mathbb{Z}$ and $b \in \mathbb{N} \setminus \{0\}$.
By Euclidean division, $a = bq_0 + r_0$ where $0 \le r_0 < b$. Hence $\frac{a}{b} = q_0 + \frac{r_0}{b}$. Note that $q_0 = \left\lfloor \frac{a}{b} \right\rfloor$.
And we repeat: $10 r_k = b q_{k+1} + r_{k+1}$ where $0 \le r_{k+1} < b$.
Since there are only b possible remainders, the process will start looping after at most b steps.
But note that the $(q_k)_{k \ge 1}$ defines exactly the decimal expansion of x.

$\Leftarrow$ Assume that the proper decimal expansion $x = \lfloor x \rfloor + \sum_{k=1}^{+\infty} \frac{a_k}{10^k}$ is eventually periodic, i.e. $\exists r \in \mathbb{N}, \exists s \in \mathbb{N} \setminus \{0\}, \forall k \in \mathbb{N}, a_{r+k+s} = a_{r+k}$.

Then $x = \lfloor x \rfloor + \sum_{k=1}^{r} \frac{a_k}{10^k} + 10^{-r} \sum_{k=1}^{+\infty} \frac{a_{r+k}}{10^k}$. Hence it is enough to prove that $y = \sum_{k=1}^{+\infty} \frac{a_{r+k}}{10^k} \in \mathbb{Q}$.

Note that $10^s y = N + y$ where $N = \overline{a_{r+1} a_{r+2} \dots a_{r+s}}^{10} \in \mathbb{N}$. Hence $y = \frac{N}{10^s - 1} \in \mathbb{Q}$. ∎

Decimal representation of real numbers – 5

Example

We want to find the decimal expansion of $\frac{1529327}{24975}$.

1. $1529327 = 24975 \times 61 + 5852$
2. $58520 = 24975 \times 2 + 8570$
3. $85700 = 24975 \times 3 + 10775$
4. $107750 = 24975 \times 4 + 7850$
5. $78500 = 24975 \times 3 + 3575$
6. $35750 = 24975 \times 1 + 10775$

And we start to loop. Therefore $\frac{1529327}{24975} = 61.234\underline{314}$

Decimal representation of real numbers – 6

According to the above proof,

$$\begin{aligned}
a_t a_{t-1} \dots a_0 . b_1 b_2 \dots b_r \underline{c_1 c_2 \dots c_s} &= \overline{a_t a_{t-1} \cdots a_0}^{10} + \sum_{k=1}^{r} \frac{b_k}{10^k} + 10^{-r} \frac{\overline{c_1 c_2 \dots c_s}^{10}}{10^s - 1} \\
&= \overline{a_t a_{t-1} \dots a_0}^{10} + \frac{\overline{b_1 b_2 \dots b_r}^{10}}{10^r} + \frac{\overline{c_1 c_2 \dots c_s}^{10}}{10^{r+s} - 10^r} \\
&= \frac{\overline{a_t a_{t-1} \dots a_0 b_1 b_2 \dots b_r c_1 c_2 \dots c_s}^{10} - \overline{a_t a_{t-1} \dots a_0 b_1 b_2 \dots b_r}^{10}}{10^{r+s} - 10^r}
\end{aligned}$$

Examples

- $61.234\underline{314} = \dfrac{61234314 - 61234}{10^6 - 10^3} = \dfrac{61173080}{999000}$
- $0.\underline{3} = \dfrac{3 - 0}{10 - 1} = \dfrac{3}{9}$
- $42.\underline{012} = \dfrac{42012 - 42}{10^3 - 1} = \dfrac{41970}{999}$

$\sqrt{2} \notin \mathbb{Q}$ – 1

Proof 1 (Fundamental Theorem of Arithmetic).
Assume by contradiction that $\sqrt{2} = \frac{a}{b} \in \mathbb{Q}$. Then $2b^2 = a^2$.
The prime factorization of the LHS has an odd number of primes (counted with exponents) whereas the RHS has an even number of primes (counted with exponents).
Which is impossible since the prime factorization is unique up to order. ∎

$\sqrt{2} \notin \mathbb{Q}$ – 2

Proof 2 (Euclid's lemma).
Assume by contradiction that $\sqrt{2} = \frac{a}{b} \in \mathbb{Q}$ written in lowest form.
Then $2b^2 = a^2$. Therefore $2|a^2$. By Euclid's lemma, $2|a$, so $a = 2k$.
Thus $2b^2 = 4k^2$, from which we get $b^2 = 2k^2$. By Euclid's lemma, $2|b$.
Hence $2|\gcd(a,b) = 1$, which is a contradiction. ∎

$\sqrt{2} \notin \mathbb{Q}$ – 3

Proof 3 (Gauss' lemma).

Assume by contradiction that $\sqrt{2} = \frac{a}{b} \in \mathbb{Q}$ written in lowest form.
Then $2b^2 = a^2$. Therefore $b|a^2$.
Since $\gcd(a, b) = 1$, by Gauss' lemma (applied twice), $b|1$ and hence $b = 1$ (since $b \in \mathbb{N} \setminus \{0\}$).
Hence $a^2 = 2$.
Which is impossible (2 is not a perfect square: $\forall x \in \mathbb{Z}$, $x^2 \equiv 0 \mod 3$ or $x^2 \equiv 1 \mod 3$). ∎

Proof 4 (proof by infinite descent).
Assume by contradiction that $\sqrt{2} = \frac{a}{b} \in \mathbb{Q}$ where $a \in \mathbb{N}$ and $b \in \mathbb{N} \setminus \{0\}$.
Then $2b^2 = a^2$. Then $a(a-b) = a^2 - ab = 2b^2 - ab = b(2b-a)$. Hence $\sqrt{2} = \frac{a}{b} = \frac{2b-a}{a-b}$.
Note that $1 < \sqrt{2} = \frac{a}{b}$, thus $0 < a - b$. Therefore $0 < 2b - a$, so $a - b < b$.
Therefore we obtained another expression of $\sqrt{2}$ with a smaller positive denominator.
By repeating this process, we may construct an infinite sequence $\sqrt{2} = \frac{a}{b} = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots$ such that $a_k > 0$ and $0 < b_{k+1} < b_k$.
Which is a contradiction since there is no decreasing infinite sequence of natural numbers. ■

$\sqrt{2} \notin \mathbb{Q} - 5$

Proof 5 (by congruences).
Assume by contradiction that $\sqrt{2} = \frac{a}{b} \in \mathbb{Q}$ written in lowest form. Then $2b^2 = a^2$.
We can't have $a \equiv 0 \mod 3$ and $b \equiv 0 \mod 3$ simulatenously (otherwise $3 | \gcd(a, b) = 1$).

- Either $a \equiv \pm 1 \mod 3$ and $b \equiv 0 \mod 3$, then $a^2 - 2b^2 \equiv 1 \mod 3$,
- or $a \equiv 0 \mod 3$ and $b \equiv \pm 1 \mod 3$, then $a^2 - 2b^2 \equiv 1 \mod 3$,
- or $a \equiv \pm 1 \mod 3$ and $b \equiv \pm 1 \mod 3$, then $a^2 - 2b^2 \equiv 2 \mod 3$.

Therefore $a^2 - 2b^2 \not\equiv 0 \mod 3$ and so $a^2 - 2b^2 \neq 0$. Which is a contradiction. ∎

Proof 6 (by the well-ordering principle).
Assume by contradiction that $\sqrt{2} = \frac{a}{b} \in \mathbb{Q}$. Then $a = \sqrt{2}b$.
Therefore $E = \left\{ n \in \mathbb{N} \ : \ n\sqrt{2} \in \mathbb{N} \setminus \{0\} \right\}$ is not empty since it contains $|b|$ as $\sqrt{2}|b| = |a|$.
By the well-ordering principle, E admits a least element p. Then $p\sqrt{2} \in \mathbb{N} \setminus \{0\}$.
Set $q = p\sqrt{2} - p$. Then $q \in \mathbb{Z}$. Besides $q = p(\sqrt{2} - 1)$ so that $0 < q < p$.
But $q\sqrt{2} = 2p - p\sqrt{2} = p - q \in \mathbb{N} \setminus \{0\}$. So $q \in E$.
Which is a contradiction since p is the least element of E and $q < p$. ∎

Proof 7 (by the rational root theorem).
Assume by contradiction that $\sqrt{2} = \frac{a}{b} \in \mathbb{Q}$ written in lowest form.
Since $\sqrt{2} = \frac{a}{b}$ is a root of $x^2 - 2 = 0$, we deduce from the rational root theorem that $a|2$ and $b|1$.
So either $\sqrt{2} = \pm 1$ or $\sqrt{2} = \pm 2$.
We obtain a contradiction in both cases since $(\pm 1)^2 = 1 \neq 2$ and $(\pm 2)^2 = 4 \neq 2$. ∎

Proof 8 (by the archimedean property).

For $n \in \mathbb{N}$, set $u_n = \left(\sqrt{2} - 1\right)^n$. We may prove either by induction or using the binomial formula, that for every n, there exist $a_n, b_n \in \mathbb{Z}$ such that $u_n = a_n + b_n\sqrt{2}$.

Since[1] $0 < \sqrt{2} - 1 < \frac{1}{2}$, we may also prove that $0 < u_n \leq \frac{1}{2^n}$.

Assume by contradiction that $\sqrt{2} = \frac{p}{q} \in \mathbb{Q}$, then

$$u_n = a_n + b_n\sqrt{2} = a_n + b_n\frac{p}{q} = \frac{qa_n + pb_n}{q}$$

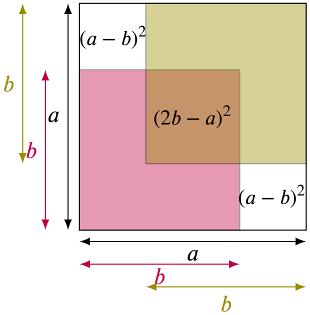
Since $u_n > 0$ we get that $|qa_n + pb_n| \geq 1$ and that $u_n \geq \frac{1}{|q|}$.

Therefore $\forall n \in \mathbb{N}$, $0 < \frac{1}{|q|} \leq u_n \leq \frac{1}{2^n}$. Which contradicts the archimedean property. ■

[1] Use the fact that $(0, +\infty) \ni x \to x^2 \in \mathbb{R}$ is increasing and that $2 \leq \left(\frac{3}{2}\right)^2$ to conclude that $\sqrt{2} \leq \frac{3}{2}$.

$\sqrt{2} \notin \mathbb{Q} - 9$

Proof 9 (geometric version of proof 4).
Assume by contradiction that $\sqrt{2} = \frac{a}{b} \in \mathbb{Q}$ where $a \in \mathbb{N}$ and $b \in \mathbb{N} \setminus \{0\}$. Then $a = \sqrt{2}b > b$.



By a direct computation of the side length, the square at the center has an area of $\mathcal{A} = (2b-a)^2$.
But, by inclusion-exclusion, $\mathcal{A}$ also satisfies $2(a-b)^2 + 2b^2 - \mathcal{A} = a^2$.
So $\mathcal{A} = 2(a-b)^2 + 2b^2 - a^2 = 2(a-b)^2$ since $a^2 = 2b^2$.
Therefore $2(a-b)^2 = \mathcal{A} = (2b-a)^2$. Thus $2 = \frac{(2b-a)^2}{(a-b)^2}$.
Hence $\sqrt{2} = \frac{2b-a}{a-b}$ with[a] $0 < 2b-a$ and $0 < a-b < b$.

[a] See Proof 4 for $0 < 2b - a$.

By repeating this process, we may construct an infinite sequence $\sqrt{2} = \frac{a}{b} = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots$ such that $a_k > 0$ and $0 < b_{k+1} < b_k$.
Which is a contradiction since there is no decreasing infinite sequence of natural numbers. ■

Proof 10 (Pythagoras flavored).

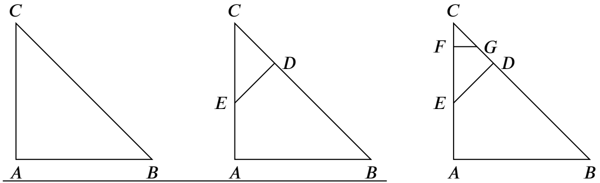
Let ABC be a isosceles right triangle in A. By the Pythagorean theorem $\frac{\overline{BC}}{\overline{AB}} = \sqrt{2}$.

Assuming that $\sqrt{2}$ is rational means geometrically that $\overline{BC}$ and $\overline{AB}$ are commensurable, i.e. they are both integral multiple of a another length d[2].

Put D on $[BC]$ such that $\overline{BD} = \overline{AB}$.
Define E as the intersection of (AC) with the line through D which is perpendicular to (BC).

Note that[3] $\overline{AE} = \overline{ED} = \overline{DC}$.
Thus $\overline{CD} = \overline{BC} - \overline{AB}$ and $\overline{EC} = \overline{AC} - \overline{AE} = \overline{AB} - (\overline{BC} - \overline{AB}) = 2\overline{AB} - \overline{BC}$.

Therefore $\overline{CD}$ and $\overline{EC}$ are integral multiple of d.
Besides DEC is a isosceles right triangle in D, therefore we may repeat this construction on the triangle DEC in order to construct an infinite sequence of segment lines ($AC, EC, FC, \ldots$, see below) which are all integral multiple of d and with decreasing length.
Which is impossible.



[2] That is the geometric version of *irrationality* used by ancient Greeks:
if $\sqrt{2} = \frac{a}{b}$, set $d = \frac{\overline{AB}}{b}$ then $\overline{AB} = bd$ and $\overline{BC} = \sqrt{2} \times \overline{AB} = \frac{a}{b} bd = ad$.

[3] Compare the triangles BAE and BDE which are respectively right in A and D with common hypotenuse and $\overline{AB} = \overline{DB}$, so, by the Pythagorean theorem, $\overline{AE} = \overline{ED}$.
Besides the triangle CDE is isosceles right in A by angle considerations, thus $\overline{ED} = \overline{DC}$.

$\sqrt{2} \notin \mathbb{Q}$ – 11

Proof 11 (my favorite one).
The proof is left as an exercise to the reader. $\blacksquare$

Before leaving $\sqrt{2}$, I would like to show you a funny proof relying on the *tertium non datur*.

Proposition

There exist $a, b > 0$ irrational numbers such that $a^b \in \mathbb{Q}$.

Proof.

- Assume that $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$. Then we can take $a = b = \sqrt{2}$.
- Assume that $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$. Then we can take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$.

  Indeed, $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2$.

$\blacksquare$

Recall from your first year calculus that $e = \sum_{n=0}^{+\infty} \frac{1}{n!}$.

Proof 1.
Assume by contradiction that $e = \frac{a}{b}$ where $a, b \in \mathbb{N} \setminus \{0\}$. Note that $b > 1$ since $e \notin \mathbb{N}$. Besides

$$b!\left(e - \sum_{n=0}^{b} \frac{1}{n!}\right) = b!\left(\sum_{n \geq b+1} \frac{1}{n!}\right)$$

Note that the LHS is an integer.
We are going to derive a contradiction by proving that the RHS is not an integer. Indeed

$$0 < b!\left(\sum_{n \geq b+1} \frac{1}{n!}\right) \leq \sum_{n \geq 1} \frac{1}{(b+1)^n} = \frac{1}{b} < 1$$

$\blacksquare$

Proof 2.

For $n \in \mathbb{N}$, set $u_n = \int_0^1 x^n e^x \mathrm{d}x$.

Using an induction and integration by part, we can prove that for $n \in \mathbb{N}$, there exist $a_n, b_n \in \mathbb{Z}$ such that $u_n = a_n + eb_n$.

Assume by contradiction that $e = \frac{p}{q}$ where $p, q \in \mathbb{N} \setminus \{0\}$. Then $0 < u_n = a_n + b_n \frac{p}{q} = \frac{qa_n + pb_n}{q}$.

Since $u_n > 0$ we get that $qa_n + pb_n \geq 1$ and that $u_n \geq \frac{1}{q}$.

Therefore $\forall n \in \mathbb{N},\ 0 < \frac{1}{q} \leq u_n \leq \int_0^1 x^n e \mathrm{d}x = \frac{e}{n+1}$.

Which is impossible. ∎