Real numbers – 1

University of Toronto

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Dedekind-completeness – 1

The following results concerning $\mathbb{R}$ from your first year calculus course are equivalent:

- The Least Upper Bound principle
- The Monotone Convergence Theorem for sequences
- The Extreme Value Theorem
- The Intermediate Value Theorem
- Rolle's Theorem/The Mean Value Theorem
- A continuous function on a segment line is Riemann-integrable
- *Bolzano-Weierstrass Property*: a bounded sequence in $\mathbb{R}$ admits a convergent subsequence
- Cut property:

$$\forall A, B \subset \mathbb{R}, \quad \left.\begin{array}{r} A, B \neq \varnothing \\ \mathbb{R} = A \cup B \\ \forall a \in A, \forall b \in B, a < b \end{array}\right\} \implies \exists! c \in \mathbb{R}, \forall a \in A, \forall b \in B, a \leq c \leq b$$

- ...

We say that $\mathbb{R}$ is Dedekind-complete to state that the above statements hold.

Dedekind-completeness – 2

Intuitively, the Dedekind-completeness of the real line tells us two things:

1. *Archimedean property:* there is no infinitely small positive real number (already true for $\mathbb{Q}$):

$$\forall \varepsilon > 0, \forall A > 0, \exists n \in \mathbb{N}, n\varepsilon > A$$

Dedekind-completeness – 2

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2. There is no gap in the real line. That's the difference with $\mathbb{Q}$.
   See for instance the following examples involving $\sqrt{2} \notin \mathbb{Q}$:
   - LUB: $\sqrt{2} = \sup \left\{ x \in \mathbb{Q} \ : \ x^2 < 2 \right\}$.
   - MCT: define a sequence by $x_0 = 1$ and $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$.
     Then (x_n) converges to some limit l by the MCT. But this limit must satisfy $l^2 = 2$.
   - IVT: let $f(x) = x^2 - 2$. Then $f(0) < 0$ and $f(2) > 0$.
     Hence we deduce from the IVT that f has a root, i.e. $\exists x \in \mathbb{R}, x^2 - 2 = 0$.

Dedekind-completeness – 3

The Dedekind-completeness of the real line has several consequences that you already know:

- The various results connecting the sign of f' to the monotonicity of f.
- $ACV \implies CV$ (for series and improper integrals).
- The Fundamental Theorem of Calculus.
- L'Hôpital's rule.
- The BCT and the LCT (for series and improper integrals).
- Cauchy-completeness of $\mathbb{R}$: any Cauchy sequence converges.
  Beware, despite very close names, without the Archimedean property Cauchy-completeness is strictly weaker than Dedekind-completeness.
- ...

Hence a first year calculus course is basically about the Dedekind-completeness of $\mathbb{R}$ and its consequences.

Infima and suprema – 1

Recall that a binary relation $\leq$ on a set E is an *order* if

1. $\forall x \in E,\ x \leq x$ *(reflexivity)*
2. $\forall x, y \in E,\ \left(x \leq y \text{ and } y \leq x\right) \implies x = y$ *(antisymmetry)*
3. $\forall x, y, z \in E,\ \left(x \leq y \text{ and } y \leq z\right) \implies x \leq z$ *(transitivity)*

Definitions: least/greatest element

Let $(E, \leq)$ be an ordered set and $A \subset E$.

- We say that $m \in A$ is the *least element of* A if $\forall a \in A,\ m \leq a$.
- We say that $M \in A$ is the *greatest element of* A if $\forall a \in A,\ a \leq M$.

Infima and suprema – 2

Remark

Note that, if it exists, the least element (resp. greatest element) of A is in A by definition.

Infima and suprema – 2

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The least (resp. greatest) element may not exist, but if it exists then it is unique.
For instance $\{n \in \mathbb{Z} \ : \ n \leq 0\} \subset \mathbb{Z}$ and $\{x \in \mathbb{Q} \ : \ 0 < x < 1\}$ have no least element.

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Proof of the uniqueness.
Assume that m, m' are two least elements of A, then

- $m \leq m'$ since m is *a* least element of A and $m' \in A$, and,
- $m' \leq m$ since m' is *a* least element of A and $m \in A$.

Hence $m = m'$. ■

Infima and suprema – 3

Definitions: upper/lower bounds

Let $(E, \leq)$ be an ordered set and $A \subset E$.

- We say that A is *bounded from below* if it admits a *lower bound*, i.e.

$$\exists c \in E, \forall a \in A, c \leq a$$

- We say that A is *bounded from above* if it admits an *upper bound*, i.e.

$$\exists C \in E, \forall a \in A, a \leq C$$

- We say that A is *bounded* if it is bounded from below and from above.

Infima and suprema – 4

Definitions: infimum/supremum

Let $(E, \leq)$ be an ordered set and $A \subset E$.

- If the greatest lower bound of A exists, we denote it $\inf(A)$ and call it the *infimum of* A.
- If the least upper bound of A exists, we denote it $\sup(A)$ and call it the *supremum of* A.

Infima and suprema – 4

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- If the least upper bound of A exists, we denote it $\sup(A)$ and call it the *supremum of* A.

Remarks

If it exists, the greatest element of the set of lower bounds of A is unique, therefore the infimum is unique (if it exists). And similarly for the supremum.
However, it may not exist:

- If $A = \{n \in \mathbb{Z} \ : \ n \leq 0\} \subset \mathbb{Z}$ then the set of lower bounds of A is empty, so A has no infimum.
- If $A = \left\{x \in \mathbb{Q} \ : \ x > 0 \text{ and } x^2 > 2\right\} \subset \mathbb{Q}$ then the set of lower bounds of A is not empty but has no greatest element, so A has no infimum.

Note that the infimum (resp. supremum) may not be an element of A, but if it is then it is the least (resp. greatest) element of A.
For instance, the infimum of $A = \{x \in \mathbb{Q} \ : \ 0 < x < 1\} \subset \mathbb{Q}$ is $0 \notin A$.

The set of real numbers – 1

Theorem

Up to a bijection preserving the addition, the multiplication and the order, there exists a unique (totally) ordered field $(\mathbb{R},+,\times,\leq)$ which is Dedekind-complete, i.e. such that:

- $+$ is associative: $\forall x,y,z\in\mathbb{R},\ (x+y)+z=x+(y+z)$
- 0 is the unit of $+$: $\forall x\in\mathbb{R},\ x+0=0+x=x$
- Existence of the additive inverse: $\forall x\in\mathbb{R},\ \exists(-x)\in\mathbb{R},\ x+(-x)=(-x)+x=0$
- $+$ is commutative: $\forall x,y\in\mathbb{R},\ x+y=y+x$
- $\times$ is associative: $\forall x,y,z\in\mathbb{R},\ (xy)z=x(yz)$
- $\times$ is distributive with respect to $+$: $\forall x,y,z\in\mathbb{R},\ x(y+z)=xy+xz$ and $(x+y)z=xz+yz$
- 1 is the unit of $\times$: $\forall x\in\mathbb{R},\ 1\times x=x\times 1=x$
- Existence of the multiplicative inverse: $\forall x\in\mathbb{R}\setminus\{0\},\ \exists x^{-1}\in\mathbb{R},\ xx^{-1}=x^{-1}x=1$
- $\times$ is commutative: $\forall x,y\in\mathbb{R},\ xy=yx$
- $\leq$ is reflexive: $\forall x\in\mathbb{R},\ x\leq x$
- $\leq$ is antisymmetric: $\forall x,y\in\mathbb{R},\ \left(x\leq y \text{ and } y\leq x\right)\implies x=y$
- $\leq$ is transitive: $\forall x,y,z\in\mathbb{R},\ \left(x\leq y \text{ and } y\leq z\right)\implies x\leq z$
- $\leq$ is total: $\forall x,y\in\mathbb{R},\ x\leq y$ or $y\leq x$
- $\forall x,y,r,s\in\mathbb{R},\ (x\leq y \text{ and } r\leq s)\Rightarrow x+r\leq y+s$
- $\forall x,y,z\in\mathbb{R},\ \left(x\leq y \text{ and } z>0\right)\Rightarrow xz\leq yz$
- $\mathbb{R}$ *is Dedekind-complete:* a non-empty subset A of $\mathbb{R}$ which is bounded from above admits a supremum.

The set of real numbers – 2

Remark

$\mathbb{Q} \subset \mathbb{R}$ and $+, \times, <$ for $\mathbb{R}$ are compatible with the ones for $\mathbb{Q}$.

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Sketch of proof:

1. $\mathbb{N} \subset \mathbb{R}$: if $n \in \mathbb{N}$ then $n = 1 + 1 + \cdots + 1 \in \mathbb{R}$. So $\mathbb{N} \subset \mathbb{R}$.
2. $\mathbb{Z} \subset \mathbb{R}$: if $n \in \mathbb{N}$ then $-n \in \mathbb{R}$. So $\mathbb{Z} \subset \mathbb{R}$.
3. $\mathbb{Q} \subset \mathbb{R}$: if $(a, b) \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ then $\frac{a}{b} := ab^{-1} \in \mathbb{R}$. So $\mathbb{Q} \subset \mathbb{R}$. ■

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About the proof of the previous theorem:

- **Existence:** there are several ways to construct $\mathbb{R}$ (for instance via Dedekind cuts, or via Cauchy sequences).
- **Uniqueness:** if we have two such fields $\mathbb{R}$ and $\tilde{\mathbb{R}}$, each of them contains a copy of $\mathbb{Q}$.
  Therefore there is a natural bijection $\mathbb{R} \supset \mathbb{Q} \to \mathbb{Q} \subset \tilde{\mathbb{R}}$.
  Using Dedekind-completeness, we may extend it to a bijection $\mathbb{R} \to \tilde{\mathbb{R}}$ preserving $+, \times, <$. ■

Proposition

- $\forall x, y, z \in \mathbb{R},\ x \leq y \Rightarrow x + z \leq y + z$
- $\forall x, y, z \in \mathbb{R},\ (x \leq y$ and $0 \leq z) \Rightarrow xz \leq yz$
- $\forall x, y, u, v \in \mathbb{R},\ (x \leq y$ and $u \leq v) \Rightarrow x + u \leq y + v$
- $\forall x \in \mathbb{R},\ 0 < x \Leftrightarrow 0 < \frac{1}{x}$
- $\forall x, y \in \mathbb{R},\ \forall z \in \mathbb{R}_+^*,\ x \leq y \Leftrightarrow xz \leq yz$
- $\forall x, y, u, v \in \mathbb{R},\ (0 \leq x \leq y$ and $0 \leq u \leq v) \Rightarrow xu \leq yv$
- $\forall x, y \in \mathbb{R},\ 0 < x < y \Leftrightarrow \frac{1}{y} < \frac{1}{x}$

Absolute value – 1

Definition: absolute value

We define the *absolue value* by $|\cdot|$: $\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & |x| \coloneqq \begin{cases} x & \text{si } x \geq 0 \\ -x & \text{si } x \leq 0 \end{cases} \end{array}$

Proposition

- $\forall x \in \mathbb{R},\ |x| = \max(x, -x)$
- $\forall x \in \mathbb{R},\ |x| \geq 0$
- $\forall x \in \mathbb{R},\ x = 0 \Leftrightarrow |x| = 0$
- $\forall x, y \in \mathbb{R},\ |x| = |y| \Leftrightarrow (x = y$ or $x = -y)$
- $\forall x, y \in \mathbb{R},\ |xy| = |x||y|$
- $\forall x \in \mathbb{R} \setminus \{0\},\ \left|\frac{1}{x}\right| = \frac{1}{|x|}$
- $\forall x, y \in \mathbb{R},\ |x + y| \leq |x| + |y|$ (triangle inequality)
- $\forall x, y \in \mathbb{R},\ ||x| - |y|| \leq |x - y|$ (reverse triangle inequality)

Absolute value – 2

Proposition

For $x, a \in \mathbb{R}$,

- $|x| \leq a \Leftrightarrow -a \leq x \leq a$
- $|x| < a \Leftrightarrow -a < x < a$
- $|x| \geq a \Leftrightarrow (x \geq a$ or $x \leq -a)$
- $|x| > a \Leftrightarrow (x > a$ or $x < -a)$
- If $a \geq 0$ then $|x| = a \Leftrightarrow (x = a$ or $x = -a)$