Positional numeral system

From the appendix of Chapter 4

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Introduction

In our everyday life, we usually use a base ten positional notation (decimal numeral system).

It allows us to write all natural numbers using only 10 digits although $\mathbb{N}$ is infinite.
The idea is that the position of a digit changes its value:

$$590743 = 5 \times 10^5 + 9 \times 10^4 + 0 \times 10^3 + 7 \times 10^2 + 4 \times 10^1 + 3 \times 10^0$$

But it is not the only base that we can use:

- Base 2 (binary) and base 16 (hexadecimal) are quite common nowadays in computer sciences.
- The first known positional numeral system is the Babylonian one using a base 60 (sexagesimal), circa 2000BC.
- In our everyday life, we can observe the influence of bases 60 (1 hour is 60 minutes) or 20 (in French 96 is litteraly pronounced $4 \times 20 + 16$).

Positional numeral system with base $b-1$

Theorem

Let $b \geq 2$ be an natural number. Then any natural number $n \in \mathbb{N}$ admits a unique expression

$$n = \sum_{k \geq 0} a_k b^k$$

where $a_k \in \{0, 1, \ldots, b-1\}$ and $a_k = 0$ for all but finitely many $k \geq 0$.

We write $\overline{a_r a_{r-1} \ldots a_1 a_0}^b$ for $\sum_{k=0}^{r} a_k b^k$.

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Question 2 of PS1 was about the existence of the binary numeral system (i.e. when $b = 2$).

Positional numeral system with base b – 2

Remark

In order to pass from a base 10 expression to a base b expression, we can perform successive Euclidean divisions by b.

Example: from base 10 to base 2

$$\begin{aligned}
42 &= 2 \times 21 + 0 \\
&= 2 \times (2 \times 10 + 1) + 0 \\
&= 2 \times (2 \times (2 \times 5 + 0) + 1) + 0 \\
&= 2 \times (2 \times (2 \times (2 \times 2 + 1) + 0) + 1) + 0 \\
&= 2 \times (2 \times (2 \times (2 \times (2 \times 1 + 0) + 1) + 0) + 1) + 0 \\
&= 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0
\end{aligned}$$

Hence $\overline{42}^{10} = \overline{101010}^{2}$.

Positional numeral system with base b – 3

Proof. ***Existence.*** Let $b \geq 2$. We are going to prove by strong induction that for $n \geq 0$, there exist $a_k \in \{0, 1, \dots, b-1\}$ such that $a_k = 0$ for all but finitely many $k \geq 0$ and $n = \sum_{k \geq 0} a_k b^k$.

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  Hence,
  $$n + 1 = bq + r = \sum_{k\geq 0} a_k b^{k+1} + r b^0$$
  which ends the induction step.

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WLOG, we may assume that $a_\ell < a'_\ell$. Then

$$0 = \sum_{k\geq 0} a_k b^k - \sum_{k\geq 0} a'_k b^k = \sum_{k\geq 0} (a_k - a'_k) b^k = \sum_{k=0}^{\ell} (a_k - a'_k) b^k$$

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So that

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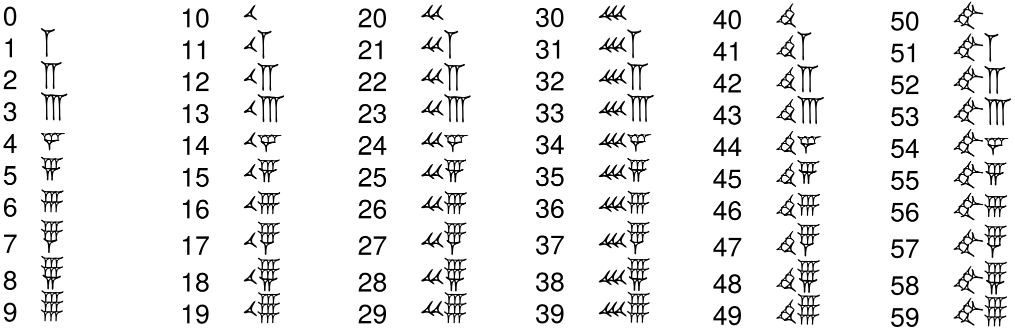
Therefore

$$(a'_\ell - a_\ell) b^\ell \leq \sum_{k=0}^{\ell-1} |a_k - a'_k| b^k \leq \sum_{k=0}^{\ell-1} (b-1) b^k = b^\ell - 1 < b^\ell \leq (a'_\ell - a_\ell) b^\ell$$

Hence a contradiction. ∎

Babylonian cuneiform numerals (circa 2000BC)

The first known positional numeral system is the Babylonian one (circa 2000BC) whose base is 60 and whose digits are:



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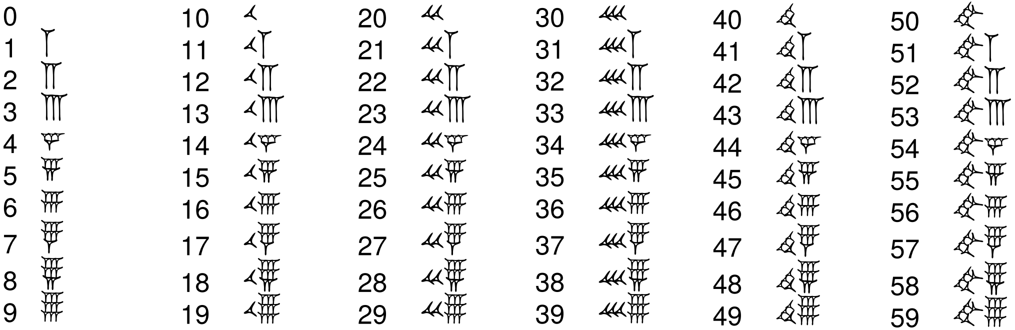

We want to write 13655 using Babylonian cuneiform numerals.
We perform successive Euclidean divisions by 60 as follows:
$13655 = 60 \times 227 + 35 = 60 \times (60 \times 3 + 47) + 35 = 3 \times 60^2 + 47 \times 60^1 + 35 \times 60^0$.

Hence it was written: 

YBC 7289, clay tablet, between 1800BC and 1600BC.

Original picture from `https://commons.wikimedia.org/wiki/File:YBC-7289-OBV-REV.jpg`

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It shows (extremely accurate) approximations of

$$\sqrt{2} \simeq 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3}$$

and of

$$30\sqrt{2} \simeq 42 + \frac{25}{60} + \frac{35}{60^2}$$

(diagonal of the square of side length 30, see the  above de square)

Original picture from https://commons.wikimedia.org/wiki/File:YBC-7289-OBV-REV.jpg