Proofs by induction

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Proof by induction: the formal statement

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Theorem

Let $\mathcal{P}(n)$ be a statement depending on $n \in \mathbb{N}$.
If $\mathcal{P}(0)$ is true and if $\mathcal{P}(n) \implies \mathcal{P}(n+1)$ is true for all $n \in \mathbb{N}$, then $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$. Formally,

$$\begin{cases} \mathcal{P}(0) \\ \forall n \in \mathbb{N},\ \big(\mathcal{P}(n) \implies \mathcal{P}(n+1)\big) \end{cases} \implies \forall n \in \mathbb{N},\ \mathcal{P}(n)$$

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Proof. We define the set $A = \{n \in \mathbb{N} \ : \ \mathcal{P}(n) \text{ is true}\}$. Then:

- $A \subset \mathbb{N}$ by definition of A.
- $0 \in \mathbb{N}$ since $\mathcal{P}(0)$ is true.
- $s(A) \subset A$
  Indeed, let $n \in s(A)$. Then $n = s(m) = m + 1$ for some $m \in A$.
  By definition of A, $\mathcal{P}(m)$ is true. But by assumption $\mathcal{P}(m) \implies \mathcal{P}(m+1)$ is also true.
  Hence $\mathcal{P}(m+1)$ is true, meaning that $n = m + 1 \in A$.

By the *induction principle* we get $A = \mathbb{N}$. Hence, for every $n \in \mathbb{N}$, we have that $\mathcal{P}(n)$ is true. ∎

Proof by induction: in practice

When writing a proof by induction, there are several steps that you should make appear clearly.

- What statement are you proving? What is your $\mathcal{P}(n)$?
  Particularly, on which parameter are you doing the induction?
  You should make everything clear for the reader!
- **Base case:** prove that $\mathcal{P}(0)$ is true.
- **Induction step:** prove that if $\mathcal{P}(n)$ is true for some $n \in \mathbb{N}$ then $\mathcal{P}(n+1)$ is also true.
  It is important to clearly write the induction hypothesis and what you want to prove in this step (the reader shouldn't have to guess).
  Make sure that you used the induction hypothesis somewhere, otherwise there is something suspicious with your proof.

Proof by induction: an example

Proposition

$$\forall n \in \mathbb{N},\ 0+1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

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$\forall n \in \mathbb{N},\ 0+1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$

Proof. We are proving the above statement by induction on n.

- **Base case at $n=0$.** Then the sum in the left hand side is equal to 0. And $\frac{n(n+1)}{2} = \frac{0\cdot 1}{2} = 0$. So the equality holds.
- **Induction step.** Assume that $0+1+2+3+\cdots+n = \frac{n(n+1)}{2}$ for some $n \in \mathbb{N}$ then

$$\begin{aligned} 0+1+2+3+\cdots+n+(n+1) &= \frac{n(n+1)}{2} + (n+1) \quad \text{by the induction hypothesis} \\ &= \frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2} \end{aligned}$$

Which proves the induction step. ∎

Proof by induction: base case at $n_0 - 1$

Theorem

Let $n_0 \in \mathbb{N}$. Let $\mathcal{P}(n)$ be a statement depending on a natural number $n \geq n_0$.
If $\mathcal{P}(n_0)$ is true and if $\mathcal{P}(n) \implies \mathcal{P}(n+1)$ is true for every natural number $n \geq n_0$, then $\mathcal{P}(n)$ is true for every natural number $n \geq n_0$. Formally,

$$\begin{cases} \mathcal{P}(n_0) \\ \forall n \in \mathbb{N}_{\geq n_0},\ \big(\mathcal{P}(n) \implies \mathcal{P}(n+1)\big) \end{cases} \implies \forall n \in \mathbb{N}_{\geq n_0},\ \mathcal{P}(n)$$

Proof by induction: base case at $n_0 - 1$

Theorem

Let $n_0 \in \mathbb{N}$. Let $\mathcal{P}(n)$ be a statement depending on a natural number $n \geq n_0$.
If $\mathcal{P}(n_0)$ is true and if $\mathcal{P}(n) \implies \mathcal{P}(n+1)$ is true for every natural number $n \geq n_0$, then $\mathcal{P}(n)$ is true for every natural number $n \geq n_0$. Formally,

$$\left\{ \begin{array}{l} \mathcal{P}(n_0) \\ \forall n \in \mathbb{N}_{\geq n_0},\ \big(\mathcal{P}(n) \implies \mathcal{P}(n+1)\big) \end{array} \right. \implies \forall n \in \mathbb{N}_{\geq n_0},\ \mathcal{P}(n)$$

Proof. For $n \in \mathbb{N}$, we define $\mathcal{R}(n)$ by

$$\mathcal{R}(n) \text{ is true} \Leftrightarrow \mathcal{P}(n+n_0) \text{ is true}$$

Then $\mathcal{R}(0)$ is true since $\mathcal{P}(n_0)$ is. And, for all $n \in \mathbb{N}$, $\mathcal{R}(n) \implies \mathcal{R}(n+1)$ is true.
By the usual induction $\mathcal{R}(n)$ is true for any $n \in \mathbb{N}$, i.e. $\mathcal{P}(n)$ is true for any $n \in \mathbb{N}_{\geq n_0}$. ∎

Proof by induction: base case at $n_0 - 2$

Proposition

For any integer $n \geq 5$, $2^n > n^2$.

Proof by induction: base case at n_0 – 2

Proposition

For any integer $n \geq 5$, $2^n > n^2$.

Proof. We are going to prove that $\forall n \geq 5,\ 2^n > n^2$ by induction on n.

- **Base case at $n = 5$:** $2^5 = 32 > 25 = 5^2$.
- **Induction step:** Assume that $2^n > n^2$ for some $n \geq 5$ and let's prove that $2^{n+1} > (n+1)^2$.
  Note that $2^{n+1} = 2 \times 2^n \geq 2n^2$ by the induction hypothesis. Hence it is enough to prove that $2n^2 > (n+1)^2$ which is equivalent to $n^2 - 2n - 1 > 0$.
  We study the sign of the polynomial $x^2 - 2x - 1$. It is a polynomial of degree 2 with positive leading coefficient and its discriminant is $(-2)^2 - 4 \times (-1) = 8 > 0$. Therefore

x	$-\infty$		$1-\sqrt{2}$		$1+\sqrt{2}$		$+\infty$
x^2-2x-1		$+$	0	$-$	0	$+$	

  Since $5 > 1 + \sqrt{2}$, we know that $n^2 - 2n - 1 > 0$ for $n \geq 5$. Which proves the induction step. ∎

Proof by induction: strong induction

Theorem

Let $\mathcal{P}(n)$ be a statement depending on $n \geq n_0$.
If $\mathcal{P}(n_0)$ is true and if $\big(P(n_0), P(n_0+1), \dots, \mathcal{P}(n)\big) \implies \mathcal{P}(n+1)$ is true for all $n \geq n_0$, then $\mathcal{P}(n)$ is true for all $n \geq n_0$. Formally,

$$\begin{cases} \mathcal{P}(n_0) \\ \forall n \geq n_0,\ \Big(\big(\mathcal{P}(n_0), \mathcal{P}(n_0+1), \dots, \mathcal{P}(n)\big) \implies \mathcal{P}(n+1)\Big) \end{cases} \implies \forall n \geq n_0,\ \mathcal{P}(n)$$

Proof by induction: strong induction

Theorem

Let $\mathcal{P}(n)$ be a statement depending on $n \geq n_0$.
If $\mathcal{P}(n_0)$ is true and if $\left(P(n_0), P(n_0+1), \dots, \mathcal{P}(n)\right) \implies \mathcal{P}(n+1)$ is true for all $n \geq n_0$, then $\mathcal{P}(n)$ is true for all $n \geq n_0$. Formally,

$$\left\{\begin{array}{l} \mathcal{P}(n_0) \\ \forall n \geq n_0,\ \left(\left(\mathcal{P}(n_0), \mathcal{P}(n_0+1), \dots, \mathcal{P}(n)\right) \implies \mathcal{P}(n+1)\right) \end{array}\right. \implies \forall n \geq n_0,\ \mathcal{P}(n)$$

Proof. For $n \geq n_0$, we define $\mathcal{R}(n)$ by

$$\mathcal{R}(n) \text{ is true} \Leftrightarrow \mathcal{P}(n_0), \mathcal{P}(n_0+1), \dots, \mathcal{P}(n) \text{ are true}$$

Assume that $\mathcal{P}(n_0)$ is true and that $\left(P(n_0), P(n_0+1), \dots, \mathcal{P}(n)\right) \implies \mathcal{P}(n+1)$ is true for all $n \geq n_0$.
Then $\mathcal{R}(0)$ is true since $\mathcal{P}(0)$ is. And, for all $n \geq n_0$, $\mathcal{R}(n) \implies \mathcal{R}(n+1)$ is true.
By the usual induction $\mathcal{R}(n)$ is true for any $n \geq n_0$.
Particularly, $\mathcal{P}(n)$ is true for any $n \geq n_0$ as expected. ∎