Concepts in Abstract Mathematics

Natural numbers – 2

January 14th, 2021

From last Tuesday: Peano's axioms

Theorem: Peano axioms

There exists a set $\mathbb{N}$ together with an element $0 \in \mathbb{N}$ "*zero*" and a function $s : \mathbb{N} \to \mathbb{N}$ "*successor*" such that:

1. 0 is not the successor of any element of $\mathbb{N}$, i.e. 0 is not in the image of s:

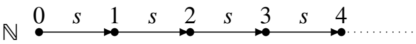
$$0 \notin s(\mathbb{N})$$

2. If the successor of n equals the successor of m then $n = m$, i.e. s is injective:

$$\forall n, m \in \mathbb{N},\ s(n) = s(m) \implies n = m$$

3. *The induction principle.* If a subset of $\mathbb{N}$ contains 0 and is closed under s then it is $\mathbb{N}$:

$$\forall A \subset \mathbb{N}, \begin{cases} 0 \in A \\ s(A) \subset A \end{cases} \implies A = \mathbb{N}$$



The addition – 1

We are going to define $\begin{array}{ccc} \mathbb{N} & \to & \mathbb{N} \\ b & \mapsto & a+b \end{array}$ for a given $a \in \mathbb{N}$.

How to do so? What would be a good definition to obtain what you intuitively know about $+$?

The idea is to define it inductively using the following properties we would like to have:

- $a + 0 = a$
- For $b \in \mathbb{N}$, $a + (b+1) = (a+b) + 1$
  So if $a + b$ is already defined, then we can define $a + (b+1)$.
  Remember that intuitively $+1$ is "taking the successor".

Formally, we prove:

Proposition

Let $a \in \mathbb{N}$. Then there exists a unique function $(a + \bullet) : \begin{array}{ccc} \mathbb{N} & \to & \mathbb{N} \\ b & \mapsto & a+b \end{array}$ such that

1. $a + 0 = a$
2. $\forall b \in \mathbb{N},\ a + s(b) = s(a+b)$

The above result is a consequence of the induction principle.

The addition – 2

Remark

Set $1 := s(0)$. Then, as expected, for $n \in \mathbb{N}$, we have

$$n + 1 = n + s(0) = s(n + 0) = s(n)$$

Hence, from now on, I will use indistinctively $n + 1$ or $s(n)$.

We can prove the following properties.

Proposition

- $\forall a, b, c \in \mathbb{N},\ a + (b + c) = (a + b) + c$ *(the addition is associative)*
- $\forall a, b \in \mathbb{N},\ a + b = b + a$ *(the addition is commutative)*
- $\forall a, b, c \in \mathbb{N},\ a + b = a + c \implies b = c$ *(cancellation)*
- $\forall a, b \in \mathbb{N},\ a + b = 0 \implies a = b = 0$

Proof that $\forall a, b, c \in \mathbb{N},\ a + (b + c) = (a + b) + c.$
Let $a, b \in \mathbb{N}$. Set $A = \{c \in \mathbb{N} \ :\ a + (b + c) = (a + b) + c\}$. Then

- $A \subset \mathbb{N}$
- $0 \in A$. Indeed, $a + (b + 0) = a + b = (a + b) + 0$.
- $s(A) \subset A$. Indeed, let $n \in s(A)$ then $n = s(c)$ for some $c \in A$. Therefore $a+(b+n) = a+(b+s(c)) = a+s(b+c) = s(a+(b+c)) = s((a+b)+c) = (a+b)+s(c) = (a+b)+n$. Hence $n \in A$.

Thus, by the induction principle, $A = \mathbb{N}$ and for any $c \in \mathbb{N}$, $a + (b + c) = (a + b) + c$. ■

Proof that $\forall a, b \in \mathbb{N},\ a + b = 0 \implies a = b = 0.$
Let $a, b \in \mathbb{N}$ be such that $a + b = 0$. Assume by contradiction that $a \neq 0$ or $b \neq 0$.
Without lost of generality, we may assume that $b \neq 0$ (using commutativity).
Then $b = s(n)$ for some $n \in \mathbb{N}$.
So $0 = a + b = a + s(n) = s(a + n)$.
Which is a contradiction since $0 \notin s(\mathbb{N})$. ■

The multiplication – 1

We define inductively $\begin{array}{ccc} \mathbb{N} & \to & \mathbb{N} \\ b & \mapsto & a \times b \end{array}$ for a given $a \in \mathbb{N}$ using the following desired properties:

- $a \times 0 = 0$
- For $b \in \mathbb{N}$, $a \times (b+1) = (a \times b) + a$
  So if $a \times b$ is already defined, then we can define $a \times (b+1)$.

Formally, we prove:

Proposition

Let $a \in \mathbb{N}$. Then there exists a unique function $(a \times \bullet) : \begin{array}{ccc} \mathbb{N} & \to & \mathbb{N} \\ b & \mapsto & a \times b \end{array}$ such that

1. $a \times 0 = 0$
2. $\forall b \in \mathbb{N},\ a \times s(b) = (a \times b) + a$

Remark

It is common to simply write ab for $a \times b$ when there is no possible confusion.

The multiplication – 2

Proposition

- $\forall a, b, c \in \mathbb{N},\ a \times (b \times c) = (a \times b) \times c$ *(the multiplication is associative)*
- $\forall a, b \in \mathbb{N},\ a \times b = b \times a$ *(the multiplication is commutative)*
- $\forall a, b, c \in \mathbb{N},\ a \times (b + c) = a \times b + a \times c$ and $(a + b) \times c = a \times c + b \times c$ *($\times$ is distributive over $+$)*
- $\forall a \in \mathbb{N},\ a \times 1 = a$
- $\forall a, b \in \mathbb{N},\ a \times b = 0 \implies \left(a = 0 \text{ or } b = 0\right)$
- $\forall a, b, c \in \mathbb{N},\ \begin{cases} a \times b = a \times c \\ a \neq 0 \end{cases} \implies b = c$ *(cancellation)*

Order – 1

Definition: binary relation

A **binary relation** $\mathcal{R}$ on a set E consists in associating a truth value to every couple $(x, y) \in E^2$.
We say that x *is related to* y *by* $\mathcal{R}$, denoted $x\mathcal{R}y$, if the value *true* is assigned to (x, y).

Examples

1. Let $E = \{a, b, c\}$. We can define a binary relation $\mathcal{R}$ using a truth table as below:

y \ x	a	b	c
a	✓	✗	✗
b	✗	✗	✓
c	✓	✓	✗

   Here $a\mathcal{R}a$, $a\mathcal{R}c$, $b\mathcal{R}c$ and $c\mathcal{R}b$.

2. For $E = \mathbb{R}$, we can define a binary relation as follows: $\forall (x, y) \in \mathbb{R}^2,\ x\mathcal{R}y \Leftrightarrow x^2 - y^2 = x - y$.

Order – 2

Definition: order

We say that a binary relation $\mathcal{R}$ on a set E is an *order* if

1. $\forall x \in E,\ x\mathcal{R}x$ *(reflexivity)*
2. $\forall x, y \in E,\ \left(x\mathcal{R}y \text{ and } y\mathcal{R}x\right) \implies x = y$ *(antisymmetry)*
3. $\forall x, y, z \in E,\ \left(x\mathcal{R}y \text{ and } y\mathcal{R}z\right) \implies x\mathcal{R}z$ *(transitivity)*

We say that the order $\mathcal{R}$ is *total* if additionaly $\forall x, y \in E,\ x\mathcal{R}y$ or $y\mathcal{R}x$.

Definition: the usual order $\leq$ on $\mathbb{N}$

We define the binary relation $\leq$ on $\mathbb{N}$ by $\forall a, b \in \mathbb{N},\ \left(a \leq b \Leftrightarrow \exists k \in \mathbb{N},\ b = a + k\right)$.

The intuition behind this definition is that $a \leq b$ if we need to add some k to a in order to reach b.

Notation

We write $a < b$ for $\left(a \leq b \text{ and } a \neq b\right)$.

Order – 3

Theorem

The set of natural numbers $\mathbb{N}$ is totally ordered for $\leq$.

Proof.

1. *Reflexivity.* Let $a \in \mathbb{N}$, then $a = a + 0$ with $0 \in \mathbb{N}$, hence $a \leq a$.
2. *Antisymmetry.* Let $a, b \in \mathbb{N}$ be such that $a \leq b$ and $b \leq a$.
   Then there exists $k, l \in \mathbb{N}$ such that $b = a + k$ and $a = b + l$.
   Therefore $a = b + l = a + k + l$. Hence $0 = k + l$ and thus $l = k = 0$ so that $a = b$.
3. *Transitivity.* Let $a, b, c \in \mathbb{N}$ be such that $a \leq b$ and $b \leq c$.
   Then there exists $k, l \in \mathbb{N}$ such that $b = a + k$ and $c = b + l$.
   Therefore $c = b + l = a + (k + l)$ with $k + l \in \mathbb{N}$, i.e. $a \leq c$.
4. *$\leq$ is total.* Let $a \in \mathbb{N}$. Set $A = \{b \in \mathbb{N} \ : \ a \leq b \text{ or } b \leq a\}$. Then
   - $A \subset \mathbb{N}$
   - $0 \in A$, indeed $a = 0 + a$ so that $0 \leq a$.
   - $s(A) \subset A$

   Indeed, let $n \in s(A)$. Then $n = s(b)$ for some $b \in A$, i.e. $a \leq b$ or $b \leq a$.
   - If $a \leq b$ then $b = a + k$ for some $k \in \mathbb{N}$, $n = s(b) = b + 1 = a + k + 1$ with $k + 1 \in \mathbb{N}$, so that $a \leq n$.
   - If $b \leq a$ then $a = b + l$ for some $l \in \mathbb{N}$. We may assume that $l \neq 0$ (since $a = b$ was in the above case).
     Then $l = \tilde{l} + 1$ for some $\tilde{l} \in \mathbb{N}$. Hence $a = b + l = b + \tilde{l} + 1 = b + 1 + \tilde{l} = n + \tilde{l}$, i.e. $n \leq a$.

   In both cases $n \in A$.
   Therefore, by the induction principle, $A = \mathbb{N}$. So, for all $b \in \mathbb{N}$, either $a \leq b$ or $b \leq a$. ∎

Order – 4

Proposition

1. $\forall a \in \mathbb{N},\ a \leq 0 \implies a = 0$
2. $\forall a, b, c \in \mathbb{N},\ a + b \leq a + c \implies b \leq c$
3. There is no $a \in \mathbb{N}$ such that $0 < a < 1$.
4. There is no $a \in \mathbb{N}$ such that $\forall b \in \mathbb{N},\ b \leq a$.
5. $\forall a, b, c \in \mathbb{N},\ a \leq b \implies ac \leq bc$

Proof.

1. Let $a \in \mathbb{N}$ be such that $a \leq 0$. Then there exists $k \in \mathbb{N}$ such that $0 = a + k$. Hence $a = k = 0$.
2. Let $a, b, c \in \mathbb{N}$. Assume that $a + b \leq a + c$. Then there exists $k \in \mathbb{N}$ such that $a + c = a + b + k$.
   Thus $c = b + k$ so that $b \leq c$ as expected.
3. Let $a \in \mathbb{N}$. Assume that $a < 1$, then there exists $l \in \mathbb{N} \setminus \{0\}$ such that $1 = a + l$.
   Since $l \neq 0$, $l = k + 1$ for some $k \in \mathbb{N}$, and $1 = a + k + 1$ so that $0 = a + k$. Therefore $a = 0$.
4. Assume by contradiction that there exists $a \in \mathbb{N}$ such that $\forall b \in \mathbb{N},\ b \leq a$. Then $a + 1 \leq a$ hence $1 \leq 0$,
   i.e. $0 = 1 + k$ for some $k \in \mathbb{N}$. Therefore $1 = 0$ which is a contradiction (otherwise $0 = s(0)$).
5. Let $a, b, c \in \mathbb{N}$. Assume that $a \leq b$. Then $b = a + k$ for some $k \in \mathbb{N}$.
   Thus $bc = (a + k)c = ac + kc$ with $kc \in \mathbb{N}$. Therefore $ac \leq bc$. ∎

The well-ordering principle

Theorem: the well-ordering principle

A nonempty subset A of $\mathbb{N}$ has a least element, i.e. there exists $n \in A$ such that $\forall a \in A,\ n \leq a$.

Proof.
Let's prove the contrapositive, i.e. if a subset $A \subset \mathbb{N}$ doesn't have a least element then it is empty.
Let $B = \{a \in \mathbb{N}\ :\ \forall i \leq a,\ i \notin A\}$.

- $B \subset \mathbb{N}$
- $0 \in B$ (otherwise 0 would be the least element of A).
- $s(B) \subset B$
  Indeed, if $n \in s(B)$, then $n = s(a)$ for $a \in B$, i.e. $\forall i \leq a,\ i \notin A$.
  Note that $n = a + 1 \notin A$ otherwise it would be the least element of A.
  Therefore $\forall i \leq n,\ i \notin A$, i.e. $n \in B$.

Thus, by the induction principle, $B = \mathbb{N}$ so A is empty. ∎