Natural numbers – 1

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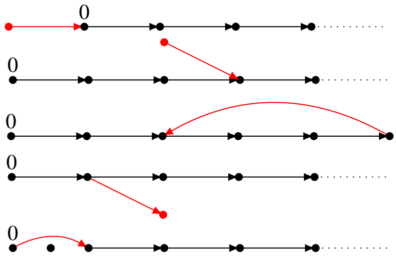
What is $\mathbb{N}$?

- There is an initial element: 0 *"zero"*.
- Every natural number n has a *successor* $s(n)$, intuitively $n+1$.

But, that's not enough:



0 is not the successor of another natural number

A natural number can't be the successor of two distinct natural numbers

The successor can't *loop back* to a previous natural number

The successor of a natural number is uniquely defined

Every natural number is obtained from 0 applying iteratively the successor

So we need extra assumptions on 0 and s.

Peano's axioms – 1

We admit the following theorem (we have to start from somewhere…).

Theorem: Peano axioms

There exists a set $\mathbb{N}$ together with an element $0 \in \mathbb{N}$ "*zero*" and a function $s : \mathbb{N} \to \mathbb{N}$ "*successor*" such that:

1. 0 is not the successor of any element of $\mathbb{N}$, i.e. 0 is not in the image of s:
$$0 \notin s(\mathbb{N})$$
2. If the successor of n equals the successor of m then $n = m$, i.e. s is injective:
$$\forall n, m \in \mathbb{N},\ s(n) = s(m) \implies n = m$$
3. *The induction principle.* If a subset of $\mathbb{N}$ contains 0 and is closed under s then it is $\mathbb{N}$:
$$\forall A \subset \mathbb{N}, \begin{cases} 0 \in A \\ s(A) \subset A \end{cases} \implies A = \mathbb{N}$$

The last point means we obtain $\mathbb{N}$ taking iteratively the successor starting at 0 (we don't miss any element).

- That's why we can define a sequence $(u_n)_{n\in\mathbb{N}}$ iteratively by fixing u_0 and setting $u_{n+1} = f(u_n)$.
- It is closely related to the notion of proof by induction.

Remarks

- We can't use $+$ and $\times$ yet, because they are still not defined.
  Currently, we can only use the three Peano's axioms from above.
- Intuitively, $s(n) = n + 1$ (which will become true after we define the addition).
- All the results about $\mathbb{N}$ will derive from these three basic properties (and even more since we will construct the other sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ from $\mathbb{N}$).

A first result

Proposition

A natural number is never its own successor, i.e.

$$\forall n \in \mathbb{N},\ n \neq s(n)$$

Proof.
Set $A = \{n \in \mathbb{N} \ :\ n \neq s(n)\}$. Then

- $A \subset \mathbb{N}$
- $0 \in A$ since $0 \neq s(0)$ (remember that $0 \notin s(\mathbb{N})$).
- $s(A) \subset A$
  Indeed, let $m \in s(A)$. Then $m = s(n)$ for some $n \in A$. So $s(n) \neq n$.
  Since s is injective, we get that $s(s(n)) \neq s(n)$, i.e. $s(m) \neq m$.
  Hence $m \in A$.

So, by the induction principle, $A = \mathbb{N}$.
Thus, for every $n \in \mathbb{N}$ we have that $n \neq s(n)$. ∎