Question: study the locus $|z-4| = 4|z|$

We recognize the equation of a circle (Apollonius, $\rho = 4 \neq 1$)

Let's find the center/radius:

$$|z-4| = 4|z|$$

$$\Leftrightarrow |z-4|^2 = 16|z|^2$$

$$\Leftrightarrow \overline{(z-4)}\,(z-4) = 16\bar{z}z$$

$$\Leftrightarrow (\bar{z}-4)(z-4) = 16\bar{z}z$$

$$\Leftrightarrow \bar{z}z - 4z - 4\bar{z} + 16 = 16\bar{z}z$$

$$\Leftrightarrow 15\bar{z}z + 4z + 4\bar{z} = 16$$

$$\Leftrightarrow \bar{z}z + \frac{4}{15}z + \frac{4}{15}\bar{z} = \frac{16}{15} \qquad \text{so } \omega = -\frac{4}{15}$$

$$\Leftrightarrow \bar{z}z + \frac{4}{15}z + \frac{4}{15}\bar{z} + \underbrace{\left(\frac{4}{15}\right)^2}_{\bar{\omega}\omega} = \frac{16}{15} + \left(-\frac{4}{15}\right)^2$$

$$\Leftrightarrow \left|z + \frac{4}{15}\right|^2 = \frac{256}{225}$$

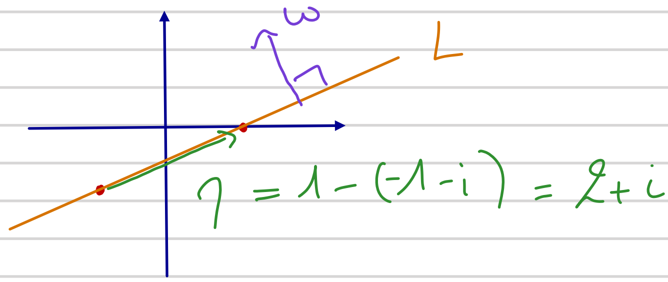
$$\left|z + \frac{4}{15}\right| = \left|z - \left(-\frac{4}{15}\right)\right|$$

$\rightsquigarrow$ circle centered at $-\frac{4}{15} + i\,0$

of radius $\sqrt{\frac{256}{225}} = \frac{16}{15}$

Question: give a complex equation of the line passing through 1 and $-1-i$

Method 1:



$\eta = 1-(-1-i) = 2+i$ is a vector director of the line L

hence $\omega = 1-2i$ is orthogonal to L $(\eta = a+ib \rightsquigarrow \omega = -b+ia$ or $\omega = b-ia)$

therefore L admits an equation of the form $\mathrm{Re}(\overline{\omega} z) = k$, for some $k \in \mathbb{R}$

Since $z=1 \in L$ then $\mathrm{Re}(\overline{\omega}\cdot 1) = k$

$$\mathrm{Re}(\overline{\omega}\cdot 1) = \mathrm{Re}(1+2i) = 1$$

hence $k = 1$

Conclusion:

$$\mathrm{Re}((1+2i)z) = 1$$

Method 2: $z_1 = 1$, $z_2 = -1-i$

$z \in L \Leftrightarrow \mathrm{Angle}(\overrightarrow{z_1z}, \overrightarrow{z_1z_2}) = 0$ or π mod 2π

$\Leftrightarrow \mathrm{Arg}\left(\frac{z-z_1}{z_2-z_1}\right) = 0$ or π mod 2π

$\Leftrightarrow \frac{z-z_1}{z_2-z_1} \in \mathbb{R}$

$\Leftrightarrow i\,\frac{z-z_1}{z_2-z_1} \in i\mathbb{R}$

$\Leftrightarrow \mathrm{Re}\left(i\,\frac{z-z_1}{z_2-z_1}\right) = 0$

$\Leftrightarrow \mathrm{Re}\left(i\,\frac{z}{z_2-z_1}\right) = \mathrm{Re}\left(i\,\frac{z_1}{z_2-z_1}\right)$

$\Leftrightarrow \mathrm{Re}\left(i\,\frac{z}{-2-i}\right) = \mathrm{Re}\left(i\,\frac{1}{-2-i}\right) = \mathrm{Re}\left(-\frac{1}{5} - \frac{2}{5}i\right) = -\frac{1}{5}$

$$\mathrm{Re}\left(i\,\frac{z}{-2-i}\right) = \mathrm{Re}\left(-\left(\frac{1}{5} + \frac{2}{5}i\right)z\right)$$

$\Leftrightarrow \mathrm{Re}((1+2i)z) = 1$