# 8 - Power series

**Definition**: A (complex) power series centered at $z_0 \in \mathbb{C}$ is a series of the form $S(z) = \sum_{n \geq 0} a_n (z - z_0)^n$ where $a_n \in \mathbb{C}$

A first question is : for which $z \in \mathbb{C}$ is $S(z)$ convergent ?

**Theorem**: Given a power series $S(z) = \sum a_n (z - z_0)^n$, there exists a unique $R \in [0, +\infty]$ ($R$ may be equal to $+\infty$) such that

$$\forall z \in \mathbb{C}, \begin{cases} |z - z_0| < R \Rightarrow \sum_{n \geq 0} a_n (z - z_0)^n \text{ absolutely convergent} \\ |z - z_0| > R \Rightarrow \sum_{n \geq 0} a_n (z - z_0)^n \text{ divergent} \end{cases}$$

We say that $R$ is the radius of convergence of the power series $\sum_{n \geq 0} a_n (z - z_0)^n$

$\Delta$ WLOG, we may assume that $z_0 = 0$

Let $E = \{ r \in [0, +\infty) : (a_n r^n) \text{ is bounded} \}$

then $E \neq \emptyset$ (since $0 \in E$) so we set $R = \sup E$ → possibly $R = +\infty$ if $E$ unbounded.

- if $|z| < R$, then $\exists r$ s.t. $0 < r < R$ and $(a_n r^n)$ bounded, i.e. $|a_n r^n| \leq M$

  then $\sum_{n \geq 0} |a_n z^n| = \sum_{n \geq 0} |a_n r^n| \left( \frac{|z|}{r} \right)^n \leq M \sum_{n \geq 0} \left( \frac{|z|}{r} \right)^n$ convergent (geometric series with $\frac{|z|}{r} < 1$)

  hence $\sum a_n z^n$ is ACV

- if $|z| > R$ then $(a_n z^n)$ is not bounded hence $\lim_{n \to \infty} a_n z^n \neq 0$ and $\sum a_n z^n$ is DV.

Obviously such a $R$ is unique □

**Comment**:

- $R = 0 \iff S(z) = \sum_{n \geq 0} a_n (z - z_0)^n$ CV only for $z = z_0$
- $R = +\infty \iff S(z) = \sum_{n \geq 0} a_n (z - z_0)^n$ CV $\forall z \in \mathbb{C}$
- $R = \sup \{ r \in [0, +\infty) : (a_n r^n) \text{ is bounded} \}$

  $= \sup \{ |z - z_0| : \sum a_n (z - z_0)^n \text{ CV} \}$

  $= \inf \{ |z - z_0| : \sum a_n (z - z_0)^n \text{ DV} \}$

**Remark**: we can't conclude when $|z|=R$.

**Eg**: • $\sum_{n\geqslant 0} z^n$, $R=1$ but $\sum z^n$ DV for all $|z|=1$

• $\sum_{n\geqslant 1} \frac{z^n}{n}$, $R=1$ but $\sum \frac{z^n}{n}$ diverges for $z=1$

and $\sum \frac{z^n}{n}$ CV for $|z|=1$ and $z\neq 1$

• $\sum_{n\geqslant 1} \frac{z^n}{n^2}$, $R=1$ but $\sum \frac{z^n}{n^2}$ ACV for $|z|=1$

**Theorem**: (d'Alembert ratio test for power series)

Let $S(z)=\sum a_n (z-z_0)^n$ be a power series s.t. $a_n\neq 0$ for $n$ big enough.

Assume that $\ell = \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists or is $+\infty$

Then the radius of convergence of $\sum a_n (z-z_0)^n$ is $R=\frac{1}{\ell}$ $\left(\text{where } \frac{1}{0}=\infty \text{ and } \frac{1}{\infty}=0\right)$

$\Delta$ $\left|\frac{a_{n+1} z^{n+1}}{a_n z^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |z| \longrightarrow \ell |z|$ (or $+\infty$ if $\ell=+\infty$)

then by d'Alembert ratio test if $|z|<\frac{1}{\ell}$, $\sum a_n z^n$ ACV

if $|z|>\frac{1}{\ell}$, $\sum a_n z^n$ DV

Hence $R=\frac{1}{\ell}$

$\square$

**Theorem**: Let $S(z)=\sum_{n\geqslant 0} a_n (z-z_0)^n$ be a power series.

If $\ell = \lim_{n\to+\infty} |a_n|^{1/n}$ exists or is $+\infty$ then the radius of CV of $S$ is $R=\frac{1}{\ell}$.

$\Delta$ • Let $z\in\mathbb{C}$ be s.t. $|z|<1/\ell$ then $\lim_{n\to+\infty} |a_n z^n|^{1/n} < 1$

take $r$ s.t. $\lim_{n\to+\infty} |a_n z^n|^{1/n} < r < 1$ then $|a_n z^n|^{1/n} < r$ for $n\geqslant N$ ↳ big enough

then $\sum_{n\geqslant 0} |a_n z^n| < \sum_{n\geqslant 0} r^n$ CV (geometric series)

ie $\sum a_n z^n$ is ACV

• Let $z\in\mathbb{C}$ s.t. $|z|>1/\ell$ then $\lim |a_n z^n| \neq 0$ so $\sum a_n z^n$ DV

• CCL: $R=1/\ell$

$\square$

**Proposition** : (sum of power series)

Let $S_A(z) = \sum_{n\geq 0}' a_n z^n$ and $S_B(z) = \sum_{n\geq 0}' b_n z^n$ be two power series respectively of radii $R_A$ and $R_B$

Define $S_{A+B}(z) := \sum'(a_n+b_n) z^n$ and denote its radius of convergence by $R_{A+B}$

**Then** : ① $R_{A+B} \geq \min(R_A, R_B)$

② if $R_A \neq R_B$ then $R_{A+B} = \min(R_A, R_B)$

③ if $|z| < \min(R_A, R_B)$ then $S_{A+B}(z) = S_A(z) + S_B(z)$

ie $\sum_{n\geq 0}' (a_n+b_n) z^n = \sum_{n\geq 0}' a_n z^n + \sum_{n\geq 0}' b_n z^n$

$\Delta$ Take $z$ st. $|z| < \min(R_A, R_B)$ then $\sum a_n z^n$ and $\sum b_n z^n$ are ACV, so

$\sum (a_n+b_n) z^n = \sum' (a_n z^n + b_n z^n)$ is ACV proving ①

$= \sum a_n z^n + \sum b_n z^n$ proving ③

For ② : assume that $R_A < R_B$. Take $r$ s.t. $R_A < r < R_B$

then $\sum a_n r^n$ DV
and $\sum b_n r^n$ CV
} hence $\sum (a_n+b_n) r^n$ DV

$\square$

**Proposition**: (Product of power series)

Let $S_A(z) = \sum_{n\geq 0}' a_n z^n$ and $S_B(z) = \sum_{n\geq 0}' b_n z^n$ be two power series respectively of radii $R_A$ and $R_B$.

Define $S_{AB}(z) = \sum_{n\geq 0}' c_n z^n$ where $c_n = \sum_{k=0}^{n}' a_k b_{n-k}$ and denote its radius of convergence by $R_{AB}$

**Then** : ① $R_{AB} \geq \min(R_A, R_B)$

② $|z| < \min(R_A, R_B) \Rightarrow S_{AB}(z) = S_A(z)\, S_B(z)$

ie $\sum_{n\geq 0} c_n z^n = \left(\sum_{n\geq 0} a_n z^n\right)\left(\sum_{n\geq 0} b_n z^n\right)$

$\Delta$ Same as before with Cauchy product $\square$

**Proposition**: Let $S(z)=\sum_{n\geq 0} a_n (z-z_0)^n$ be a power series with radius of convergence $R$ then $S$ is $\mathbb{C}$-differentiable / holomorphic / analytic on $D_R(z_0)$ and moreover $S'(z) = \sum_{n\geq 1} n a_n (z-z_0)^{n-1}$ whose radius of convergence is $R$ too.

$\Delta$ • The radius of convergence of $\sum n a_n (z-z_0)^{n-1}$ is $R$:

we denote the radius of convergence of $\sum n a_n (z-z_0)^{n-1}$ by $R'$ and we want to prove that $R'=R$

$\rightarrow$ $\sum n a_n (z-z_0)^n$ and $\sum n a_n (z-z_0)^{n-1}$ have same radius of convergence and $|a_n(z-z_0)^n| \leq |n a_n (z-z_0)^n|$

Hence if $\sum n a_n (z-z_0)^n$ is ACV so is $\sum a_n (z-z_0)^n$ so $R \geq R'$

$\rightarrow$ Let $z\in\mathbb{C}$ s.t. $|z-z_0|<R$ then $\exists \rho$ s.t. $|z-z_0|<\rho<R$

and $|n a_n (z-z_0)^n| = |a_n \rho^n| \left( n \left(\frac{|z-z_0|}{\rho}\right)^n\right)$

since $0<\rho<R_a$, $a_n\rho^n$ is bounded, ie $\exists M$ s.t. $|a_n \rho^n| \leq M$

hence $|n a_n (z-z_0)^n| \leq M n \left(\frac{|z-z_0|}{\rho}\right)^n \xrightarrow[n\to\infty]{} 0$

so $n a_n (z-z_0)^n$ is bounded $\Rightarrow |z-z_0| \leq R'$

and $R\leq R'$ ↳ remember that $R = \sup\{|z-z_0| : a_n(z-z_0)^n \text{ bounded}\}$

$\rightarrow$ Ccl: $R=R'$

• $S'(z) = \sum_{n\geq 1} n a_n z^{n-1}$

if you already know uniform convergence, it is not difficult to check that $S(z)$ and $S'(z)$ are uniformly convergent on $D_r(z_0)$ for $0\leq r<R$ which is enough to conclude... ↳ even normally convergent

otherwise, below is a "elementary proof": (You can skip it)

WLOG: $z_0=0$

Since $a^n - b^n = (a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\dots+ab^{n-2}+b^{n-1})$

$(z+h)^n - z^n = h\left((z+h)^{n-1}+(z+h)^{n-2}z+\dots+(z+h)z^{n-2}+z^{n-1}\right)$

Take $|z|<R$ and $h$ s.t. $|z+h|<R$

then $S(z)$ and $S(z+h)$ are ACV so that

$$\frac{S(z+h)-S(z)}{h} = \sum_{n\geq 1} a_n \left( (z+h)^{n-1} + (z+h)^{n-2} z + \dots + z^{n-1} \right)$$

and we also know that $\sum n a_n z^{n-1}$ is ACV (same radius)

so $$\frac{S(z+h)-S(z)}{h} - \sum_{n\geq 1} n a_n z^{n-1}$$

$$= \sum_{n=1}^{\infty} a_n \sum_{k=0}^{n-1} \left( (z+h)^{n-1-k} z^k - z^{n-1} \right)$$

$$= \sum_{n=1}^{\infty} a_n \sum_{k=0}^{n-1} z^k \left( (z+h)^{n-1-k} - z^{n-1-k} \right)$$

$$= \sum_{n=1}^{\infty} a_n \sum_{k=0}^{n-1} z^k h \left( (z+h)^{n-2-k} + (z+h)^{n-3-k} z + \dots + z^{n-2-k} \right)$$

so $$\left| \frac{S(z+h)-S(z)}{h} - \sum_{n\geq 1} n a_n z^n \right|$$

$$\leq |h| \sum_{n=1}^{\infty} |a_n| \sum_{k=0}^{n-1} \rho^k (n-1-k) \rho^{n-2-k}$$

pick $\rho$ s.t.

$|z|<\rho<R$

$|z+h|<\rho<R$

$$\leq |h| \sum_{n=1}^{\infty} |a_n| \binom{n}{2} \rho^{n-2}$$

But $\sum_{n=2}^{\infty} n(n-1) a_n \rho^n$ is CV by the first part of the proof twice

so $$\left| \frac{S(z+h)-S(z)}{h} - \sum n a_n z^{n-1} \right| \xrightarrow[h\to 0]{} 0$$

$\square$

Corollary: Let $S(z)=\sum_{m\geq 0} a_m (z-z_0)^m$ be a power series of radius $R$ then $S$ is infinitly many times $\mathbb{C}$-differentiable on $D_R(z_0)$ and

$$S^{(k)}(z) = \sum_{m=k}^{+\infty} \frac{m!}{(m-k)!} a_m (z-z_0)^{m-k}$$ whose radius of convergence is $R$

Particularly $a_m = \frac{S^{(m)}(z_0)}{m!}$

Eg: $e^z = \sum_{m\geq 0} \frac{z^m}{m!}$ on $\mathbb{C}$

① $$\left| \frac{\frac{z^{m+1}}{(m+1)!}}{\frac{z^m}{m!}} \right| = \frac{|z|}{m+1} \xrightarrow[m\infty]{} 0$$

so the radius of CV of $\sum \frac{z^m}{m!}$ is $\infty$

② $$\left(\sum_{m\geq 0} \frac{z^m}{m!}\right)' = \sum_{m\geq 1} \frac{z^{m-1}}{(m-1)!} = \sum_{m\geq 0} \frac{z^m}{m!}$$

③ $$\frac{\partial}{\partial z}\left(e^{-z} F(z)\right) = -e^{-z} F(z) + e^{-z} F'(z)$$ where $F(z) = \sum_{m\geq 0} \frac{z^m}{m!}$

$$= 0$$

④ hence $e^{-z} F(z) = \lambda$ is constant, take $z=0$ then $\lambda = 1$

↳ $\mathbb{C}$ connected

⑤ $F(z) = e^z$

Eg: $$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \frac{1}{2}\left(\sum_{m=0}^{\infty} \frac{(iz)^m}{m!} + \sum_{m=0}^{\infty} \frac{(-iz)^m}{m!}\right)$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m}}{(2m)!}$$

series are ACV for $z \in \mathbb{C}$