7- Holomorphic functions

<u>Definition</u>: $U \subset \mathbb{C}$ open, $f: U \longrightarrow \mathbb{C}$, $z_0 \in U$

We say that f is $\mathbb{C}$-differentiable at z_0 (or holomorphic at z_0, or analytic* at z_0) if $\lim_{h\to 0} \dfrac{f(z_0+h)-f(z_0)}{h}$ exists and is finite (ie $\in \mathbb{C}$)

(or equivalently $\lim_{z\to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}$ exists and is finite)

then we set $f'(z_0) := \lim_{h\to 0} \dfrac{f(z_0+h)-f(z_0)}{h} = \lim_{z\to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}$

(* I don't like the term "analytic" because it means that f can be locally expressed as a power series: we will see later that holomorphic $\Leftrightarrow$ analytic, but we don't know that yet. But that's the word used in the textbook...)

The next theorem is VERY important because it highlights the fact that $\mathbb{C}$-differentiability is more rigid that $\mathbb{R}$-differentiability. (which is due to the fact that the complex multiplication plays a role in the definition of $\mathbb{C}$-differentiability)

<u>Theorem</u>: $U \subset \mathbb{C}$ open, $f: U \longrightarrow \mathbb{C}$, $z_0 = x_0 + iy_0 \in U$.

We assume that $f(x+iy) = u(x,y) + i v(x,y)$ and we set

$\tilde{U} = \{(x,y) \in \mathbb{R}^2 : x+iy \in U\}$, $\tilde{f}: \tilde{U} \longrightarrow \mathbb{R}^2$, $\tilde{f}(x,y) = (u(x,y), v(x,y))$

Then the following are equivalent:

1. f is $\mathbb{C}$-differentiable/holomorphic/analytic at z_0
2. $\tilde{f}$ is $\mathbb{R}$-differentiable at (x_0,y_0) and its partial derivatives satisfy the Cauchy-Riemann equations:

$$\begin{cases} \dfrac{\partial u}{\partial x}(x_0,y_0) = \dfrac{\partial v}{\partial y}(x_0,y_0) \\ \dfrac{\partial u}{\partial y}(x_0,y_0) = -\dfrac{\partial v}{\partial x}(x_0,y_0) \end{cases}$$

$\triangle$ Let $z_0 = x_0 + iy_0$, $z = x+iy$, $h = r+is$

$\Rightarrow$: then $f'(z_0) = \lim_{h\to 0} \dfrac{f(z_0+h) - f(z_0)}{h}$

is equivalent to $f(z_0+h) = f(z_0) + h f'(z_0) + |h| \varepsilon(h)$ where $\varepsilon(h) \xrightarrow[h\to 0]{} 0$

that we may rewrite

$$\tilde{f}(x_0+r, y_0+s) = \tilde{f}(x_0,y_0) + \begin{pmatrix} \mathrm{Re}(f'(z_0)) & -\mathrm{Im}(f'(z_0)) \\ \mathrm{Im}(f'(z_0)) & \mathrm{Re}(f'(z_0)) \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} + \sqrt{r^2+s^2}\, \varepsilon(r,s)$$

hence $\tilde{f}$ is $\mathbb{R}$-differentiable and

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \mathrm{Re}(f'(z_0)) & -\mathrm{Im}(f'(z_0)) \\ \mathrm{Im}(f'(z_0)) & \mathrm{Re}(f'(z_0)) \end{pmatrix}$$

$\Leftarrow$: assume that

$$\tilde{f}(x_0+r, y_0+s) = \tilde{f}(x_0,y_0) + \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} + \sqrt{r^2+s^2}\, \varepsilon(r,s)$$

then $f(z_0+h) = f(z_0) + (a+ib)h + |h|\varepsilon(h)$

so $\lim_{h\to 0} \dfrac{f(z_0+h) - f(z_0)}{h} = a+ib$ $\square$

In the above proof, we also showed that, if f is holomorphic/analytic at $z_0 = x_0 + iy_0$ then

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i \frac{\partial v}{\partial x}(x_0,y_0)$$

The Cauchy–Riemann equations

The Cauchy–Riemann equations come in various equivalent flavours:

1. $\begin{cases} \dfrac{\partial u}{\partial x}(x_0, y_0) = \dfrac{\partial v}{\partial y}(x_0, y_0) \\ \dfrac{\partial u}{\partial y}(x_0, y_0) = -\dfrac{\partial v}{\partial x}(x_0, y_0) \end{cases}$ And in this case, $f'(z_0) = \dfrac{\partial u}{\partial x}(x_0, y_0) + i\dfrac{\partial v}{\partial x}(x_0, y_0)$.

2. $\mathrm{Jac}_{\tilde{f}}(x_0, y_0) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for some $a, b \in \mathbb{R}$. Then $f'(z_0) = a + ib$.

3. $\dfrac{\partial f}{\partial y}(z_0) = i\dfrac{\partial f}{\partial x}(z_0)$.

4. $\dfrac{\partial f}{\partial \bar{z}}(x_0, y_0) = 0$.

(3 and 4:) Then $f'(z_0) = \dfrac{\partial f}{\partial x}(z_0) = -i\dfrac{\partial f}{\partial y}(z_0) = \dfrac{\partial f}{\partial z}(z_0)$.

Remember that $\frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$ *and* $\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$.

Examples:

① $f : \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = \bar{z}$

- Let $z_0 \in \mathbb{C}$, we want to compute $\lim\limits_{h \to 0} \dfrac{f(z_0+h) - f(z_0)}{h} = \lim\limits_{h\to 0} \dfrac{\bar{h}}{h}$

Take $h = \rho e^{i\theta}$ then $\dfrac{\bar{h}}{h} = \dfrac{\rho e^{-i\theta}}{\rho e^{i\theta}} = e^{-2i\theta} \xrightarrow[\rho \to 0]{} e^{-2i\theta}$

hence $\lim\limits_{h\to 0} \dfrac{\bar{h}}{h}$ DNE and $z \mapsto \bar{z}$ is nowhere holomorphic

- $\tilde{f}(x,y) = (x,-y)$ is of course differentiable but doesn't satisfy the Cauchy-Riemann equation:

$$\operatorname{Jac}_{\tilde{f}}(x_0,y_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

② $f(z) = \operatorname{Re}(z)$

- $\lim\limits_{h\to 0} \dfrac{f(z_0+h) - f(z_0)}{h} = \lim\limits_{h\to 0} \dfrac{\operatorname{Re}(h)}{h}$

take $h = \rho \in \mathbb{R}$ then $\lim\limits_{\rho\to 0} \dfrac{\operatorname{Re}(h)}{h} = 1$

$\neq$ so $\lim\limits_{h\to 0} \dfrac{f(z_0+h)-f(z_0)}{h}$ DNE

take $h = i\rho$, $\rho \in \mathbb{R}$ then $\lim\limits_{\rho \to 0} \dfrac{\operatorname{Re}(h)}{h} = 0$

and f is nowhere holomorphic

- $\tilde{f}(x,y) = (x,0)$ is $\mathbb{R}$-differentiable

$\operatorname{Jac}\tilde{f} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ doesn't satisfy Cauchy-Riemann Eqn

③ $f(z) = |z|^2$

- $\underline{z_0 = 0}$: $\lim\limits_{h\to 0} \dfrac{f(h) - f(0)}{h} = \lim\limits_{h\to 0} \dfrac{|h|^2}{h} = 0$ so $f'(0) = 0$

- $\underline{z_0 \neq 0}$: $\lim\limits_{h\to 0} \dfrac{f(z_0+h) - f(z_0)}{h} = \lim\limits_{h\to 0} \dfrac{\bar{z}_0 h - z_0 \bar{h} + |h|^2}{h}$

$h = \rho \in \mathbb{R}$: $\lim\limits_{\rho\to 0} \dfrac{f(z_0+h) - f(z_0)}{h} = -\bar{z}_0 - z_0$

$\neq$

$h = i\rho$, $\rho \in \mathbb{R}$: $\lim\limits_{\rho\to 0} \dfrac{f(z_0+h) - f(z_0)}{h} = -\bar{z}_0 + z_0$

hence $\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$ DNE when $z_0 \neq 0$

<u>Conclusion</u>: f is holomorphic only at 0 and $f'(0)=0$

- $\tilde{f}(x,y) = (x^2+y^2, 0)$

$\text{Jac}_{\tilde{f}}(x_0,y_0) = \begin{pmatrix} 2x_0 & 2y_0 \\ 0 & 0 \end{pmatrix}$ which satisfies the Cauchy-Riemann conditions if and only if $x_0 = y_0 = 0$.

- $f(z) = e^z$, so that $f(x+iy) = e^{x+iy} = e^x \cos y + i e^x \sin y$.

$\tilde{f}(x,y) = (e^x \cos y, e^x \sin y)$ is differentiable

$\text{Jac}_{\tilde{f}}(x,y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$

hence f is holomorphic on $\mathbb{C}$ and $f'(z_0) = e^{x_0}\cos y_0 + i e^{x_0} \sin y_0 = e^{z_0}$

$z_0 = x_0 + i y_0$

<u>CCL</u>: exp is holomorphic on $\mathbb{C}$ and $\exp' = \exp$

<u>Proposition</u>: f, g holomorphic at z_0 and $\lambda \in \mathbb{C}$

1. λf is holomorphic at z_0 and $(\lambda f)'(z_0) = \lambda f'(z_0)$
2. $f+g$ holomorphic at z_0 and $(f+g)'(z_0) = f'(z_0) + g'(z_0)$
3. fg holomorphic at z_0 and $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$
4. If $f(z_0) \neq 0$ then $1/f$ is holomorphic at z_0 and $\left(\frac{1}{f}\right)' = -\frac{f'(z_0)}{f(z_0)^2}$

<u>Proposition</u>: f holomorphic at z_0, g holomorphic at $f(z_0)$ then $g \circ f$ is holomorphic at z_0 and $(g\circ f)'(z_0) = g'(f(z_0)) \cdot f'(z_0)$.

Eg: ① A polynomial function is holomorphic on $\mathbb{C}$ and

if $f(z) = \sum_{k=0}^{m} a_k z^k$ then $f'(z) = \sum_{k=1}^{m} k a_k z^{k-1}$

② $f(z) = e^{z^2}$ is holomorphic on $\mathbb{C}$ and $f'(z) = 2z e^{z^2}$

③ $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2}$ is holomorphic on $\mathbb{C}$ and

$$\cos'(z) = \frac{ie^{iz} - ie^{-iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin(z)$$

④ $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$ is holomorphic on $\mathbb{C}$ and

$$\sin'(z) = \frac{ie^{iz} + ie^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos(z)$$

Def: $U \subset \mathbb{R}^2$ open, $f: U \to \mathbb{R}$ C^2

The Laplacian of f is $\Delta f := \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$

Def: $U \subset \mathbb{R}^2$ open, $f: U \to \mathbb{R}$.

We say that f is harmonic if f is C^2 and $\Delta f = 0$ on U

Proposition: $U \subset \mathbb{C}$ open, $f: U \to \mathbb{C}$, $\tilde{U} = \{(x,y) \in \mathbb{R}^2 : x+iy \in U\}$

$f(x+iy) = u(x,y) + i v(x,y)$

If f is holomorphic on U then u and v are harmonic on $\tilde{U}$

Δ We will see later that if f is holomorphic then u, v are C^2

then $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial v}{\partial y}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{\partial v}{\partial x}\right) = 0$ by Clairaut theorem

Cauchy-Riemann equation

$\square$

Definition: Let $u, v: U \to \mathbb{R}$ be two harmonic functions.

We say that v is harmonic conjugate to u if $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

Cauchy-Riemann Equations

A natural question is: assume $f = u + iv$ is holomorphic, knowing u up to what extent is f determined? or equivalently, up to what extent is v determined?

open + connected ⚠

Theorem: $U \subset \mathbb{R}^2$ domain, $u, v_1, v_2 : U \to \mathbb{R}$ harmonic functions

If v_1 and v_2 are harmonic conjugate to u then they differ by a constant, ie $v_2 - v_1 = \lambda \in \mathbb{R}$

Δ $\partial_x(v_1 - v_2) = \partial_x v_1 - \partial_x v_2 = -\partial_y u + \partial_y u = 0$

similarly $\partial_y(v_1 - v_2) = 0$

Since U is connected $v_1 - v_2$ is constant $\square$

Remark: a harmonic function may not admit a harmonic conjugate

Eg: $u : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$, $u(x,y) = \log(x^2+y^2)$

Assume for the sake of contradiction that v is harmonic conjugate to u on $\mathbb{R}^2 \setminus \{0\}$

then $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic and $f'(x+iy) = \frac{2x}{x^2+y^2} - i\frac{2y}{x^2+y^2} = \frac{2}{x+iy}$

ie $f'(z) = \frac{2}{z}$, hence $\int_{S^1} f'(z)\,dz = 4i\pi$

but by Green's theorem $\int_{S^1} f'(z)\,dz = i\iint_{D_1(0)} 0 = 0$. Contradiction $\square$

Nonetheless we have the following theorem:

Theorem: $U \subset \mathbb{R}^2$ open and star-shaped. Let $u : U \to \mathbb{R}$ be harmonic.

Then there exists $v : U \to \mathbb{R}$ harmonic conjugate to u.

Δ $F = \left(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}\right)$ satisfies $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ since u is harmonic.

Hence by Poincaré lemma, $\exists v : U \to \mathbb{R}$ C^2 s.t. $F = Dv$

ie: $\left(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}\right) = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$ $\square$

We may actually weaken the assumptions of the statement and only assume that U is open and simply-connected (ie U has no hole) (see the slides)

By the way, we deduce from the above theorem that any harmonic function on an open simply connected domain is the real part of a holomorphic function.

Theorem: $U \subset \mathbb{C}$ a domain, $f = u + iv$ holomorphic on U.
If either u, or v, or u^2+v^2 is constant on U then f is constant on U.

Δ ① Assume that u is constant then $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$
By Cauchy-Riemann eqn: $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 0$
By connectedness, u and v are constant

② If v is constant: same proof

③ Assume that $\sqrt{u^2+v^2}$ is constant
1st case: $u^2+v^2 \equiv 0$ then $f \equiv 0$
2nd case: $u^2+v^2 \equiv c > 0$ then $c = |f|^2 = f\bar{f} \Rightarrow \bar{f} = \frac{c}{f}$
which implies that $\bar{f}$ is holomorphic
hence $\frac{f+\bar{f}}{2} = \mathrm{Re}(f)$ is holomorphic with constant
imaginary, hence $f+\bar{f}$ is constant by ②
hence $\mathrm{Re}(f)$ is constant and so is f by ① $\square$

Hence if the range of a holomorphic function defined on a domain lies on a horizontal line, or on a vertical line, or on a circle then this function is actually constant.
That's a first example of the rigidity of the holomorphic funct°; we will see more examples of this rigidity in the next lectures.